MATH101

Calculus & Analytic Geometry I

Lecture Notes

Chapter 2: Limits and Continuity

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Chapter 2 : Limits and Continuity

2.1. Limits (An Intuitive Approach)

Motivation: Instantaneous Velocity

Let a particle be moving along the s-axis, so that its position at time t is given by $s = s(t)$. Then the **average velocity** of the particle on the interval $[t_0, t_1]$ is

$$v_{\text{ave}} = \frac{\Delta s}{\Delta t} = \frac{s_1 - s_0}{t_1 - t_0} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}.$$

In order to *estimate* the (**instantanous**) **velocity** of the particle at $t = t_0$, we may consider the average velocity of the particle on intervals $[t_0, t]$ or $[t, t_0]$, where t is *very close* to t_0.

Example *Suppose a ball is thrown vertically upwards, so that its height in feet at time t is given by*

$$s(t) = -16t^2 + 29t + 6, \; 0 \le t \le 2.$$

To estimate the (*instantaneous*) *velocity of the particle at* $t = \frac{1}{2}$ *sec, we make the following list*

	t	Δt	$v_{ave} = \frac{\Delta s}{\Delta t}$
	0.5010	0.0010	12.9840
	0.5005	0.0005	12.9920
	0.5001	0.0001	12.9984
	0.5	0	*Undefined*
	0.4999	-0.0001	13.0016
	0.4995	-0.0005	13.0080
	0.4990	-0.0010	13.0160

From the list above one may conjecture that the (*instantaneous*) *velocity of the particle at* $t_0 = \frac{1}{2}$ *is* 13 *ft/sec*. *However this conjecture still needs a* ***Corroboration Evidence***!!

Tow-Sided Limits
General Definition

Let $f\ x$ be a function and a be a real number, such that $f\ x$ is defined on some open interval containing a (possibly $a \quad$ Domain f). If we can make the values of $f\ x$ *as close as we wish to L* by choosing x *sufficiently close to a* (from both sides), then we say: "the **(two-sided) limit** of $f\ x$ as x approaches
"the limit of $f\ x$ as x approaches x is L" and write

$$\lim_{x \quad a} f\ x \qquad L.$$

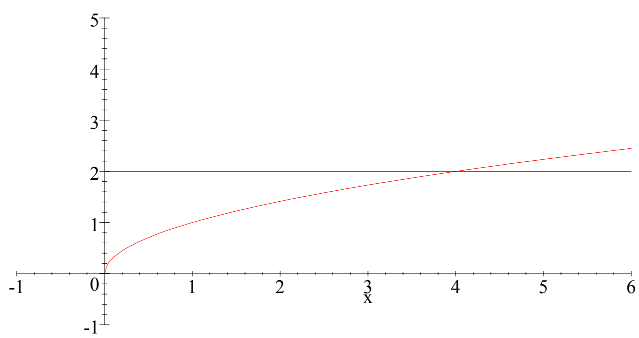
Example *Consider* $f\ x \qquad \sqrt{x}$ *at* $x_0 \qquad 4$. *In order to find* $\lim_{x \quad 4} f\ x$ *we consider the values of* $f\ x$ *at points very close to* $x_0 \qquad 4$ (*from both sides*):

x	$\sqrt{x}$
3.9	1.974841766
3.99	1.997498436
3.999	1.999749984
3.9999	1.999975
3.99999	1.9999975
4.00001	2.0000025
4.0001	2.000025
4.001	2.000249984
4.01	2.002498439
4.1	2.024845673

So one may conjecture that

$$\lim_{x \quad 4} \sqrt{x} \qquad 2.$$

This conjecture is supported by the graph of $f\ x \qquad \sqrt{x}$



$$f\ x \qquad \sqrt{x}\ ;\ y \qquad 2$$

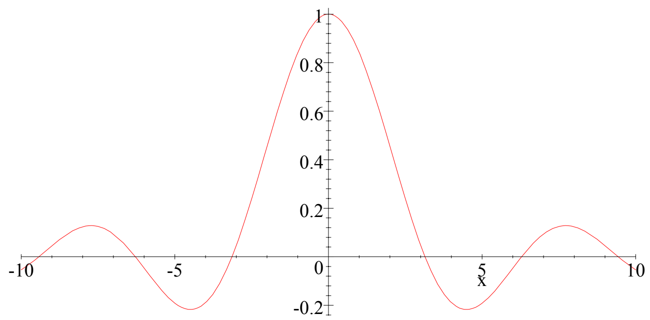
Example *Consider*

$$f(x) = \frac{\sin x}{x},\ x \neq 0.$$

Although $f(x)$ *is not defined at* $x = 0$, *one may ask if the (two-sided) limit of* $f(x)$ *as* x *approaches* 0 *exists? To make a conjecture about this we make a list of the values of the functions at points very close to* $x = 0$ *(from both sides):*

x	$\frac{\sin x}{x}$
0.1	0.9983341665
0.01	0.9999833334
0.001	0.9999998333
0.0001	0.9999999983

So one may conjecture that $\lim_{x \to 0} \frac{\sin x}{x} = 1$. *This conjecture is supported by the graph of the function:*



$$f(x) = \frac{\sin x}{x}$$

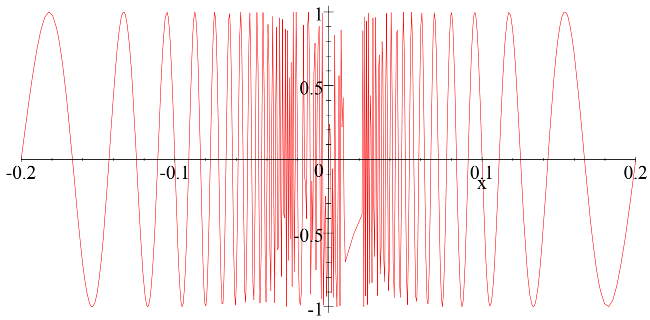
Example *Consider the function*

$$f(x) = \sin\frac{\pi}{x},\ x \neq 0.$$

In order to find $\lim_{x\to 0} f(x)$, *we consider, as usual, the values of* $f(x)$ *at points very close to* $x_0 = 0$ (*from both sides*):

x	$\sin\frac{\pi}{x}$
0.1	0
0.01	0
0.001	0
0.0001	0

On the basis of this table one may conjecture that $\lim_{x\to 0}\sin\frac{\pi}{x} = 0$. *However this conjecture is FALSE and it follows from the graph of* $f(x)$, *that* $\lim_{x\to 0}\sin\frac{\pi}{x}$ *DOES NOT EXIST*:



$$f(x) = \sin\frac{\pi}{x}$$

One Sided Limits

Sometimes one may be interested on the behavior of a function $f(x)$ as x a approaches $x = a$ from the left (i.e. the left-hand limit $\lim_{x \to a^-} f(x)$) or as x approaches a from the right (i.e. the right-hand limit $\lim_{x \to a^+} f(x)$).

Example *Consider the function* $f(x) = \sqrt{x}$. *The functions is NOT defined to the left of* $a = 0$. *So one is just interested in* $\lim_{x \to 0^+} \sqrt{x}$, *which is easily seen to be* 0.

Example *Consider the function*

$$f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0. \end{cases}$$

One easily sees that

$$\lim_{x \to 0^+} f(x) = 1, \text{ while } \lim_{x \to 0^-} f(x) = -1.$$

Obviously the two-sided limit $\lim_{x \to 0} f(x)$ *Does Not Exist.*

Definition 1. *Let* $f(x)$ *be a function, defined on* (x_0, a). *Then the* ***left-hand limit*** *of* $f(x)$ *as* x *approaches* a *from the left is* L, *if the values of* $f(x)$ *can be made as close as we like to* L *by taking the values of* x *sufficiently close to* a (*but less that* a).

2. *Let* $f(x)$ *be a function, defined on* (a, x_0). *Then the* ***right-hand limit*** *of* $f(x)$ *as* x *approaches* a *from the right is* L, *if the values of* $f(x)$ *can be made as close as we like to* L *by taking the values of* x *sufficiently close to* a (*but larger that* a).

Theorem *Let* $f(x)$ *be a function defined on an open interval* (x_1, x_2) *with* $x_1 < a < x_2$ (*with the possible exception of* $x = a$ *itself*). *Then*

$$\lim_{x \to a} f(x) \text{ exists} \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x);$$

(*i.e. the two-sided limit of* $f(x)$ *at* $x = a$ *exists if and only if, the one-sided limits exist and are equal*). *If this is the case, then*

$$\lim_{x \to a} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$

Example *Consider*

$$f(x) = \begin{cases} x^2 + 1, & x < 1 \\ 1 - x & x \geq 1 \end{cases}$$

Then

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 1) = 2;$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (1 - x) = 0.$$ *Now* $\lim_{x \to 1} f(x)$ *does not exists, since* $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$.

Example *Consider*

$$f\ x \quad \begin{cases} x^2 \quad 1, & x \quad 2 \\ x \quad 1 & x \quad 2 \end{cases}$$

Then

$$\lim_{x\ 2} f\ x \quad \lim_{x\ 2} \ x^2 \quad 1 \quad 3;$$

$$\lim_{x\ 2} f\ x \quad \lim_{x\ 2} \ x \quad 1 \quad 3.$$

Hence $\lim_{x\ 2} f\ x \quad 3$, *since*

$$\lim_{x\ 2} f\ x \quad 3 \quad \lim_{x\ 2} f\ x\ .$$

Vertical & Horizontal Asymptotes

Summary *If the values of* $f\ x$ *increase without bound as x approaches a from the left or from the right, then we write*

$$\lim_{x\quad a} f\ x \qquad \text{or} \quad \lim_{x\quad a} f\ x \qquad .$$

If he values of $f\ x$ *increase without bound as x approaches a from both sides, then we write* $\lim_{x\quad a} f\ x$.

