MATH101
Calculus & Analytic Geometry I

Lecture Notes

Chapter 3: The Derivative

Dr. Jawad Y. Abuihlail

Department of Mathematical Science
King Fahd University of Petroleum & Minerals
Box # 281, KFUPM
31261 Dhahran
Saudi Arabia

abuhlail@kfupm.edu.sa
http://faculty.kfupm.edu.sa/math/abuhlail

Chapter 3

The Derivative

3.1. Slopes & Rates of Change

Let $s = s(t)$ be the position of an object moving along the s-axis at time t. Then the **average velocity** of the object between times t_0 and t_1 is

$$v_{ave} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

and represents *geometrically* the **slope of the secant** joining $(t_0, s(t_0))$ and $(t_1, s(t_1))$.
The (**instantaneous**) **velocity** of the particle at $t = t_0$ is given by

$$v_{\mathbf{inst}} = \lim_{t \mapsto t_0} \frac{s(t) - s(t_0)}{t - t_0} \left(= \lim_{h \mapsto 0} \frac{s(t_0+h) - s(t_0)}{h} \right).$$

and represents *geometrically* the **slope of the tangent** at $t = t_0$.

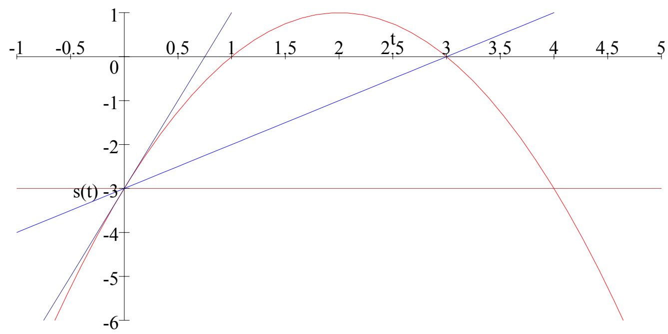
Example *Let* $s(t) = 1 - (t-2)^2$ *ft/sec*.

1. *The average velocity on* $[0,3]$ *is*

$$v_{ave} = \frac{s(3) - s(0)}{3 - 0} = \frac{0 - (-3)}{3 - 0} = 1 ft/sec.$$

2. *The instantaneous velocity at* $t = 0$ *is the slope of the tangent to the graph at* $(0,-3)$. *It's clear from the graph that the tangent passes also through* $(\frac{3}{4}, 0)$, *hence*

$$v_{inst} = \frac{s(\frac{3}{4}) - s(0)}{\frac{3}{4} - 0} = \frac{0 - (-3)}{\frac{3}{4} - 0} = 4\,ft/sec.$$



$s = 1 - (t-2)^2$; $s = t - 3$; $s = -3$; $s = 4t - 3$

Definition *Let $f(x)$ be a function with domain $\mathbb{D}$, and let $x_0, x_1 \in \mathbb{D}$. The* ***average rate of change*** *of $f(x)$, when x changes from x_0 to x_1 is given by*

$$r_{ave} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

and represents geometrically the ***slope of the secant*** *joining $(x_0, f(x_0))$ and $(x_1, f(x_1))$.*

Definition *Let $f(x)$ be a function with domain $\mathbb{D}$, and let x_0 be an* <u>*interior point*</u> *of $\mathbb{D}$. The (****instantaneous****)* ***rate of change*** *of $f(x)$ w.r.t. x at x_0 is*

$$r_{inst} = \lim_{x \mapsto x_0} \frac{f(x) - (x_0)}{x - x_0} \left(= \lim_{h \mapsto 0} \frac{f(x_0 + h) - f(x_0)}{h} \right)$$

and represents geometrically the ***slope of the curve*** *(****the tangent****) at the point $(x_0, f(x_0))$.*

Example *Let $f(x) = x^2$.*

1. *The average rate of change, when x changes from $x = 1$ to $x = 1.1$ is given by*

$$r_{ave} = \frac{(1.1)^2 - (1)^2}{1.1 - 1} = \frac{0.21}{0.1} = 2.1$$

2. *The instantaneous rate of change of $f(x)$ at $x = 1$ is given by*

$$\begin{aligned} r_{inst} &= \lim_{x \mapsto 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \mapsto 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x \mapsto 1} \frac{(x-1)(x+1)}{x - 1} = \lim_{x \mapsto 1} (x + 1) \\ &= 2. \end{aligned}$$

3. *The instantaneous rate of change of $f(x)$ at arbitrary x_0 is given by*

$$\begin{aligned} r_{inst} &= \lim_{x \mapsto x_0} \frac{f(x) - f(x_0)}{x - 1} \\ &= \lim_{x \mapsto x_0} \frac{x^2 - x_0^2}{x - x_0} \\ &= \lim_{x \mapsto x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} \\ &= \lim_{x \mapsto x_0} (x + x_0) \\ &= 2x_0. \end{aligned}$$

3.2. The Derivative

Definition *Let $f(x)$ be a function with domain $\mathbb{D}$ and x_0 be an interior point of $\mathbb{D}$. We say $f(x)$ is* ***differentiable at*** x_0*, if the following limit exists*

$$f'(x_0) =: \lim_{x \mapsto x_0} \frac{f(x)-f(x_0)}{x-x_0} \left(= \lim_{h \mapsto 0} \frac{f(x_0+h)-f(x_0)}{h}\right). \quad \#$$

Definition *If $\lim_{x \mapsto x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists, then $f'(x_0)$ is called the* ***slope of the graph of*** *$f(x)$ at $x = x_0$ (or at the point $(x_0, f(x_0))$ and the equation of the* ***tangent line to the graph of*** *$f(x)$* ***at*** *$(x_0, f(x_0))$ is given by*

$$y - f(x_0) = f'(x_0)(x - x_0).$$

If this limit does not exist, then the slope of the graph at $x = x_0$ is ***undefined***.

