King Fahd University of Petroleum & Minerals
Department of Mathematical Sciences

# MATH-533: Complex Variables I
Spring Semester 2004 (032)

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**Final Exam**

**Name:** **ID:**

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**Q1. (10 Points - Suggested Time: 10 Minutes)** State if each of the following statements is TRUE or FALSE:

1. Any analytic function $f(z) : \Omega \to \mathbb{R}$ (where $\Omega \subseteq \mathbb{C}$ is a region) is constant on $\Omega$.
2. The radius of convergence for $\sum\limits_{n=1}^{\infty} \frac{n! z^n}{n^n}$ is $\frac{1}{e}$.
3. $\oint\limits_{|z-a|=1} \frac{1}{(z-a)^n} dz = 0$ for all $n \geq 2$.
4. $\oint\limits_{|z|=1} |z-1| \, |dz| = 4$.
5. The inversion $T(z) = \frac{1}{z}$ maps lines not passing through the origins onto circles.
6. The residue of $\Gamma(z)$ at $z = -n$ for $n = 0, 1, 2, ...$ is $\frac{(-1)^n}{n!}$.
7. The equation $az + b\overline{z} + c = 0$ $(a, b, c \in \mathbb{C})$ represents a straight line.
8. $(\mathbb{C}_\infty, d)$, where $d$ is the chordal metric, is a complete metric space.
9. If $f(z) : \Omega \to \mathbb{C}$ is analytic, where and $\Omega$ is a region, then
$$\oint\limits_{\gamma} f(z) dz = 0,$$
where $\gamma \subset \Omega$ is a cycle.
10. The infinite product $\prod\limits_{n=1}^{\infty} (1 + \frac{(-1)^{n+1}}{(n+1)\ln(n+1)})$ converges absolutely.

**Q2. (20 Points - Suggested Time: 30 Minutes)** Give a counter example to each of the following **false** statements:

1. If $f(z)$ is meromorphic but not entire on $\mathbb{C}$, then $e^{f(z)}$ is meromorphic.

2. If $u(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is harmonic, then $g(x, y) = \bigtriangledown u \bullet \bigtriangledown u$ is harmonic.

**Q3. (40 Points - Suggested Time: 75 Minutes)** Prove **any 5** of the following statements.

1. $f(z) = \frac{1}{z}$ is not uniformly continuous in the region $\Omega := \{z \in \mathbb{C} : 0 < |z| < 1\}$.

2. If $f(z)$ is an analytic function with a zero of order $h$ at $z_0$, then $f(z) = g(z)^h$, where $g(z)$ is analytic near $z_0$ and satisfies $g'(z_0) \neq 0$.

3. $f(z) = \sqrt{z^2 - \frac{1}{z}}$ can be defined as a single-valued continuous function outside the unit disk.

4. The image of the closed region

$$\Omega := \{z = x + iy \in \mathbb{C} : |x| \le \frac{1}{2}, y \ge 0\}$$

under the mapping $T(z) = e^{2\pi i z}$ is the closed unit disk minus the origin.

5. Any analytic function $f(z) : \Omega \to \mathbb{C}$,

$$\Omega := \{z \in \mathbb{C} : R_1 < |z - a| < R_2\}, R_2 > R_1 > 0,$$

has a Laurent's series expansion $f(z) = \sum\limits_{n=-\infty}^{\infty} A_n (z - z_0)^n$, where $C := C(a, \rho)$ is any circle with center at $z = a$, radius $R_1 < \rho < R_2$ and

$$A_n = \frac{1}{2\pi i} \oint\limits_C \frac{f(\zeta)}{(\zeta - a)^{n+1}} d\zeta.$$

6. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of analytic functions $f_n : \Omega_n \to \mathbb{C}$, where $\Omega_n \subseteq \Omega_{n+1}$ for all $n \geq 1$ and $\{f_n\}_{n=1}^{\infty}$ converges uniformly to $f(z)$ on every compact subset of $\Omega := \bigcup_{n=1}^{\infty} \Omega_n$, then $f(z)$ is analytic on $\Omega$ and $\{f'_n\}_{n=1}^{\infty}$ converges uniformly to $f'(z)$ on every compact subset of $\Omega$.

7. A series $\sum_{n=1}^{\infty} z_n$ with $\lim_{n\mapsto\infty} \frac{|z_{n+1}|}{|z_n|} = \rho$ converges absolutely, if $\rho < 1$ and diverges if $\rho > 1$.

**Q4. (10 Points - Suggested Time: 15 Minutes)** Consider the series $\sum_{n=1}^{\infty} z^n(1-z)$.

1. Prove that the series is absolutely convergent to $f(z) = z$ for $|z| < 1$.

2. Prove that the series converges uniformly to its sum for $|z| \leq \rho$, where $0 < \rho < 1$ and explain why the series does not converge uniformly for $|z| \leq 1$.

**Q5. (20 Points - Suggested Time: 30 Minutes)** Evaluate the following integrals:

1. $\oint\limits_{|z|=4} \frac{\sin z}{z(z^2-\pi^2)} dz$

2. $\oint\limits_{|z|=5} \frac{e^z}{z^2(z-1)} dz$

3. $\oint\limits_{|z|=5} \frac{\cos z}{z^3+9z} dz$

4. $\int_0^\infty \frac{\sin x}{x} dx$