If the values of $f\ x$ *decrease without bound as x approaches a from both sides, then we write*

$$\lim_{x\quad a} f\ x \qquad \text{or} \quad \lim_{x\quad a} f\ x \qquad .$$

If the values of $f\ x$ *decrease without bound as x approaches a from the left or from the right, then we write* $\lim_{x\quad a} f\ x$.

Remark $\lim_{x\quad a} f\ x$ (*respectively* $\lim_{x\quad a} f\ x$) *does not mean that the function has a limit as x approaches a. It just tells us that the values of* $f\ x$ *are increasing (respectively decreasing) indefinitely as x approached a.*

Definition *If*

$$\lim_{x\quad a} f\ x \qquad \text{or} \quad \lim_{x\quad a} f\ x \qquad ,$$

then we say the graph of $f\ x$ *has a* ***vertical asymptote*** $x \quad a$.

Definition *If*

$$\lim_{x} f\ x \quad L \text{ or } \lim_{x} f\ x \quad L,$$

then we say the graph of $f\ x$ *has a* ***horizontal asymptote*** $y \quad L$.

Remarks

1. If $f\ x \quad \frac{p\ x}{q\ x}$ ($p\ x$, $q\ x$ polynomials) is a rational function, then the zeros of $q\ x$ are *candidates* for the values of x at which the graph of $f\ x$ has vertical asymptotes.
2. An asymptote line to the graph of some function *may* intersect the graph of that function.
3. The graph of a function $f\ x$ can have *at most* two horizontal asymptotes, while it can have infinite number of vertical asymptotes (e.g. $f\ x \quad \tan x$).

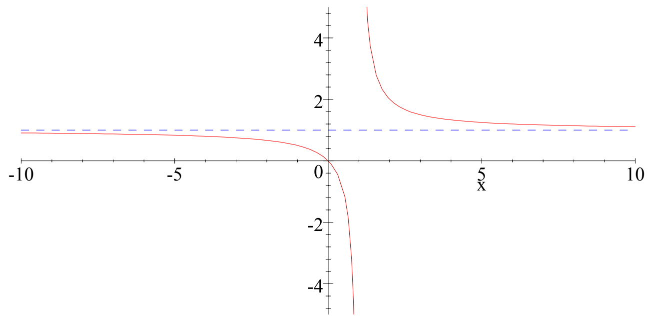
Example *To find the vertical asymptotes for the graph of the rational function* $f\;x \qquad \frac{x\;x\;\;1}{x^2\;\;1}$ *we find the one sides limits of* $f\;x$ *as* x *approaches* 1 *and* 1. *We get*

$$\lim_{x\quad 1} f\;x \qquad \frac{1}{2} \qquad \lim_{x\quad 1} f\;x\;,$$

while

$$\lim_{x\;\;1} f\;x \qquad\qquad \text{and} \;\; \lim_{x\;\;1} f\;x \qquad\quad .$$

So the graph of $f\;x$ *has* ***one*** *vertical asymptote at* $x \qquad 1$ (*there is no vertical asymptote at* $x \qquad 1$).



$$f\;x \qquad \frac{x\;x\;\;1}{x^2\;\;1};\; y \qquad 1$$

Summary *If the values of* $f\;x$

increase without bound as x increases without bound, then we write $\lim_{x} f\;x \qquad\quad ;$

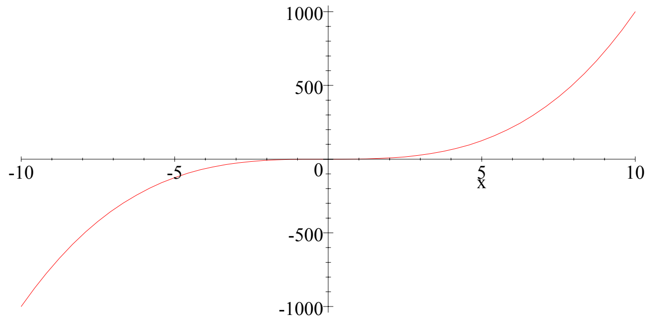
increase without bound as x decreases without bound, then we write $\lim_{x} f\;x \qquad\quad ;$

decrease without bound as x increases without bound, then we write $\lim_{x} f\;x \qquad\quad ;$

decrease without bound as x decreases without bound, then we write $\lim_{x} f\;x \qquad\quad .$

Example *Consider* $f\;x \qquad x^3$. *From the graph of* $f\;x$ *it's clear that*

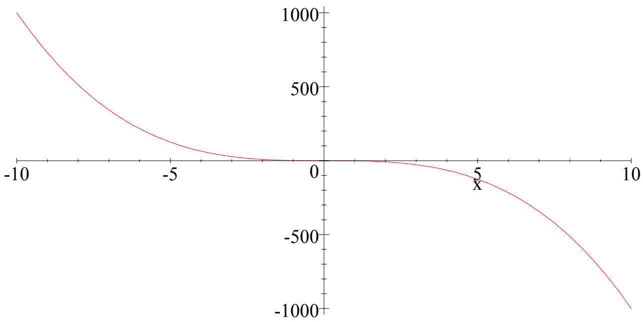
$$\lim_{x} f\;x \qquad\quad \text{and} \quad \lim_{x} f\;x \qquad\quad .$$



$$f\;x \qquad x^3$$

Example *Consider* $g\ x \quad x^3$*. It's clear from the graph of* $g\ x$ *that*

$$\lim_{x} g\ x \qquad \text{and} \quad \lim_{x} g\ x \qquad .$$

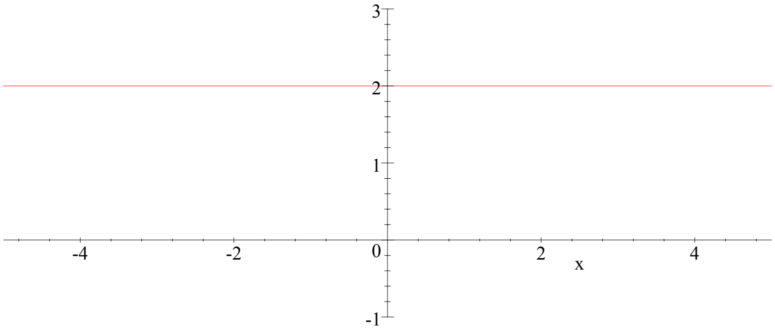


$g\ x \quad x^3$

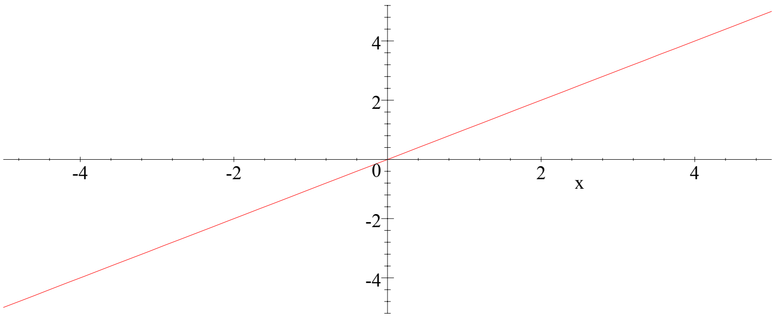
2.2. Computing Limits

Theorem *Let a and k be real numbers. Then:*

1. $\lim_{x \to a} k = k$.
2. $\lim_{x \to a} x = a$.



$y = 2$



$y = x$

Theorem *Let* $a \in \mathbb{R}$ *and suppose that*

$$\lim_{x \to a} f(x) = L_1 \ \& \ \lim_{x \to a} g(x) = L_2.$$

Then:

1. $\lim_{x \to a} (f + g)(x) = L_1 + L_2.$
2. $\lim_{x \to a} (f - g)(x) = L_1 - L_2.$
3. $\lim_{x \to a} (f \cdot g)(x) = L_1 \cdot L_2.$
4. $\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{L_1}{L_2}, L_2 \neq 0.$
5. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{L_1}$ (*provided* $L_1 \geq 0$ *if* n *is even*).

Moreover these statements remain true for the one-sided limits as $x \to a^{+}$ *or as* $x \to a^{-}$.

Remark *The converse of the previous theorem is not necessarily true*!!

Corollary *Let* $a, k \in \mathbb{R}$.