Definition *The* ***normal*** *to the graph of $y = f(x)$ at (x_0, y_0) is the line that is perpendicular to the tangent line of the curve at (x_0, y_0).*

Remark *If $f(x)$ is differentiable at $x = x_0$ with $f'(x_0) \neq 0$ then the slope of the normal to the curve of $y = f(x)$ at $(x_0, y_0))$ is $-\frac{1}{f'(x_0)}$ and the equation of the normal is given by*

$$y - y_0 = \frac{-1}{f'(x_0)}(x - x_0).$$

If $f'(x_0) = 0$ then the tangent line the curve $y = f(x)$ is the horizontal line $y = y_0$ and the normal is the vertical line $x = x_0$.

Example *Let*

$$f(x) = x^3 - x + 1.$$

1. *Then the derivative of* $f(x)$ *at* $x = 1$ *is*

$$\begin{aligned} f'(1) &= \lim_{x\mapsto 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\mapsto 1} \frac{(x^3-x+1)-1}{x-1} \\ &= \lim_{x\mapsto 1} \frac{x^3-x}{x-1} = \lim_{x\mapsto 1} \frac{x(x+1)(x-1)}{x-1} \\ &= \lim_{x\mapsto 1} x(x+1) = 2. \end{aligned}$$

2. *The slope of the graph of* $f(x)$ *at* $x = 1$ *is* $f'(1) = 2$.

   The equation of the tangent line *to the graph of* $f(x)$ *at* $x = 1$ *is*

$$y - f(1) = f'(1)(x-1) \Rightarrow y - 1 = 2(x-1),$$

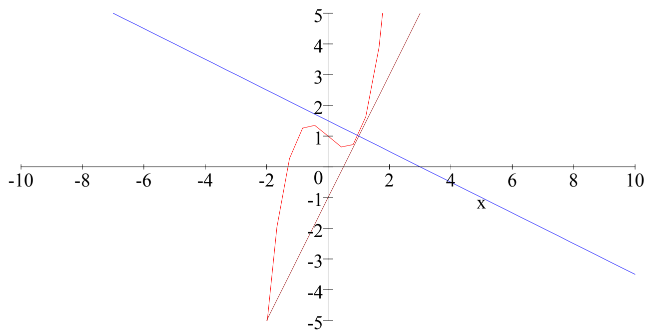
i.e. $y = 2x - 1$.

3. *The slope of the* perpendicular line *to the tangent of the graph at* $x = 1$ *is* $\frac{-1}{f'(1)} = \frac{-1}{2}$.

   The equation of the normal line *at* $x = 1$ *is*

$$y - f(1) = \frac{-1}{2}(x-1) \Rightarrow y - 1 = \frac{-1}{2}(x-1),$$

i.e. $y = \frac{-1}{2}x + \frac{3}{2}$.



$$y = x^3 - x + 1;\ y = 2x - 1;\ y = \frac{-1}{2}x + \frac{3}{2}$$

Remark *In contrast to tangents of circles, a tangent line to the graph of a function* $f(x)$ ***may*** *intersect the graph of* $f(x)$ *at points other than the point of tangency.*

Definition *Let $f(x)$ be a function with domain $\mathbb{D}$. The **derivative** of $f(x)$ is defined to be the function*

$$f'(x) = \lim_{w \mapsto x} \frac{f(w) - f(x)}{w - x} \left(= \lim_{h \mapsto 0} \frac{f(x+h) - f(x)}{h}\right).$$

with domain $D(f') = \{c \in \mathbb{D} \mid f'(c) \text{ exists}\}$.

Remark *Let $f(x)$ be a function with domain $\mathbb{D}$. The domain of $f'(x)$ is a subset of $\mathbb{D}$. We have in fact three cases:*

1. *$Domain(f'(x)) = \mathbb{D}$ (e.g. $f(x) = x^2$, $\mathbb{D} = \mathbb{R} = Domain(f'(x))$).*
2. *$\emptyset \neq Domain(f'(x)) \subsetneq \mathbb{D}$ (e.g. $f(x) = |x|$, $\mathbb{D} = \mathbb{R}$, $Domain(f'(x)) = \mathbb{R} - \{0\}$).*
3. *$Domain(f'(x)) = \emptyset$ (e.g.*

$$f(x) = \begin{cases} 1, & \text{for } x \in \mathbb{Q}, \\ 0, & \text{for } x \notin \mathbb{Q}. \end{cases}$$

$\mathbb{D} = \mathbb{R}$, $Domain(f'(x)) = \emptyset$). In this case we say $f(x)$ is nowhere differentiable.

Example *Let*

$$f(x) = x^3 - x + 1.$$

Then the derivative of $f(x)$ is

$$\begin{aligned} f'(x) &= \lim_{w \mapsto x} \frac{f(w)-f(x)}{w-x} \\ &= \lim_{w \mapsto x} \frac{(w^3-w+1)-(x^3-x+1)}{w-x} \\ &= \lim_{w \mapsto x} \frac{(w^3-x^3)-(w-x)}{w-x} \\ &= \lim_{w \mapsto x} \frac{(w-x)((w^2+wx+x^2)-1)}{w-x} \\ &= \lim_{w \mapsto x} (w^2 + wx + x^2 - 1) \\ &= 3x^2 - 1. \end{aligned}$$

The limit exists for each $x \in \mathbb{R}$, so the domain of $f'(x)$ is $\mathbb{R}$.

Example *Let $f(x) = |x|$. Then*

$$\lim_{w \mapsto 0^+} \frac{f(w) - f(0)}{w - 0} = \lim_{w \mapsto 0^+} \frac{w - 0}{w - 0} = 1,$$

whereas

$$\lim_{w \mapsto 0^-} \frac{f(w) - f(0)}{w - 0} = \lim_{w \mapsto 0^-} \frac{-w - 0}{w - 0} = -1,$$

so $f(x) = |x|$ is not differentiable at $x_0 = 0$. In fact $f(x) = |x|$ is differentiable on $\mathbb{R} \backslash \{0\}$.

Definition *Let $f(x)$ be a function with domain $\mathbb{D}$.*

1. *Let b be a right endpoint in $\mathbb{D}$ (e.g. $\mathbb{D} = (a,b]$, $\mathbb{D} = [a,b]$, $\mathbb{D} = (-\infty,b]$). Then $f(x)$ is said to be* ***differentiable at*** $x = b$ ***from the left****, if the following limit exists*

$$\lim_{x\to b^-} \frac{f(x)-f(b)}{x-b} \left(= \lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h}\right).$$ #

2. *Let a be a left endpoint in $\mathbb{D}$ (e.g. $\mathbb{D} = [a,b)$, $\mathbb{D} = [a,b]$, $\mathbb{D} = [a,\infty)$). Then $f(x)$ is said to be* ***differentiable at*** $x = a$ ***from the right****, if the following limit exists*

$$\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} = \left(\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}\right).$$ #

Remark *We say a function f is* ***differentiable on an interval*** *of the form $[a,b]$, $[a,b)$, $(a,b]$, $[a,\infty)$, $(-\infty,b]$, if it's differentiable at each interior point in that interval and differentiable at the endpoint(s) that it contains, from the left or the right, as applicable.*

The derivative $f'(x)$ of $f(x)$ can be interpreted as a function, whose value at each x_0 in its domain is:
the *slope* of the graph of $y = f(x)$ at $(x_0, f(x_0))$;
the *slope* of the tangent to the graph of $y = f(x)$ at $(x_0, f(x_0))$;
the *instantanous rate of change* of $y = f(x)$ with respect to x at x_0.

If $s = s(t)$ describes the position of an object moving along a straight line, then $s'(t)$ describes the (*instantaneous*) *velocity* $v = v(t)$ of the object at time t.

Example *Assume a ball is thrown vertically upward, so that its height (in feet) from the ground is given by*

$$s(t) = -16t^2 + 29t + 6, 0 \leq t \leq 2.$$

1. *the (instantaneous) velocity of the ball at* $t = 0.5$ *seconds is*

$$\begin{aligned}
v(0.5) = s'(0.5) &= \lim_{t \mapsto 0.5} \frac{s(t) - s(0.5)}{t - \frac{1}{2}} \\
&= \lim_{t \mapsto 0.5} \frac{(-16t^2 + 29t + 6) - 16.5}{t - \frac{1}{2}} \\
&= \lim_{t \mapsto 0.5} \frac{-16t^2 + 29t - 10.5}{t - \frac{1}{2}} \\
&= \lim_{t \mapsto 0.5} \frac{-16t^2 + 29t - 10.5}{\frac{2wt-1}{2}} \\
&= \lim_{t \mapsto 0.5} \frac{-32t^2 + 58t - 21}{2t - 1} \\
&= \lim_{t \mapsto 0.5} \frac{(2t-1)(-16t+21)}{2t - 1} \\
&= \lim_{t \mapsto 0.5} (-16t + 21) \\
&= 13\, ft/sec.
\end{aligned}$$

2. *the* (*instantaneous*) *velocity of the ball at t seconds is*

$$\begin{aligned} s'(t) &= \lim_{w \mapsto t} \frac{s(w)-s(t)}{w-t} \\ &= \lim_{w \mapsto t} \frac{(-16w^2+29w+6)-(-16t^2+29t+6)}{w-t} \\ &= \lim_{w \mapsto t} \frac{-16(w^2-t^2)+29(w-t)}{w-t} \\ &= \lim_{w \mapsto t} \frac{(w-t)(-16(w+t)+29)}{w-t} \\ &= \lim_{w \mapsto t} (-16(w+t)+29) \\ &= -32t + 29 \; ft/sec. \end{aligned}$$

3. *The initial velocity of the balls is*

$$v(0) = f'(0) = \lim_{t \mapsto 0^+} (-32t + 29) = 29 \; ft/sec.$$

4. *The final velocity of the ball is*

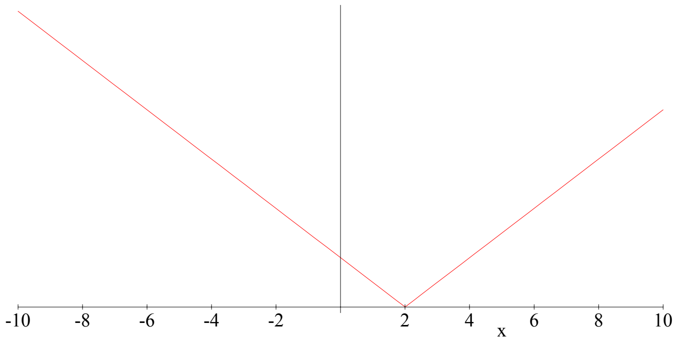
$$v(2) = f'(2) = \lim_{t \mapsto 2^-} (-32t + 29) = -35 \; ft/sec.$$

The negative sign mean that the ball is moving in the opposite direction (*downwards*).

When doesn't a function have a derivative at a point?

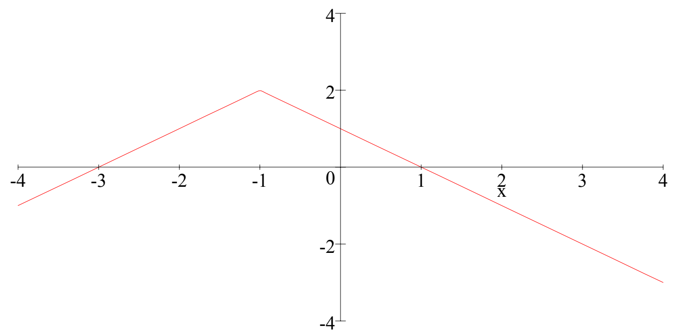
1. **Corners**

   Example: $f(x) = |x-2|$ is not differentiable at $x = 2$.



$$y = |x-2|$$

Example: $f(x) = 2-|1+x|$ is not differentiable at $x_0 = -1$.