1. *If* $f(x)$ *is such that* $\lim_{x \to a} f(x) = L$, *then*

$$\lim_{x \to a} (kf)(x) = k \cdot L.$$

2. *If* $n \in \mathbb{N}$, *then*

$$\lim_{x \to a} x^n = a^n.$$

Theorem *For any polynomial*

$$p(x) = c_0 + c_1 x + \ldots + c_{n-1} x^{n-1} + c_n x^n$$

and any real number $a \in \mathbb{R}$, *we have*

$$\lim_{x \to a} p(x) = c_0 + c_1 \cdot a + \ldots. + c_{n-1} \cdot a^{n-1} + c_n \cdot a^n$$
$$= p(a).$$

Example

$$\lim_{x \to -2} (x^3 - 3x + 4) = (-2)^3 - 3(-2) + 4 = 2.$$

Theorem *Consider the rational function*

$$f\ x \qquad \frac{p\ x}{q\ x} \text{ (where } p\ x \text{ and } q\ x \text{ are polynomials).}$$

For any a $\quad R$:

$q\ a$	$p\ a$	$\lim_{x\ a} f\ x$
0	*any real number*	$\frac{p\ a}{q\ a}$
0	0	*Doesn't Exist* (*of*)
0	0	$\lim_{x\ a} \frac{p\ x\ /\ x\ a}{q\ x\ /\ x\ a}$

Example

$$\lim_{x\ 2} \frac{x^3 \quad 3}{x^2 \quad 1} \qquad \frac{2^{\ 3} \quad 3}{2^{\ 2} \quad 1} \qquad \frac{11}{3}.$$

Example

$$\lim_{x\ 2} \frac{1 \quad x^2}{x \quad 2} \qquad \text{and} \quad \lim_{x\ 2} \frac{1 \quad x^2}{x \quad 2} \qquad .$$

Example

$$\lim_{x\to 2}\frac{x^3-8}{x^2-4}=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}=\lim_{x\to 2}\frac{x^2+2x+4}{x+2}=\frac{12}{4}=3.$$

Example

$$\lim_{x\to 0}\frac{x}{\sqrt{x+1}-1}=\lim_{x\to 0}\frac{x(\sqrt{x+1}+1)}{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}=\lim_{x\to 0}\frac{x(\sqrt{x+1}+1)}{x+1-1}=\lim_{x\to 0}\frac{x(\sqrt{x+1}+1)}{x}=\lim_{x\to 0}(\sqrt{x+1}+1)=2.$$

Example

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt{x}-1}=\lim_{x\to 1}\frac{(x^{\frac{1}{3}}-1)(x^{\frac{2}{3}}+x^{\frac{1}{3}}+1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)(x^{\frac{2}{3}}+x^{\frac{1}{3}}+1)}=\lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{(x-1)(x^{\frac{2}{3}}+x^{\frac{1}{3}}+1)}=\lim_{x\to 1}\frac{\sqrt{x}+1}{x^{\frac{2}{3}}+x^{\frac{1}{3}}+1}=\frac{2}{3}.$$

Example *Let*

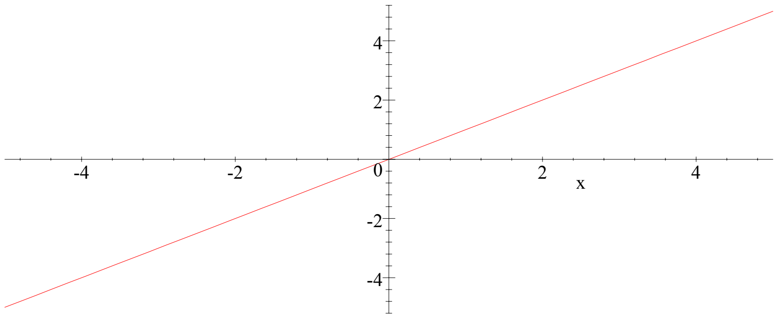
$$f(x)=\begin{cases}\frac{1}{x-1}, & x<1\\ x^3-x+1, & 1\le x<4\\ \sqrt{x+12} & x\ge 4\end{cases}$$

1. $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\frac{1}{x-1}=-\infty$.
2. $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(x^3-x+1)=(1)^3-1+1=1.$
3. $\lim_{x\to 1} f(x)$ *Doesn't Exist.*
4. $\lim_{x\to 4^-} f(x)=\lim_{x\to 4^-}(x^3-x+1)=(4)^3-4+1=61.$
5. $\lim_{x\to 4^+} f(x)=\lim_{x\to 4^+}\sqrt{x+12}=\sqrt{4+12}=4$
6. $\lim_{x\to 4} f(x)$ *Doesn't Exist.*

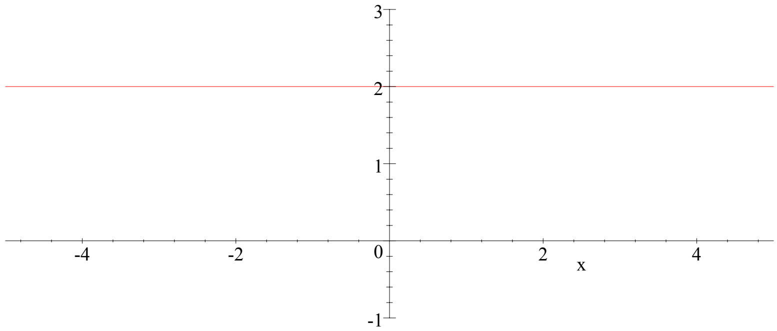
2.3. Computing Limits (End Behavior)

Theorem *Let k* R.

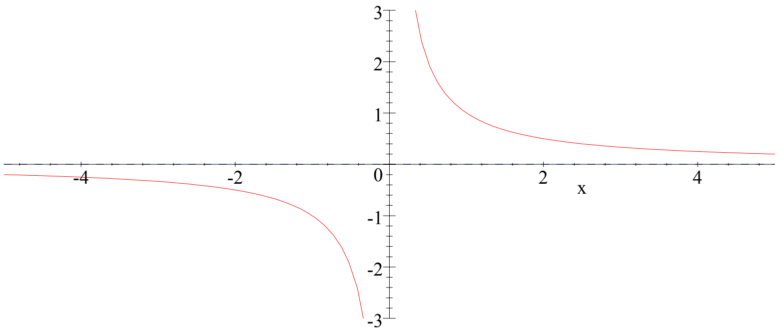
1. $\lim_{x} k \quad k$ *and* $\lim_{x} k \quad k$.
2. $\lim_{x} x \quad$ *and* $\lim_{x} x \quad$.
3. $\lim_{x} \frac{1}{x} \quad 0$ *and* $\lim_{x} \frac{1}{x} \quad 0$.



$y \quad x$



$y \quad 2$



$y \quad \frac{1}{x}$

Theorem *Suppose that*

$$\lim_{x} f\ x \quad L_1 \ \& \ \lim_{x} g\ x \quad L_2.$$

Then:

1. $\lim_{x} \ f \quad g \quad x \qquad L_1 \quad L_2.$
2. $\lim_{x} \ f \quad g \quad x \qquad L_1 \quad L_2.$
3. $\lim_{x} \ f \quad g \quad x \qquad L_1 \quad L_2.$
4. $\lim_{x} \ \frac{f}{g} \quad x \qquad \frac{L_1}{L_2},\ L_2 \quad 0.$
5. $\lim_{x} \sqrt[n]{f\ x} \qquad \sqrt[n]{L_1}$ (*provided* $L_1 \quad 0$ *if* n *is even*).

Moreover these statements remain true for limits as x .

Remark *The converse of the previous theorem is not necessarily true*!!

Corollary *Let* $p\ x \quad c_0 \quad c_1 x \quad .... \quad c_n x^n$ (*where* $c_n \quad 0$). *Then*

$$\lim_{x} \ p\ x \qquad \lim_{x} \ c_n x^n \ \& \ \lim_{x} \ p\ x \qquad \lim_{x} \ c_n x^n.$$

Theorem *Let*

$$f\ x \qquad \frac{c_n x^n \quad ... \quad c_1 x \quad c_0}{d_m x^m \quad ... \quad d_1 x \quad d_0},\ c_n \quad 0,\ d_m \quad 0.$$

Then

$$\lim_{x} \ f\ x \quad \lim_{x} \frac{c_n x^n}{d_m x^m},$$

namely

	$n \quad m$	$m \quad n$	$m \quad n$
$\lim_{x} \ f\ x$	$\frac{c_n}{d_m}$	*or*	0

Example

$$\lim_{x} \frac{3x^2 \quad 4x \quad 2}{2x^2 \quad 5} \qquad \lim_{x} \frac{3x^2}{2x^2} \qquad \frac{3}{2}.$$

Example

$$\lim_{x} \frac{2x^2 \quad 5x \quad 2}{x^3 \quad 5x^2 \quad 3} \qquad \lim_{x} \frac{2x^2}{x^3} \qquad \lim_{x} \frac{2}{x} \qquad 0.$$

Example

$$\lim_{x} \frac{2x^3 \quad 5x \quad 2}{5x^2 \quad 5x \quad 3} \qquad \lim_{x} \frac{2x^3}{5x^2}$$
$$\lim_{x} \frac{2}{5}x$$

and

$$\lim_{x} \frac{2x^3 \quad 5x \quad 2}{5x^2 \quad 5x \quad 3} \qquad \lim_{x} \frac{2x^3}{5x^2}$$
$$\lim_{x} \frac{2}{5}x$$

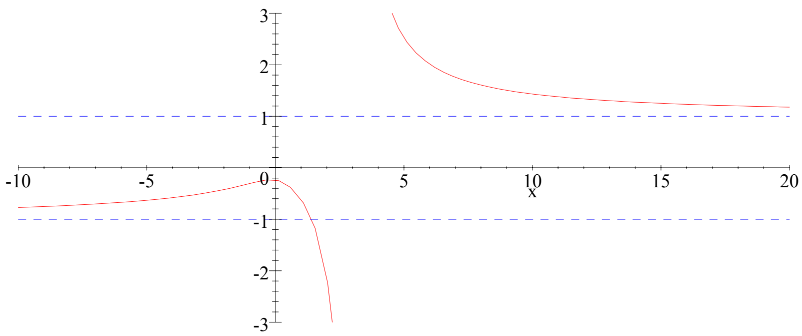
.