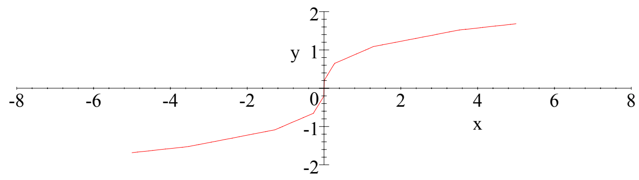
$$y = 2-|1+x|$$

2. **Vertical Tangent**

$$\lim_{x \to x_0} f'(x) = \infty \text{ or } -\infty.$$

Example: $f(x) = \sqrt[3]{x}$ is not differentiable at $x = 0$.

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^{\frac{1}{3}} - 0}{x - 0} = \lim_{x \to 0} \frac{1}{x^{\frac{2}{3}}} = \infty.$$



$y = \sqrt[3]{x}$

3. **Cusp**

$$\lim_{x\mapsto x_0^-} f'(x) = \infty, \quad \lim_{x\mapsto x_0^+} f'(x) = -\infty$$

or

$$\lim_{x\mapsto x_0^-} f'(x) = -\infty, \quad \lim_{x\mapsto x_0^+} f'(x) = \infty$$

Example: $f(x) = \sqrt[3]{x^2}$ is not differentiable at $x = 0$:

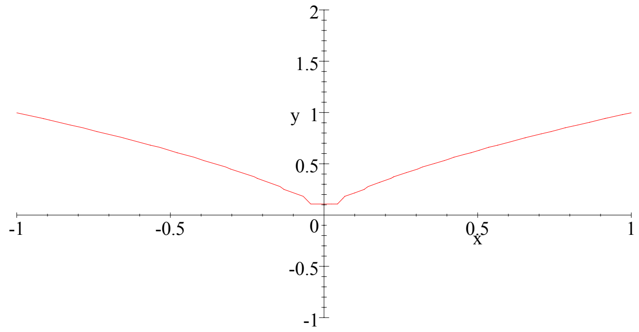
$$\frac{f(x) - f(0)}{x - 0} = \frac{x^{\frac{2}{3}} - 0}{x - 0} = \frac{1}{x^{\frac{1}{3}}}.$$

So

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{1}{x^{\frac{1}{3}}} = -\infty,$$

whereas

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{1}{x^{\frac{1}{3}}} = \infty,$$



$$y = \sqrt[3]{x^2}$$

4. **Discontinuity**

Example: The *unit step function*

$$U(x) = \begin{cases} 0, & 0 < x, \\ 1, & x \geq 0 \end{cases}$$

is not differentiable at $x_0 = 0$.

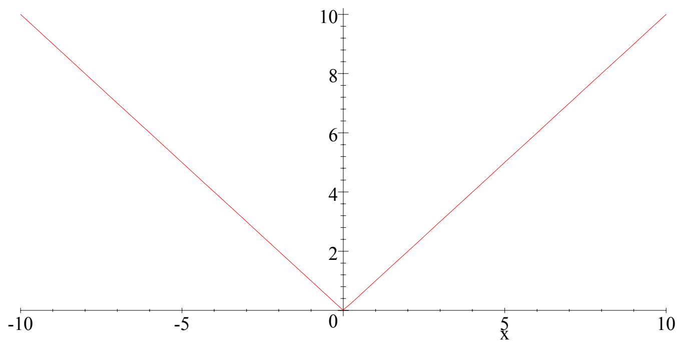
Theorem *If $f(x)$ is differentiable at $x = x_0$, then f is continuous at x_0, i.e.*

Differentiable $\Rightarrow$ *Continuous*.

Counter Example The following statement is false:

Continuous $\Rightarrow$ Differentiable

For example $f(x) = |x|$ is *continuous* at $x_0 = 0$, however it's *not differentiable* at $x_0 = 0$.



$y = |x|$

3.3. Techniques of Differentiation

Theorem *Let $f(x)$ and $g(x)$ be differentiable functions.*

y	$y' = \frac{dy}{dx}$
c (*constant*)	0
x^n $(n \in \mathbb{N})$	nx^{n-1}
$cf(x)$	$cf'(x)$
$f(x) \pm g(x)$	$f'(x) \pm g'(x)$
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
$\frac{f(x)}{g(x)}$ $(g(x) \neq 0)$	$\frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$ $(g(x) \neq 0)$

Example

y	$y' = \frac{dy}{dx}$
5	0
x^3	$3x^2$
$5x^3$	$15x^2$
$x^2 + x^4$	$2x + 4x^3$
$(x^2)(x^4) = x^6$	$(2x)x^4 + x^2(4x^3) = 6x^5$
$\frac{x+1}{x^2+1}$	$\frac{1(x^2+1) - 2x(x+1)}{(x^2+1)^2} = \frac{1-(x^2+2x)}{(x^2+1)^2}$

Theorem *Let $n \in \mathbb{Z}$. Then*

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Example

y	$y' = \frac{dy}{dx}$
$\frac{1}{x} = x^{-1}$	$-1 \bullet x^{-2} = \frac{-1}{x^2}$
$\frac{1}{x^2} = x^{-2}$	$-2 \bullet x^{-3} = \frac{-2}{x^3}$
$\frac{1}{x^3} = x^{-3}$	$-3 \bullet x^{-4} = \frac{-3}{x^4}$

Example *Let $y = 2x^3 - \frac{1}{x^2} + 4$. Then*

$$\begin{aligned} y' &= 6x^2 - (-2)x^{-3} \\ &= 6x^2 + \frac{2}{x^3}. \end{aligned}$$

Example *Let $f(x) = \frac{1}{x}$. The slope of the tangent to the graph of $f(x)$ at $x = 2$ is $f'(2) = \frac{-1}{(2)^2}$. The equation of the tangent line to the graph at $x = 2$ is*

$$y - \frac{1}{2} = \frac{-1}{4}(x - 2), \ i.e.\ y = -\frac{x}{4} + 1.$$

Let $y = f(x)$ be a differentiable function. If $f'(x)$ is itself differentiable, then the **second derivative** of $f(x)$ is

$$y'' := \frac{d^2y}{dx^2} = \lim_{w \mapsto x} \frac{f'(w) - f'(x)}{w - x} \left(= \lim_{h \mapsto 0} \frac{f'(x+h) - f'(x)}{h}\right).$$

As long as we have differentiability we continue the process of differentiating the derivatives to obtain the **third derivative** $y''' := \frac{d^3y}{dx^3}$, the **fourth derivative** $y^{(4)} := \frac{d^4y}{dx^4}$, the **fifth derivative** $y^{(5)} := \frac{d^5y}{dx^5}$, ... etc.

Example

y	$=$	$x^4 + 3x^3 - 6x^2 + 12x - 8$
y'	$=$	$4x^3 + 9x^2 - 12x + 12$
y''	$=$	$12x^2 + 18x - 12$
y'''	$=$	$24x + 18$
$y^{(4)}$	$=$	24
$y^{(5)}$	$=$	$0 = y^{(n)},\ n \geq 5.$

Remark *If* $p(x) = a_n x^n + a_{n-1} x^{n+1} + ... + a_1 x + a_0,\ a_n \neq 0$ *is a polynomial of degree* n *then* $p^{(n)}(x) = n! a_n$ *and* $p^{(k)}(x) = 0$ *for all* $k \geq n + 1$.

Velocity, Speed and Acceleration

An object is moving a long a coordinate line, so that its position s on that line is given by $s = f(t)$.
The **displacement** of the object (resp. the **average velocity**) over the time interval form t_1 to $t_1 + \Delta(t)$ is given by

$$\Delta s = f(t_1+\Delta t) - f(t_1)\ (\text{resp. } v_{ave} = \frac{\Delta s}{\Delta t}).$$

The **instantaneous velocity** (resp. the **acceleration**) of the object at time t is given by

$$v(t) = \frac{ds}{dt} = s'(t)\ (\text{resp. } a(t) := \frac{dv}{dt} = s''(t)).$$

Moreover we define

$$\text{speed} := |v(t)| = \left|\frac{ds}{dt}\right|.$$

Example *A rock thrown vertically upward from the surface of earth reaches, in absence of air, a height of*

$$s(t) = 24t - 4.9t^2 \textit{ meters in t seconds.}$$

1. *The rock's velocity is*

$$v(t) := s'(t) = 24 - 9.8t\ m/sec.$$

2. *The rock's acceleration is*

$$a(t) := s''(t) = -9.8\ m/sec^2.$$

3. *The rock reaches its* <u>*maximum height*</u>, *when* $v(t) = 0$, *i.e. when* $24 - 9.8t = 0$, *i.e.* $t = \frac{24}{9.8} \approx 2.4$ *sec.*

4. *Max. height*:

$$s(\frac{24}{9.8}) = 24(\frac{24}{9.8}) - 4.9(\frac{24}{9.8})^2 \approx 29.4\ m.$$

5. *Half of maximum height is approximately* $\frac{29.4}{2} = 14.7\ m$. *The time needed to reach a height of* $14.7\ m$ *is a solution of the eqn.* $24t - 4.9t^2 - 14.7 = 0$, *i.e.* $t_1 \approx 0.7$ *sec., when going up and* $t_2 \approx 4.2$ *sec., when going down.*

6. *The total time of the flight is given by solving the eqn.* $24t - 4.9t^2 = 0$, *i.e.* $t \approx 4.9$ *sec.* (*Note that the other solution* $\{t = 0\}$ *is* ***neglected***).

3.4. Derivatives of Trigonometric Functions

Assumption: Angles are measured in *radians*.

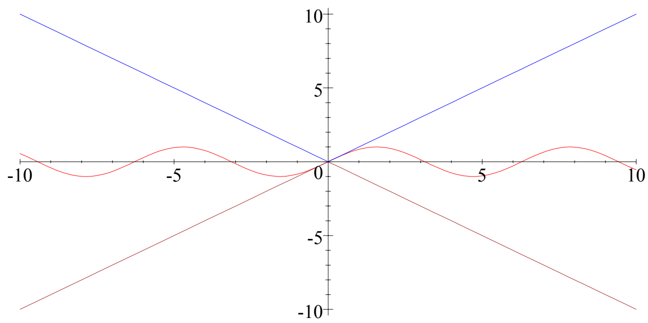
Theorem *If θ is measured in radians, then we have*

1) $-|\theta| < \sin(\theta) < |\theta|$

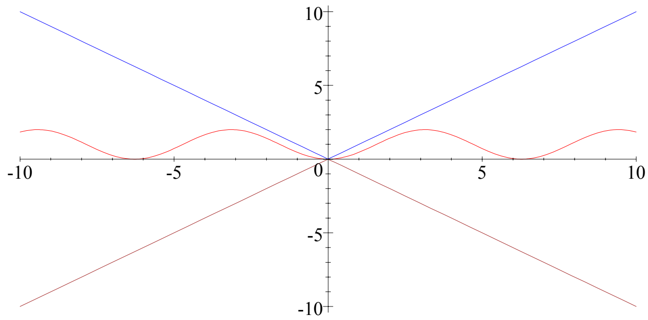
2) $-|\theta| < 1 - \cos(\theta) < |\theta|$