Example *To evaluate* $\lim_{x} \frac{\sqrt{4x^2 \quad 2}}{2x \quad 6}$ *we divide by* $\sqrt{x^2} \quad |x| \quad x$ *(since* x *) and get*

$$\lim_{x} \frac{\sqrt{4x^2 \quad 2}}{2x \quad 6} \qquad \lim_{x} \frac{\sqrt{4 \quad \frac{2}{x^2}}}{2 \quad \frac{6}{x}} \qquad \frac{\sqrt{4 \quad 0}}{2 \quad 0} \qquad 1.$$

To evaluate $\lim_{x} \frac{\sqrt{4x^2 \quad 2}}{2x \quad 6}$ *we divide by* $\sqrt{x^2} \quad |x| \quad x$ *(since* x *) and get*

$$\lim_{x} \frac{\sqrt{4x^2 \quad 2}}{2x \quad 6} \qquad \lim_{x} \frac{\sqrt{4 \quad \frac{2}{x^2}}}{2 \quad \frac{6}{x}} \qquad \frac{\sqrt{4 \quad 0}}{2 \quad 0} \qquad 1.$$



$$y \quad \frac{\sqrt{4x^2 \quad 2}}{2x \quad 6}$$

2.4. Limits (Discussed More Rigorously)

Example *Let f x 2x. Then*

$$\lim_{x \quad 1} f\,x \qquad 2.$$

To see that consider the following argument:

For 0.2 *we seek the largest possible* (?), *so that*

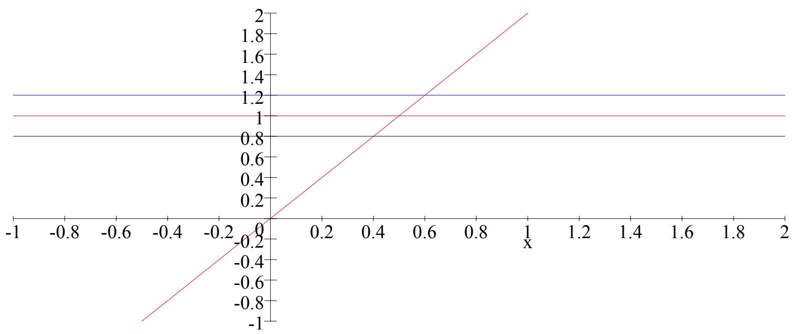
$$0 \quad d\,x,1 \qquad\qquad d\,f\,x\,,2$$

$$0 \quad |x \quad 1| \quad ? \quad |2x \quad 2| \quad 0.2$$

$$0 \quad |x \quad 1| \quad ? \quad 2|x \quad 1| \quad 0.2$$

So we should choose 0 0.1 *In general*

$$0 \qquad \overline{2}\,.$$



$y \quad 2x, \qquad 0.2$

Example *Let* $f\ x \quad x^2$. *So* $\lim_{x\ 2} f\ x \quad 4$.

For $\quad 1$ *we seek the largest possible* $\quad(\quad ?)$, *so that*

0	$d\ x, 2$				$d\ f\ x\ , 4$		
0	$\lvert x \quad 2\rvert$		?		$\lvert x^2 \quad 4\rvert$		1
0	$\lvert x \quad 2\rvert$		?		$\lvert x \quad 2\rvert\lvert x \quad 2\rvert$		1

$\lvert x^2 \quad 4\rvert \quad 1 \qquad 1 \quad x^2 \quad 4 \quad 1 \qquad 3 \quad x^2 \quad 5$, *so* $\sqrt{3} \quad x \quad \sqrt{5}$ (*ignore* $\sqrt{5} \quad x \quad \sqrt{3}$).

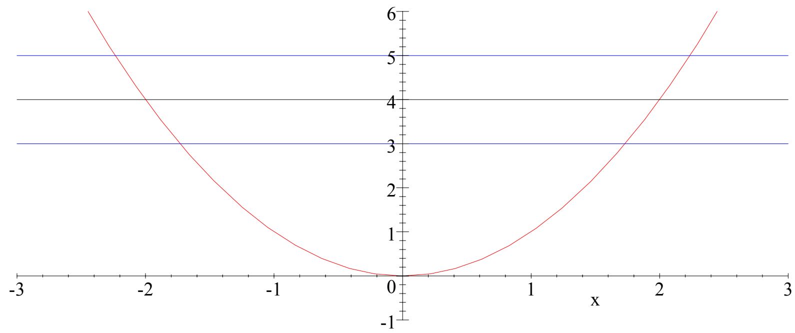
To get this we should have

$$\sqrt{3} \quad 2 \qquad\qquad x \quad 2 \qquad \sqrt{5} \quad 2$$

$$0.26795 \qquad\qquad\qquad 0.23607$$

Let $_1 : \left|\sqrt{3} \quad 2\right|$ *and* $_2 : \left|\sqrt{5} \quad 2\right|$ *and choose*

$$\min\ _1,\ _2 \qquad _2 \quad \sqrt{5} \quad 2.$$



$y \quad x^2, \qquad 1$

Definition *Let f x be defined in some open interval containing the real number c (f may not be defined at x c itself!!). Then*

$$\lim_{x \quad c} f\, x \quad L,$$

if given any number $\quad 0,$ *there exists a number* $\quad 0$ *such that*

$$0 \quad |x \quad c| \qquad\qquad |f\,x \quad L| \quad .$$

Example *Let f x* $\sqrt{x}$. *Then* $\lim_{x \quad 4} f\,x \quad 2.$

Given $\quad 0,$ *we need to find* $\quad ?,$ *such that*

$$0 \quad |x \quad 4| \qquad\qquad |\sqrt{x} \quad 2|$$

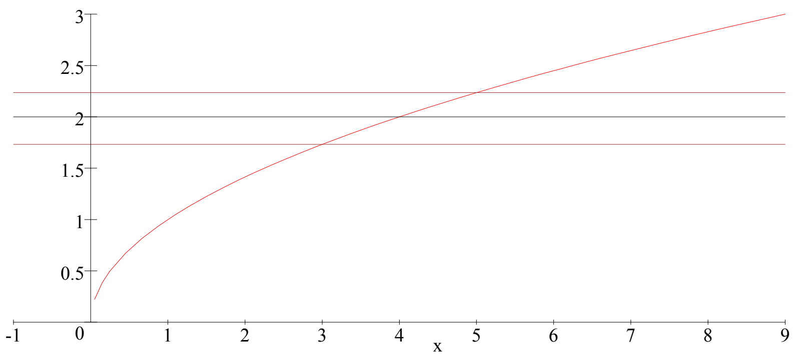
So if we restrict ourselves to x $\quad 3,5$, *then* $|\sqrt{x} \quad 2| \quad m$ *where m* $: \quad \sqrt{3} \quad 2$ *and so*

$$|x \quad 4| \quad |\sqrt{x} \quad 2||\sqrt{x} \quad 2| \quad m|\sqrt{x} \quad 2|.$$

Choosing $\quad min \;\; 1, m \quad$, *we get*

$$0 \quad |x \quad 4| \qquad\qquad |\sqrt{x} \quad 2| \quad .$$

If $|\sqrt{x} \quad 2| \quad$, *then* $|x \quad 4| \quad |\sqrt{x} \quad 2||\sqrt{x} \quad 2| \quad m|\sqrt{x} \quad 2| \quad m \qquad$ (*a contradiction*).



$f\,x \quad \sqrt{x}\,;\; y \quad \sqrt{3}\,;\; y \quad \sqrt{5}$

Definition *Let a* R *and f x be a function defined in the open interval* $\;a, b\;$ *for some real number b (f may not be defined at x a). Then*

$$\lim_{x \quad a} f\,x \quad L,$$

if given any number $\quad 0,$ *there exists a number* $\quad 0$ *such that*

$$a \quad x \quad a \qquad\qquad |f\,x \quad L| \quad .$$

Example *Let f x* $\sqrt{x}$. *Then* $\lim_{x \quad 0} f\,x \quad 0.$

Given $\quad 0,$ *take* $\quad ?,$ *such that*

$$0 \quad x \quad 0 \qquad\qquad |\sqrt{x} \quad 0| \quad .$$

Choose $0 \qquad\qquad {}^2.$

Definition *Let* $b \in \mathbb{R}$ *and* $f(x)$ *be a function defined in the open interval* (a, b) *for some real number* a *(*f *may not be defined at* $x = b$*). Then*

$$\lim_{x \to b^-} f(x) = L,$$

if given any number $\epsilon > 0$*, there exists a number* $\delta > 0$ *such that*

$$b - \delta < x < b \Rightarrow |f(x) - L| < \epsilon.$$

Example *Let* $f(x) = \sqrt{1 - x}$*. Then*

$$\lim_{x \to 1^-} f(x) = 0.$$

Given $\epsilon > 0$*, we seek the largest possible* $\delta > 0$*, so that*

$$1 - \delta < x < 1 \Rightarrow \left|\sqrt{1 - x} - 0\right| < \epsilon.$$

Notice that

$$1 - \delta < x < 1 \Leftrightarrow -1 < -x < \delta - 1$$

$$\Leftrightarrow 0 < 1 - x < \delta \Leftrightarrow 0 < \sqrt{1 - x} < \sqrt{\delta}.$$

So we may choose $0 < \delta \le \epsilon^2$.