3) $\lim_{\theta \mapsto 0} \frac{\sin\theta}{\theta} = 1$

4) $\lim_{\theta \mapsto 0} \frac{1 - \cos\theta}{\theta} = 0.$



$y = \sin(\theta);\ y = |\theta|;\ y = -|\theta|$



$y = 1 - \cos(\theta);\ y = |\theta|;\ y = -|\theta|$

Theorem

$y = f(x)$	$y' = f'(x)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\cot(x)$	$-\csc^2(x)$

Example *Let* $f(x) = \sin(x)$. *Using definintion we find that*

$$
\begin{aligned}
f'(x) &= \lim_{h\mapsto 0} \frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\mapsto 0} \frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\
&= \lim_{h\mapsto 0} \frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h} \\
&= \lim_{h\mapsto 0} \frac{\sin(x)(\cos(h)-1)}{h} \\
&= + \lim_{h\mapsto 0} \frac{\cos(x)\sin(h)}{h} \\
&= \lim_{h\mapsto 0} \sin(x) \cdot \lim_{h\mapsto 0} \frac{\cos(h)-1}{h} \\
&= + \lim_{h\mapsto 0} \cos(x) \cdot \lim_{h\mapsto 0} \frac{\sin(h)}{h} \\
&= \sin(x) \cdot 0 + \cos(x) \cdot 1 = \cos(x).
\end{aligned}
$$

Example *Let* $f(x) = cos(x)$*. Using* <u>*definintion*</u> *we find that*

$$
\begin{aligned}
f'(x) &= \lim_{h\mapsto 0} \frac{cos(x+h)-cos(x)}{h} \\
&= \lim_{h\mapsto 0} \frac{cos(x)\,cos(h)-sin(x)\,sin(h)-cos(x)}{h} \\
&= \lim_{h\mapsto 0} \frac{cos(x)(cos(h)-1)-sin(x)\,sin(h)}{h} \\
&= \lim_{h\mapsto 0} \frac{cos(x)(cos(h)-1)}{h} \\
&= -\lim_{h\mapsto 0} \frac{sin(x)\,sin(h)}{h} \\
&= \lim_{h\mapsto 0} cos(x) \cdot \lim_{h\mapsto 0} \frac{cos(h)-1}{h} \\
&= -\lim_{h\mapsto 0} sin(x) \cdot \lim_{h\mapsto 0} \frac{sin(h)}{h} \\
&= cos(x) \cdot 0 - sin(x) \cdot 1 = -sin(x).
\end{aligned}
$$

Example

y	$=$	$tan(x) - csc(x) + x^3 - 2x^2 + x - 1.$
y'	$=$	$sec^2(x) + csc(x)\,cot(x) + 3x^2 - 4x + 1.$

3.5 The Chain Rule

Theorem *Let $g(x)$ be differentiable at $x = x_0$ and $f(x)$ be differentiable at $g(x_0)$. Then $(f \circ g)(x)$ is differentiable at $x = x_0$ and moreover*

$$\boxed{(f \circ g)'(x_0) = f'(g(x_0))g'(x_0).}$$ #

With $u = g(x)$ and $y = f(g(x)) = f(u)$, we can restate the chain rule as

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}.}$$ #

Example $y = sec(tan x)$.

Put $u := tan(x)$, so that $y = sec(u)$. Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \bullet \frac{du}{dx} \\ &= sec(u)\, tan(u) \bullet sec^2(x) \\ &= sec(tan x)\, tan(tan x) \bullet sec^2(x). \end{aligned}$$

Example $y = \frac{1}{(sin x + cos x)^2}$.

Put $u := sin x + cos x$, so that $y = \frac{1}{u^2}$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \bullet \frac{du}{dx} \\ &= \frac{-2}{u^3} \bullet (cos x - sin x) \\ &= \frac{-2(cos x - sin x)}{(sin x + cos x)^3}. \end{aligned}$$

Corollary *If $u(x)$ is a differentiable function and n is an integer, then u^n is differentiable and*

$$\boxed{\frac{d}{dx} u^n = n u^{n-1} \bullet \frac{du}{dx}.}$$ #

Example $r = (csc\,\theta + cot\,\theta)^{-1}$.

Put $u := csc\,\theta + cot\,\theta$, so that $r = u^{-1}$. Then

$$\begin{aligned} \frac{dr}{d\theta} &= \frac{dr}{du} \bullet \frac{du}{d\theta} \\ &= (-1)u^{-2} \bullet (-csc\,\theta\, cot\,\theta - csc^2\theta) \\ &= \frac{csc\,\theta\, cot\,\theta + csc^2\theta}{(csc\,\theta + cot\,\theta)^2} \\ &= \frac{csc\,\theta(cot\,\theta + csc\,\theta)}{(csc\,\theta + cot\,\theta)(csc\,\theta + cot\,\theta)} \\ &= \frac{csc\,\theta}{csc\,\theta + cot\,\theta}. \end{aligned}$$

3.6 Implicit Differentiation

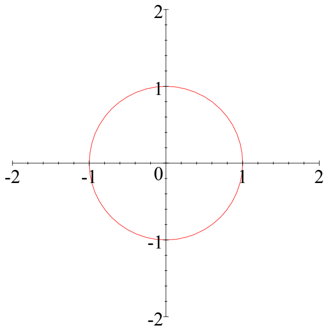
Definition *We say the a given equation in x and y* ***defines the function*** $y = f(x)$ ***implicitly****, if the graph of* $y = f(x)$ *coincides with a portion of the graph of that equation.*

Example *The equation*

$$x^2 + y^2 = 1$$

represents two functions implicitly:

$$f(x) = \sqrt{1 - x^2} \text{ and } g(x) = -\sqrt{1 - x^2}.$$



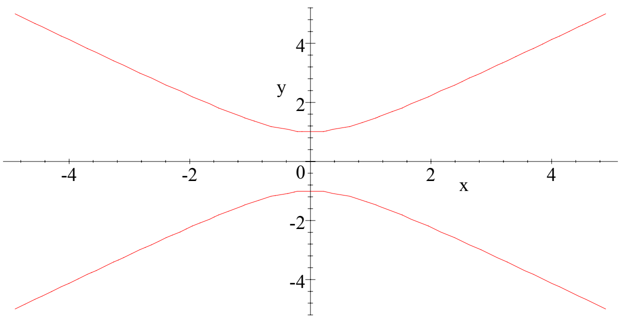
$$x^2 + y^2 = 1$$

Example *The equation*

$$y^2 - x^2 = 1$$

represents two functions implicitly:

$$f(x) = \sqrt{1 + x^2} \text{ and } g(x) = -\sqrt{1 + x^2}.$$



$$y^2 - x^2 = 1$$

How to differentiate Implicitly?

Given. An equation of the form

$$F(x,y) = c. \qquad \#$$

The Problem Find $\frac{dy}{dx}$.

The Procedure.

1. Differentiate both sides of the equation with respect to x (treating y as an explicit function of x).
   Use the various differentiation formulas wherever applicable.
   When finished, we will have an equation involving $x,y,$ and $\frac{dy}{dx}$, symbolically,
   $$G(x,y,\frac{dy}{dx}) = 0. \qquad \#$$
2. Solve the resulted equation for $\frac{dy}{dx}$.

Example $x^2y + xy^2 = 6.$

Differentiating bot sides of the eqn. we get

$$(2x \cdot y + x^2 \cdot \frac{dy}{dx}) + (1 \cdot y^2 + x \cdot 2y\frac{dy}{dx}) = 0.$$

Hence $(2xy + y^2) + (x^2 + 2xy)\frac{dy}{dx} = 0.$

We solve now for $\frac{dy}{dx}$ *and get*

$$\frac{dy}{dx} = -\frac{2xy + y^2}{x^2 + 2xy}.$$

Theorem *If n is a rational number, then*

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Example $y = x(x^2 + 1)^{1/2}.$

$$\begin{aligned} \frac{dy}{dx} &= 1 \cdot (x^2+1)^{1/2} + x \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x \\ &= \frac{(x^2+1)+x^2}{(x^2+1)^{1/2}} \\ &= \frac{2x^2+1}{(x^2+1)^{1/2}}. \end{aligned}$$

Higher Order Derivatives

1. Given an equation: $F(x,y) = c$. Differentiate implicitly and solve for y' to obtain an equation $y' = F_1(x,y)$.
2. Differentiate both sided of $y' = F_1(x,y)$ with respect to x to get an eqn. of the form $y'' = F_2(x,y,y')$.
3. Replace y' by $F_1(x,y)$ to get an eqn. of the form $y'' = F_3(x,y)$.
4. To find y''' repeat steps (1), (2) and (3) to get an eqn. of the form $y''' = F_4(x,y)$.
5. To find higher order derivatives keep repeating steps (1), (2) and (3).

Example $2\sqrt{y} = x - y.$

$2 \cdot \frac{1}{2\sqrt{y}} \cdot y' = 1 - y',\ \textit{hence}\ y'(\frac{1}{\sqrt{y}} + 1) = 1\ \textit{and}$

$y' = \frac{1}{(1/\sqrt{y}+1)} = \frac{\sqrt{y}}{1+\sqrt{y}}.$

$$\begin{aligned} y'' &= \frac{\frac{1}{2\sqrt{y}}y' \cdot (1+\sqrt{y}) - \sqrt{y} \cdot \frac{1}{2\sqrt{y}}y'}{(1+\sqrt{y})^2} \\ &= \frac{(\frac{1}{\sqrt{y}}y' + y') - y'}{2(1+\sqrt{y})^2} \\ &= \frac{\frac{1}{\sqrt{y}}(\frac{\sqrt{y}}{1+\sqrt{y}})}{2(1+\sqrt{y})^2} \\ &= \frac{1}{2(1+\sqrt{y})^3}. \end{aligned}$$

Example $2xy + \pi \sin y = 2\pi,\ \textit{Point}\ (1, \frac{\pi}{2}).$

Differentiating both sides of the eqn. we get

$(2 \cdot y + 2x \cdot \frac{dy}{dx}) + \pi \cos y \cdot \frac{dy}{dx} = 0,\ \textit{hence}$

$$\frac{dy}{dx} = -\frac{2y}{2x + \pi \cos y}$$

and $\frac{dy}{dx}\Big|_{(1,\pi/2)} = -\frac{2(\pi/2)}{2(1)+\pi(0)} = -\frac{\pi}{2}.$

Eqn. of the tangent line is:

$y - \frac{\pi}{2} = -\frac{\pi}{2}(x-1),\ \textit{i.e.}\ y = -\frac{\pi}{2}x + \pi.$

Slope of the normal line at $(1,\pi/2)$ *is* $\frac{-1}{(dy/dx)|_{(1,\frac{\pi}{2})}} = \frac{2}{\pi}.$

The eqn. of the normal line is

$y - \frac{\pi}{2} = \frac{2}{\pi}(x-1),\ \textit{i.e.}\ y = \frac{2}{\pi}x + (\frac{\pi}{2} - \frac{2}{\pi}).$

3.7 Related Rates

1. Identify the *rates of change* that are given and those to be found.
   Interpret each rate of change as a derivative of a variable w.r.t. t and provide a description of each variable involved.
2. Find an equation relating the quantities whose rates are identified in Step 1. (Draw a graph, if applicable).
3. Differentiate the equation in Step 2 with respect to t.
4. Evaluate the equation found in Step 3 using the known values for the quantities and the rates of change at the moment in question.
5. Solve for the value of the remaining rate of change at this moment.