Definition *Let* $f(x)$ *be defined on* (a, ∞) *for some* $a \in \mathbb{R}$*. Then*

$$\lim_{x \to \infty} f(x) = L,$$

if given any $\epsilon > 0$*, there exists* $N > 0$*, such that*

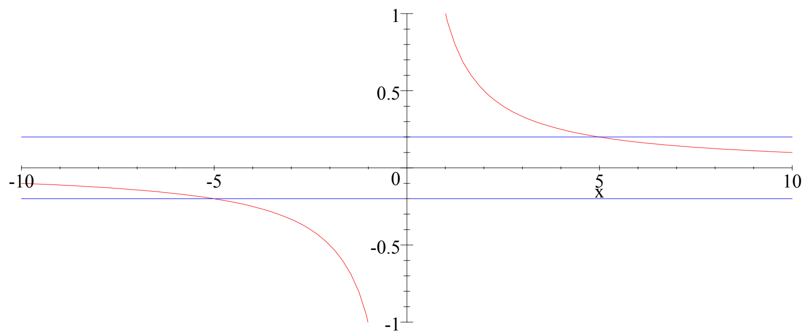
$$x > N \Rightarrow |f(x) - L| < \epsilon.$$

Example *Let* $f(x) = \frac{1}{x}$*. Then* $\lim_{x \to \infty} f(x) = 0$*.*

Given $\epsilon > 0$*, take* $N > ?$ *so that*

$$x > N \Rightarrow \left|\frac{1}{x} - 0\right| < \epsilon.$$

We may choose $N \ge \frac{1}{\epsilon}$.



$$f(x) = \frac{1}{x}, \quad \epsilon = 0.2$$

Definition *Let* $f\ x$ *be defined on* $\quad ,b$ *for some* $b \quad$ R. *Then*

$$\lim_{x} f\ x \qquad L,$$

if given any $\quad 0,$ *there exists* $N \qquad 0,$ *such that*

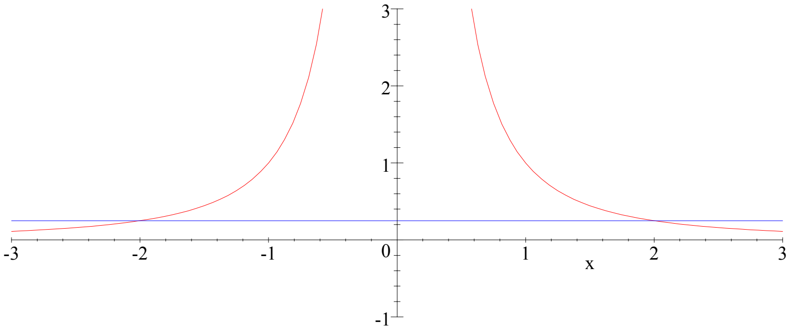
$$x \quad N \qquad |f\ x \quad L| \qquad .$$

Example *Let* $f\ x \qquad \frac{1}{x^2}$. *Then* $\lim_{x} f\ x \qquad 0.$

Given $\quad 0,$ *take* $N \qquad ?$ *so that*

$$x \quad N \qquad \left|\frac{1}{x^2} \quad 0\right| \qquad .$$

We may choose $N \qquad \frac{1}{\sqrt{\ }}.$



$$f\ x \qquad \frac{1}{x^2}, \qquad 0.25$$

Definition *Let* $f\ x$ *be defined in some open interval containing* a ($f\ x$ *may be not defined at* $x \quad a$). *Then*

$$\lim_{x \quad a} f\ x \qquad ,$$

if given any $M \quad 0,$ *there exists* $\quad M \quad 0$ *so that*

$$0 \quad |x \quad a| \qquad M \qquad f\ x \qquad M.$$

Definition *Let* $f\ x$ *be defined in some open interval containing* a ($f\ x$ *may be not defined at* $x \quad a$). *Then*

$$\lim_{x \quad a} f\ x \qquad ,$$

if given any $M \quad 0,$ *there exists* $\quad M \quad 0$ *so that*

$$0 \quad |x \quad a| \qquad M \qquad f\ x \qquad M.$$

Example *Let* $f\ x \qquad \frac{1}{x^2}$. *Then* $\lim_{x \quad 0} f\ x \qquad .$

Given $M \quad 0,$ *there exists* $\quad M \qquad ?$ *so that*

$$0 \quad |x \quad 0| \qquad M \qquad \frac{1}{x^2} \qquad M.$$

We may choose $\quad M \qquad \frac{1}{\sqrt{M}}.$

Definition *Let* $f\ x$ *be defined on* $a,$ *for some* a R. *Then*

1. $\lim_{x} f\ x$, *if given any* M 0 *there exists* $N\ M$ 0 *so that*

   $$x \quad N\ M \qquad f\ x \quad M.$$

2. $\lim_{x} f\ x$, *if given any* M 0 *there exists* $N\ M$ 0 *so that*

   $$x \quad N\ M \qquad f\ x \quad M.$$

Definition *Let* $f\ x$ *be defined on* $, b$ *for some* b R. *Then*

1. $\lim_{x} f\ x$, *if given any* M 0, *there exists* $N\ M$ 0, *such that*

   $$x \quad N\ M \qquad f\ x \quad M.$$

2. $\lim_{x} f\ x$, *if given any* M 0 *there exists* $N\ M$ 0 *so that*

   $$x \quad N\ M \qquad f\ x \quad M.$$

Example *Let* $f\ x \quad x^3.$

1. $\lim_{x} f\ x$.

   Given M 0, *find* $N\ M$? *so that*

   $$x \quad N\ M \qquad x^3 \quad M.$$

   Choose $N\ M \quad \sqrt[3]{M}.$

2. $\lim_{x} f\ x$.

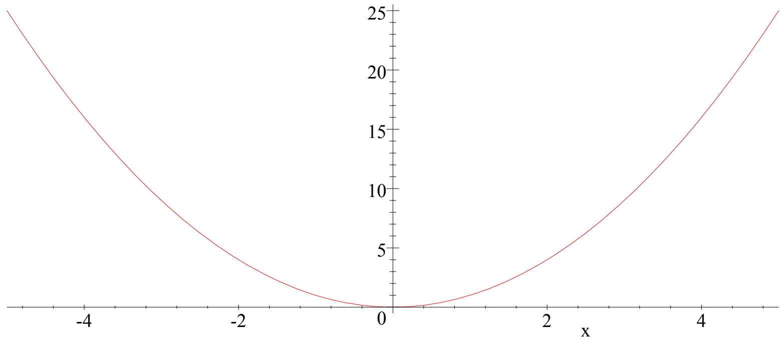
   Given M 0, *find* $N\ M$? *so that*

   $$x \quad N\ M \qquad x^3 \quad M.$$

   Choose $N\ M \quad \sqrt[3]{M}.$

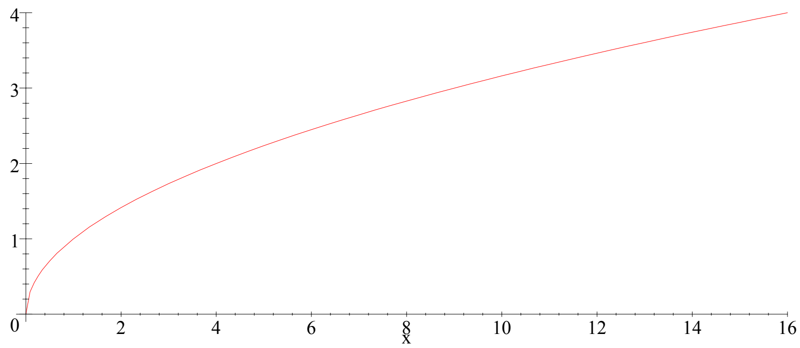
2.5. Continuity

Example $f(x) = x^2$



$y = x^2$

Example $f(x) = \sqrt{x}$



$y = \sqrt{x}$

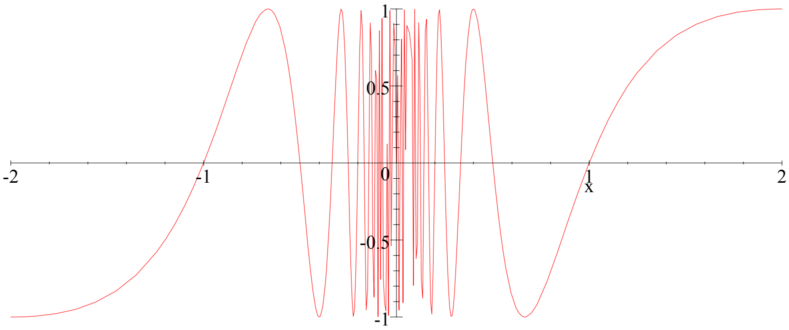
Definition *A function $f(x)$ defined on an open interval containing c is* ***continuous*** *at $x = c$, if:*

1. $f(c)$ *is defined.* 2. $\lim_{x \to c} f(x)$ *exists.*

3. $\lim_{x \to c} f(x) = f(c)$.

If one of the above conditions fails, then $f(x)$ has ***discontinuity*** *at $x = c$.*

Example $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$



$$y = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$f(x)$ is discontinuous at $x = 0$, since $\lim_{x \to 0} f(x)$ Doesn't Exist.

Theorem *Polynomials*

$$p(x) = c_0 + c_1 x + \ldots + c_n x^n, \; c_i \in \mathbb{R}$$

are continuous everywhere.

Theorem *Let $f(x)$ and $g(x)$ be defined on an open interval containing c and assume them to be continuous at $x = c$. Then:*

1. *$f \pm g$ is continuous at $x = c$.*
2. *$f \cdot g$ is continuous at $x = c$.*
3. *$f \circ g$ is continuous at $x = c$.*
4. *$\frac{f}{g}$ is continuous at $x = c$, if $g(c) \neq 0$ (If $g(c) = 0$ then $\frac{f}{g}$ is discontinuous at $x = c$).*

Remark *The converse of the previous theorem may not be true.*

Theorem *A rational function $f(x) = \frac{p(x)}{q(x)}$ (where $p(x)$ and $q(x)$ are polynomials) is continuous on $\mathbb{R} \setminus \{c : q(c) = 0\}$.*

Theorem *If*

1. $\lim_{x \to a} g(x) = L$; *and*
2. f *is continuous at* L,

then

$$\lim_{x \to a} f(g(x)) = f(L) = f\left(\lim_{x \to a} g(x)\right).$$

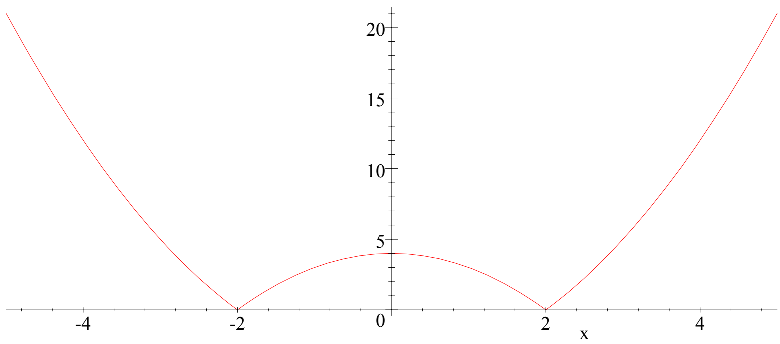
This result is also valid, if we replace $\lim_{x \to a}$ *by any one of* $\lim_{x \to a^+}$, $\lim_{x \to a^-}$, $\lim_{x \to \infty}$ *or* $\lim_{x \to -\infty}$.

Theorem *Let* f, g *be functions such that* $\text{Range}(g) \subseteq \text{Domain}(f)$.

1. *If* g *is continuous at* $x = c$ *&* f *is continuous at* $g(c)$, *then* $f \circ g$ *is continuous at* $x = c$.
2. *If* g *is continuous everywhere and* f *is continuous at each point in* $\text{Range}(g)$, *then* $f \circ g$ *is continuous everywhere.*

Remark *If* $f(x)$ *is continuous at* $x = a$, *then* $|f(x)|$ *is continuous at* $x = a$.

Example *Let* $f(x) = 4 - x^2$. *Then* $|f(x)| = |4 - x^2|$ *is continuous everywhere.*

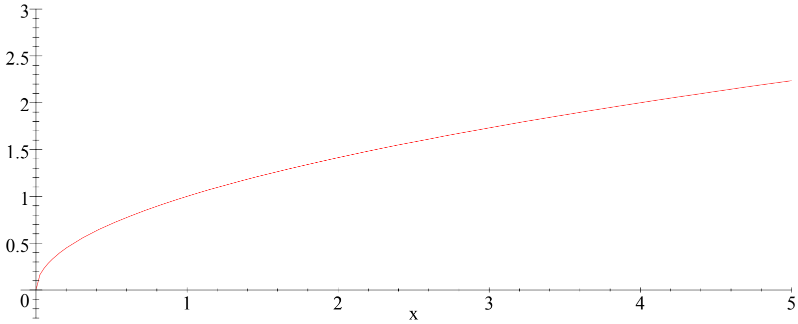


$$y = |4 - x^2|$$

Definition *Let* $c \in R$.

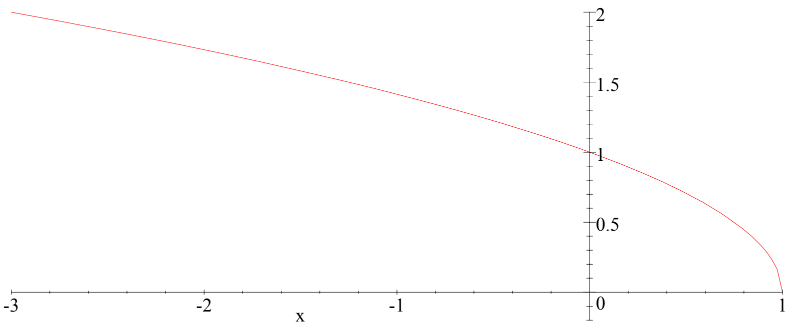
1. Let $f(x)$ be defined on $[c, b)$ for some $b \in R$. Then $f(x)$ is **continuous from the right** at c, if
$$\lim_{x \to c^+} f(x) = f(c).$$
2. Let $f(x)$ be defined on $(a, c]$ for some $a \in R$. Then $f(x)$ is **continuous from the left** at $x = c$, if
$$\lim_{x \to c^-} f(x) = f(c).$$

Example $f(x) = \sqrt{x}$ *is continuous from the right at* $x = c$.



$y = \sqrt{x}$

Example $f(x) = \sqrt{1 - x}$ *is continuous from the left at* $x = 1$.

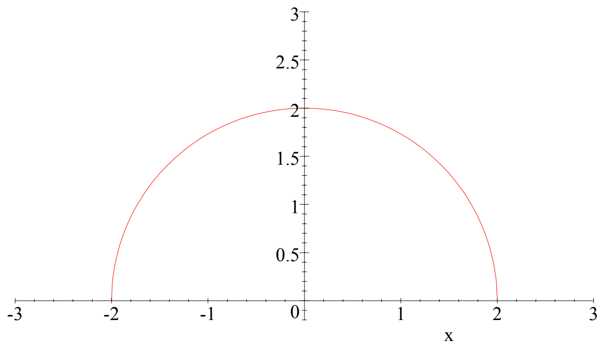


$y = \sqrt{1 - x}$

Definition *A function $f\ x$ is continuous on a,b, if it's continuous at each c a,b. It's continuous on a,b, if*

1. *f is continuous on a,b.*
2. *f is continuous from the right at x a.*
3. *f is continuous from the left at x b.*

Example $f\ x$ $\sqrt{4 \quad x^2}$ *is continuous on* $2,2$.



$f\ x$ $\sqrt{4 \quad x^2}$

Definition *A function $f\ x$ is*

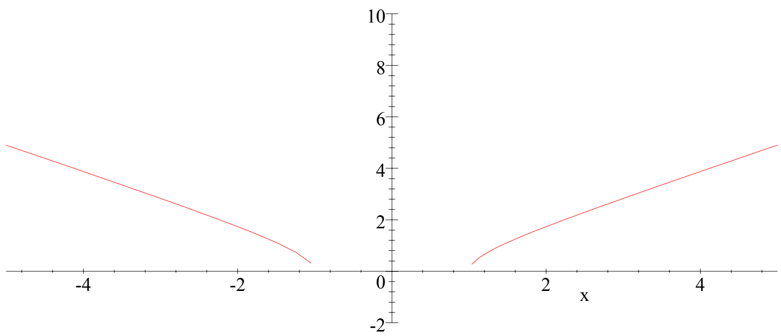
continuous on $a,$, if f is continuous at each c a.

continuous on $a,$, if f is continuous on $a,$ and f is continuous from the right at a (i.e. $\lim_{x\ a} f\ x$ $f\ a$).

continuous on $,b$, if f is continuous at each c a.

continuous on $,b$, if f is continuous at each c a and f is continuous at b from the left (i.e. $\lim_{x\ b} f\ x$ $f\ b$).

Example $f\ x$ $\sqrt{x^2 \quad 1}$ *is continuous on* $,\ 1 \quad 1,$.

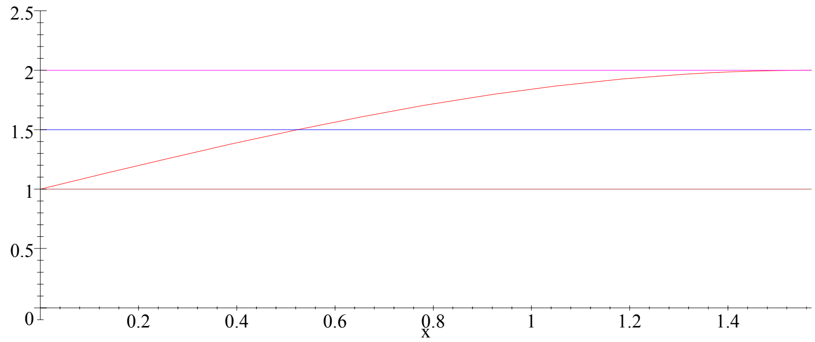


y $\sqrt{x^2 \quad 1}$

Intermediate Value Theorem

Theorem *Let $f\ x$ be continuous on a, b. If k is any real number between $f\ a$ and $f\ b$, inclusive, then there exists at least one $c\quad a, b$, such that $f\ c\quad k$.*

Example *Let $f\ x\quad \sin\ x\quad 1$ and consider the interval $0, \frac{}{2}$*



$$f\ x\quad \sin\ x\quad 1;\ y\quad 1.5$$

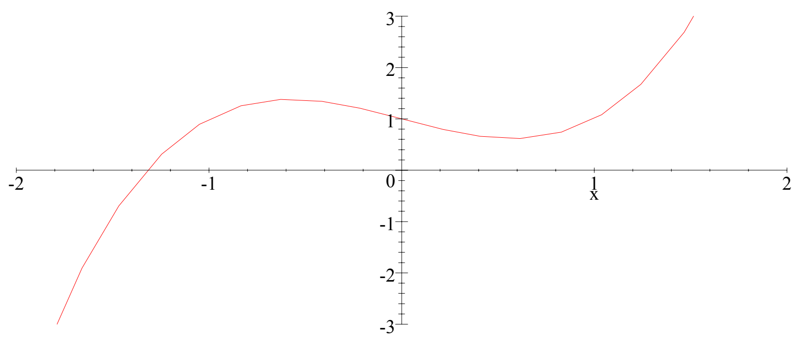
Then $f\ x$ is conditions on $0, \frac{}{2}$. Since $f\ 0\quad 1.5\quad f\ \frac{}{2}$, there exists at lest one $c\quad 0, \frac{}{2}$, such that $f\ c\quad 1.5$; indeed $c\quad \frac{}{6}$.

Corollary *Let $f\ x$ be continuous on a, b with $f\ a\quad f\ b\quad 0$ (i.e. $f\ a$ & $f\ b$ have different signs). Then there exists at least one $c\quad a, b$ such that $f\ c\quad 0$.*

Example *The function*

$$f\ x\quad x^3\quad x\quad 1$$

is continuous on the closed interval $2,\ 1$. Moreover $f\ 2\quad 5$ and $f\ 1\quad 1$. So f has at least one root in $2,\ 1$.



$$y\quad x^3\quad x\quad 1$$

In fact $x^3\quad x\quad 1\quad 0$ has exactly one real root

$$\left\{ x\quad \frac{1}{6}\sqrt[3]{\left(108\quad 12\sqrt{69}\right)}\quad \frac{2}{\sqrt[3]{\left(108\quad 12\sqrt{69}\right)}} \atop 1.3247 \right\},$$

Theorem (***Fundamental Theorem of Algebra***).