3.8 Local Linear Approximation; Differentials

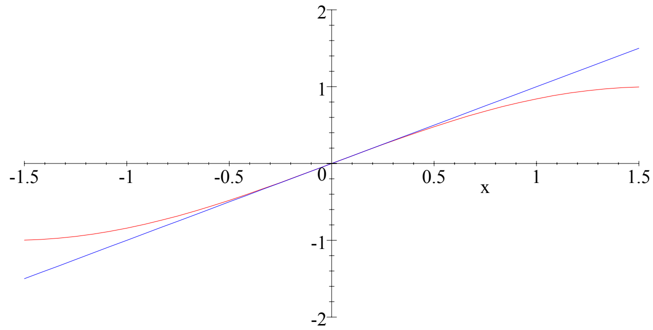
Definition *Let $f(x)$ be differentiable at $x = x_0$. The **local linear approximation of** $f(x)$ **at** $x = x_0$ is*

$$L(x) := f(x_0) + f'(x_0)(x - x_0).$$

*The approximation $f(x) \approx L(x)$ is the **standard approximation** of $f(x)$ at $x = x_0$. The point $x = x_0$ is the **center of approximation**.*

Example *Let $f(x) = sin(x)$. Then $f(x)$ is differentiable at $x = 0$ with $f'(0) = cos(0) = 1$. Hence in a small neighborhood about $x_0 = 0$ we have*

$$\begin{aligned} sin(x) &\approx f(0) + f'(0)(x-0) \\ &= 0 + 1(x-0) \\ &= x. \end{aligned}$$



$y = sin(x);\ y = x$

Example *$f(x) = \frac{1}{x}$ is diff. at $x_0 = 2$, $f'(x) = \frac{-1}{x^2}$, hence $f'(2) = \frac{-1}{4}$.*

$$\begin{aligned} \frac{1}{x} &\approx f(2) + f'(2)(x-2) \\ &= \frac{1}{2} + \frac{-1}{4}(x-2) \\ &= \frac{-1}{4}x + 1. \end{aligned}$$

Example *$f(x) = (1+x)^k$ (where $k \in \mathbb{Q}$) is differentiable at $x_0 = 0$ with $f'(0) = k$. Hence, in a small neighborhood about $x_0 = 0$ we have*

$$\begin{aligned} (1+x)^k &\approx f(0) + f'(0)x \\ &= 1 + kx. \end{aligned}$$

In particular

$$\begin{aligned} \frac{1}{1+x} &\approx 1 - x. \\ \sqrt{1+x} &\approx 1 + \frac{x}{2} \\ \sqrt[3]{1+x} &\approx 1 + \frac{x}{3} \end{aligned}$$

Example *Approximate $cos(31^\circ)$.*

$$\begin{aligned} cos(31^\circ) &= cos(\frac{31\pi}{180}) \\ &\approx cos(\frac{\pi}{6}) - sin(\frac{\pi}{6})(\frac{31\pi}{180} - \frac{\pi}{6}) \\ &= \frac{\sqrt{3}}{2} - \frac{1}{2}(\frac{\pi}{180}) \\ &= \frac{1}{2}(\sqrt{3} - \frac{\pi}{180}). \end{aligned}$$

Definition *Let $y = f(x)$ be a differentiable function. The **differential** dx is an independent variable. The **differential** dy is*

$$dy = f'(x)dx.$$

Let $y = f(x)$ be differentiable at x_0.
AC := **Absolute change**.
RC := **Relative change**
PC := **Percentage change**
As we move from x_0 to $x_0 + dx$ we can describe the change in the following ways

	Exact	**Estimated**
AC	$\Delta y = f(x + dx) - f(x_0)$	$dy = f'(x_0)dx$
RC	$\frac{\Delta y}{f(x_0)}$	$\frac{dy}{f(x_0)}$
PC	$\frac{\Delta y}{f(x_0)} \times 100\%$	$\frac{dy}{f(x_0)} \times 100\%$

Moreover we have

$$\text{Approximation Error} = |\Delta y - dy|.$$

Example $f(x) = \frac{1}{x}$, $x_0 = 0.5$, $dx = 0.1$

$f'(x) = \frac{-1}{x^2}$, $f'(0.5) = -4$.

1. *absolute change*

$$\begin{aligned}\Delta y &= f(x_0 + dx) - f(x_0) \\ &= f(0.5 + 0.1) - f(0.5) \\ &= \frac{10}{6} - \frac{10}{5} \\ &= \frac{-1}{3}\end{aligned}$$

2. *approximate change*

$$\begin{aligned}dy &= f'(x_0)dx \\ &= (-4)(0.1) \\ &= -0.4\end{aligned}$$

3. *approximation error*

$$\begin{aligned}\varepsilon &= |\Delta y - dy| \\ &= \left|\frac{-1}{3} - \left(\frac{-4}{10}\right)\right| \\ &= \frac{1}{15}.\end{aligned}$$

Derivatives in Economics

Let $c(x)$ be a function determining the **cost** of producing x units of some product. Then $\frac{\Delta c}{\Delta x}$ is the **average increase in cost** as a consequence of producing Δx more units. Moreover we set **marginal cost** := $c'(x) \approx$ the cost of producing one more unit.

If $r(x)$ is the **revenue** from selling x units, then the **marginal revenue** := $r'(x) \approx$ the increase in revenue resulting form selling one more unit.

Example *Let the cost of producing x units of some product be given by*

$$c(x) = 2000 + 100x - 0.1x^2.$$

(1) *The average cost for the first* 100 *items is*

$$\frac{\Delta c}{\Delta x} = \frac{c(100) - c(0)}{100} = 110 \textit{ dollars/items}.$$

(2) *The cost of producing the* 101*st item is* <u>*approximately*</u>

$$c'(100) = 100 - 0.2(100) = 80 \textit{ dollars}.$$