Any polynomial equation over R

$$c_0 \quad c_1 x \quad \ldots \quad c_n x^n \quad 0 \;\; c_0, \ldots, c_n \quad \mathsf{R},\; c_n \quad 0 \qquad \#$$

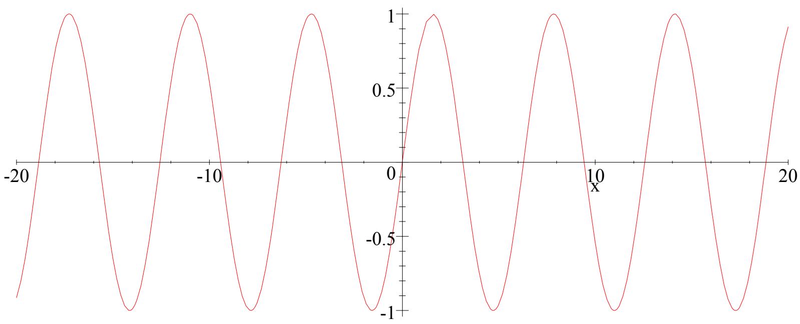
has <u>*exactly n roots*</u> (*counting multiplicity*) *in the set of complex numbers*

$$\mathsf{C} \quad a \quad bi : a, b \quad \mathsf{R} \text{ and } i \quad \sqrt{\ 1} \ .$$

Moreover, *if* $r \quad a \quad bi$ *is a root of* (*ref: n-eqn*), *then its conjugate* $\bar{r} : \quad a \quad bi$ *is also root of* (*ref: n-eqn*).

Remark *A polynomial equation of odd degree over* R *has at least one real root*.

2.6. Limits and Continuity of Trigonometric Functions



$y \quad \sin x$

Domian ,

Range 1, 1

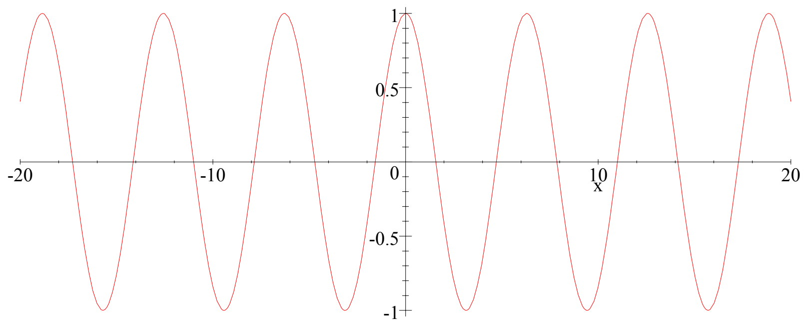
periodic with *principal period* 2 .

sin x sin x for all x R, i.e. f x sin$x$ is an *odd function* and its graph is symmetric about the origin.

sin x 0 x n where n is an integer.

Continuous at all c , :

$$\lim_{x \quad c} \sin x \quad \sin c \text{ for all } c \quad , \quad .$$



$y \quad \cos x$

Domian ,

Range $1, 1$

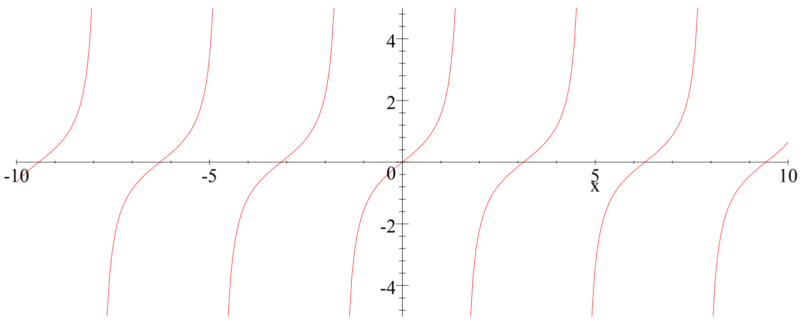
periodic with principal period 2 .

$\cos x \quad 0 \quad x \quad n\frac{}{2}$, where n is an odd integer.

$\cos \quad x \quad \cos x$ for all x R; hence $f\ x \quad \cos x$ is an *even function* and its graph is symmetric about the y-axis.

Continuous at all c , :

$$\lim_{x \quad c} \cos x \quad \cos c \text{ for all } c \quad , \quad .$$



$$y \quad \tan x \quad \frac{\sin x}{\cos x}$$

Domian R\ $n\frac{}{2}$: n is an odd integer .

Range ,

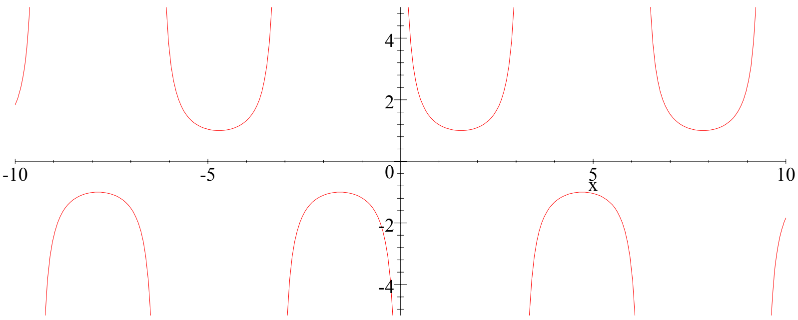
periodic with principal period .

$\tan x$ 0 $\sin x$ 0 x n , where n is an integer.

$\tan$ x $\tan x$ for all x Domian $\tan x$; hence f x $\tan x$ is an *odd function* and its graph is symmetric about the origin.

Continuous at all c R\ $n\frac{}{2}$: n is an integer :

$$\lim_{x \quad c} \tan x \quad \tan c \text{ for all } c \quad \text{Domian } \tan x \text{ .}$$



$$y \quad \csc x \quad \frac{1}{\sin x}$$

Domian $\mathsf{R}\backslash \;\; n \;\; : n$ is an integer .

Range $\quad , \; 1 \quad 1,$

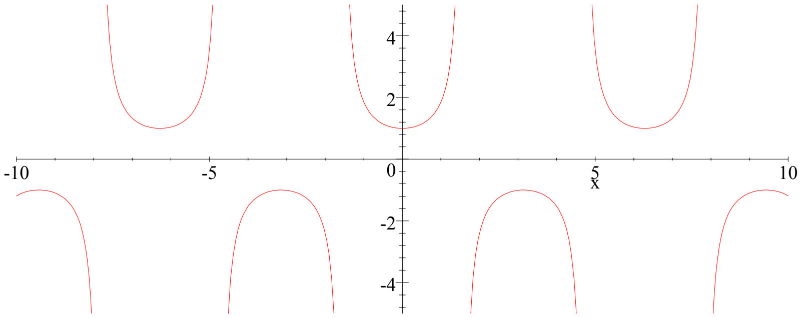
$\csc \; x \quad 0$ for all x Domian $\csc x$.

periodic with principal period 2 .

$\csc \quad x \quad \csc \; x$, hence $f \; x \quad \csc x$ is an *odd function* and its graph is symmetric about the origin.

Continuous at all c $\mathsf{R}\backslash \;\; n \;\; : n$ is an integer :

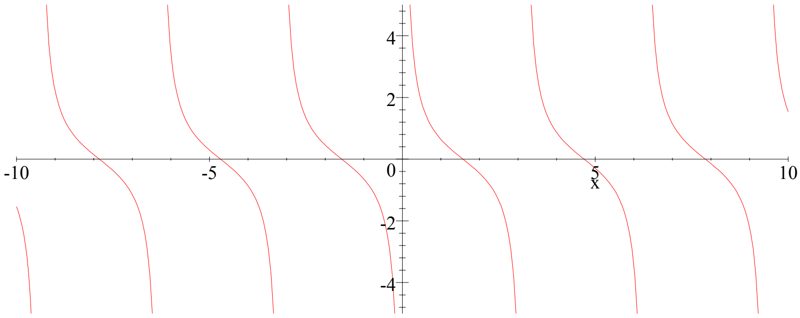
$$\lim_{x \quad c} \csc \; x \quad \csc \; c \quad \text{for all } c \quad \text{Domian } \csc x \; .$$



$$y = \sec x = \frac{1}{\cos x}$$

- Domian $= \mathbb{R} \setminus \left\{ \frac{n\pi}{2} : n \text{ is an odd integer} \right\}$.
- Range $= (-\infty, -1] \cup [1, \infty)$
- $\sec x \neq 0$ for all $x \in$ Domian$(\sec x)$.
- periodic with principal period 2π.
- $\sec(-x) = \sec(x)$, hence $f(x) = \sec x$ is an *even function* and its graph is symmetric about the y-axis.
- Continuous at all $c \in \mathbb{R} \setminus \left\{ \frac{n\pi}{2} : n \text{ is an } \underline{\text{odd}} \text{ integer} \right\}$:

$$\lim_{x \to c} \sec(x) = \sec(c) \text{ for all } c \in \text{Domian}(\sec x).$$



$$y \quad \cot x \quad \frac{\cos x}{\sin x}$$

Domian R\ n : n is an integer .
Range ,
periodic with principal period .
cot x 0 x $n\frac{}{2}$ where n is an odd integer.
Continuous at all c R\ n : n is an integer :

$$\lim_{x \quad c} \cot x \quad \cot c \ \text{ for all } c \quad \text{Domian } \cot x \ .$$

Summary

	$\sin x$	$\cos x$	$\tan x$
Domian	,	,	R\ $n\frac{\ }{2}$ n odd integer
Range	$1, 1$	$1, 1$	,
Continuity	cts on ,	cts on ,	cts on its domain
Roots x-intercepts)	n n integer	$n\frac{\ }{2}$ n odd integer	n n integer
y-**itercept**	0	1	0
Principal Period	2	2	
Symmetries	origin (odd)	y-axis (even)	origin (odd)
Vertcial Asymptotes	NONE	NONE	x $n\frac{\ }{2}$, n odd integer

	$\sec x$	$\csc x$	$\cot x$
Domian	R\ $n\frac{\ }{2}$ n odd integer	R\ n n integer	R\ n n integer
Range	, 1 1,	, 1 1,	,
Continuity	cts on its domain	cts on its domain	cts on its domain
Roots x-intercepts)	NEVER	NEVER	$n\frac{\ }{2}$ n odd integer
y-**itercept**	1	———	———
Principal Period	2	2	
Symmetries	y-axis (*even*)	origin (odd)	origin (*odd*)
Vertcial Asymptotes	x $n\frac{\ }{2}$, n odd integer	x n , n integer	x n , n integer

Example

$$\underset{x \quad 1}{lim}\ sin\ \frac{x^3 \quad 1}{x \quad 1} \qquad sin\ \underset{x \quad 1}{lim}\ \frac{x^3 \quad 1}{x \quad 1}$$

$$sin\ \underset{x \quad 1}{lim}\ \frac{x \quad 1 \quad x^2 \quad x \quad 1}{x \quad 1}$$

$$sin\ \underset{x \quad 1}{lim}\ x^2 \quad x \quad 1$$

$$sin\ 3 \qquad 0.14112.$$

Theorem (***Squeezing Theorem***)

1. *Let* (a,b) *be an open interval containing a real number c and f, g, h be functions satisfying*

$$g(x) \le f(x) \le h(x) \text{ for all } x \in (a,b)\setminus\{c\}.$$

If $\lim_{x\to c} g(x) = L = \lim_{x\to c} h(x)$, *then* $\lim_{x\to c} f(x) = L$.

2. *Let a be a (positive) real and f, g, h be functions satisfying*

$$g(x) \le f(x) \le h(x) \text{ for all } x \in (a,\infty).$$

If $\lim_{x\to\infty} g(x) = L = \lim_{x\to\infty} h(x)$, *then* $\lim_{x\to\infty} f(x) = L$.

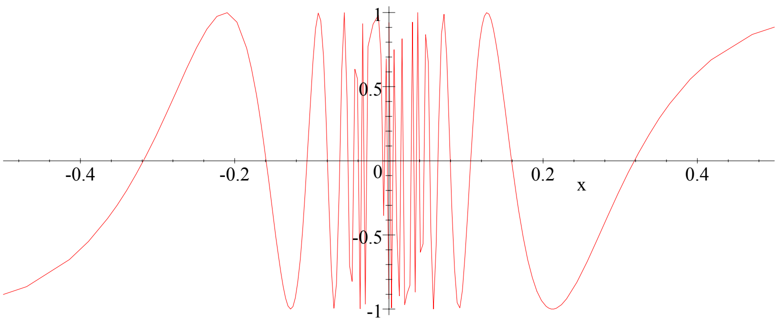
3. *Let b be a (negative) real number and f, g, h be functions satisfying*

$$g(x) \le f(x) \le h(x) \text{ for all } x \in (-\infty,b).$$

If $\lim_{x\to-\infty} g(x) = L = \lim_{x\to-\infty} h(x)$, *then* $\lim_{x\to-\infty} f(x) = L$.

Example

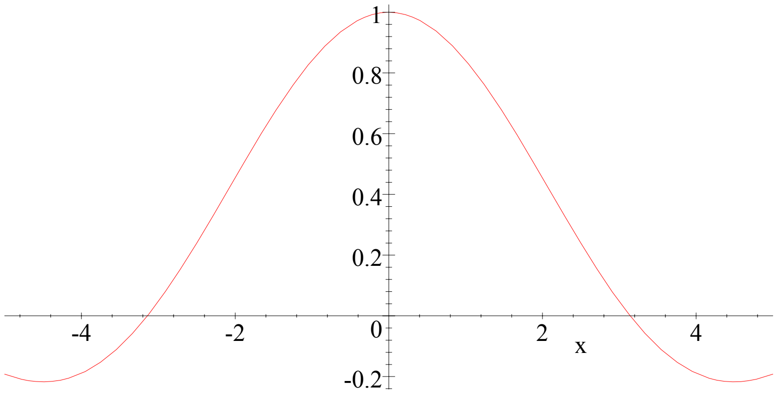
$$\lim_{x\to 0}\sin\frac{1}{x} \text{ Doesn't Exist.}$$



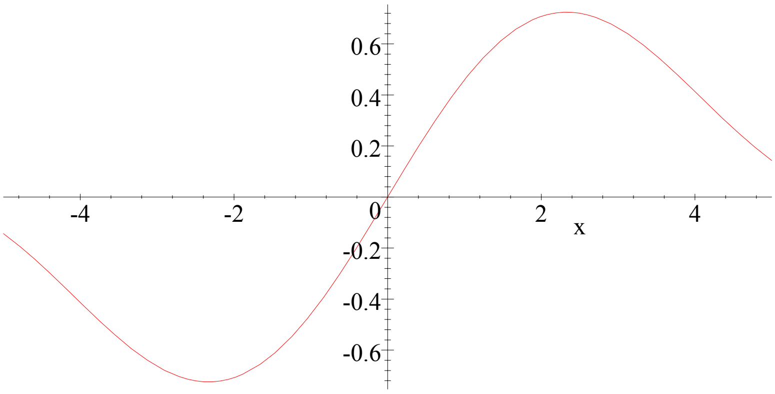
$$y = \sin\frac{1}{x}$$

Theorem

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \text{ and } \lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$$



$f(x) = \frac{\sin x}{x}$



$f(x) = \frac{1 - \cos x}{x}$

Example

$$\lim_{x \to 0} \frac{\tan 3x}{2x} = \lim_{x \to 0} \frac{3}{2} \cdot \frac{\tan 3x}{3x}$$

$$= \frac{3}{2} \cdot \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot \frac{1}{\cos 3x}$$

$$= \frac{3}{2} \cdot \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot \lim_{x \to 0} \frac{1}{\cos 3x}$$

$$= \frac{3}{2} \cdot \lim_{u \to 0} \frac{\sin u}{u} \cdot \frac{1}{\cos 0}$$

$$= \frac{3}{2} \cdot 1 \cdot 1 = \frac{3}{2}.$$

Example *For all* $x \quad 0$

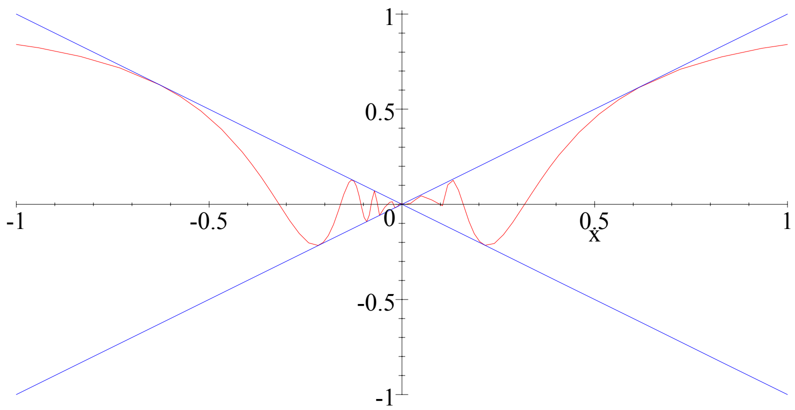
$$|x| \quad x \quad \sin\, x \quad |x|.$$

Since

$$\lim_{x \quad 0} \; |x| \quad 0 \quad \lim_{x \quad 0} |x|,$$

we conclude (*using the squeezing theorem*) *that*

$$\lim_{x \quad 0} \; x \quad \sin \frac{1}{x} \quad 0.$$



$$f\,x \quad x \sin\, \tfrac{1}{x} \;;\; y \quad |x|,\; y \quad |x|$$

Example *For all x* $\quad R\backslash\ 0$ *we have*

$$1 \quad \sin x \quad 1.$$

So

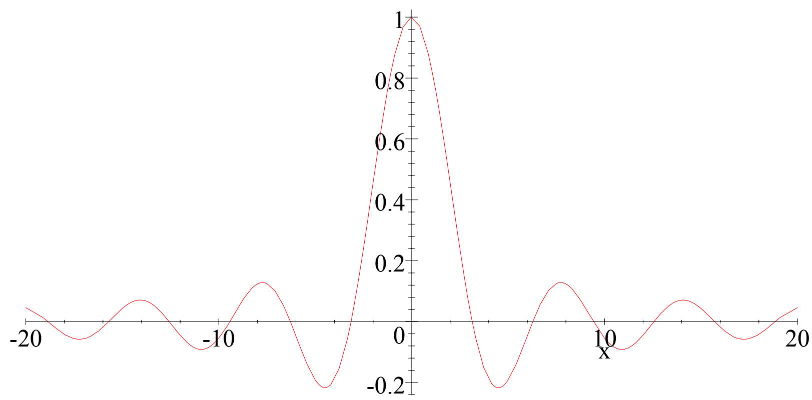
$$\frac{1}{|x|} \quad \frac{\sin x}{x} \quad \frac{1}{|x|}.$$

Since

$$\lim_{x} \frac{1}{|x|} \quad 0 \quad \lim_{x} \frac{1}{|x|},$$

we conclude that

$$\lim_{x} \frac{\sin x}{x} \quad 0.$$



$$f\,x \quad \frac{\sin x}{x}$$