King Fahd University of Petroleum & Minerals
Department of Mathematical Sciences

MATH-533: Complex Variables I
Spring Semester 2004 (032)

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First Major - Solutions

Name: **ID:**

Q1. (10 points) (Suggested time: 10 Minutes). State if each of the following statements is TRUE or FALSE:

1. Every power series represents an entire function.

   False: *A power series represents an analytic function inside its circle of convergence.*

2. The radius of convergence for $\sum_{k=1}^{\infty} \frac{2^k z^{2k}}{k^2+k}$ is $\frac{1}{\sqrt{2}}$.

   True:

$$
\begin{aligned}
R &= (\lim_{k\mapsto\infty}\sup \left|\frac{2^k}{k^2+k}\right|^{\frac{1}{2k}})^{-1} &&= 2^{\frac{-1}{2}}(\lim_{k\mapsto\infty}\sup\left|\frac{1}{k^2+k}\right|^{\frac{1}{2k}})^{-1} \\
&= \frac{1}{\sqrt{2}}(\lim_{k\mapsto\infty}\inf\left|k^2+k\right|^{\frac{1}{2k}})^{-1} &&= \frac{1}{\sqrt{2}}(\lim_{k\mapsto\infty}\inf(\exp\{\frac{\ln(k^2+k)}{2k}\}))^{-1} \\
&= \frac{1}{\sqrt{2}}(e^0) &&= \frac{1}{\sqrt{2}}.
\end{aligned}
$$

3. Two lines $z = a + bt$ and $z = c + dt$ ($a, b, c, d \in \mathbb{C}$, $b, d \neq 0$ and $t \in (-\infty, \infty)$) are perpendicular, iff $\text{Re}(\frac{d}{b}) = 0$.

   True: (Compare Ahlfors, page 17) *W.l.o.g assume* $|b| = |d| = 1$. *Let* θ *be the angle between the two lines. Then*

$$\cos(\theta) = \frac{< b, d >}{|b|\,|d|} = \text{Re}(\bar{b}d) = \text{Re}(\frac{|b|^2 d}{b}) = \text{Re}(\frac{d}{b}).$$

4. $f(z) := \sum_{k=1}^{\infty} z^k(1-z)$ is continuous on $E := \{z \in \mathbb{C} : |z| < 1\} \cup \{1\}$.

   False: $S_n(z) = z - z^{n+1}$. If $|z| < 1$, then $\lim_{n\mapsto\infty} S_n(z) = z$. If $z = 1$, then $S_n(z) = 0$, hence

$$f(z) = \begin{cases} z, & \text{if } |z| < 1 \\ 0, & \text{if } z = 1 \\ \text{Not Defined}, & \text{Otherwise} \end{cases}$$

   Clearly $f(z)$ is not continuous at $z_0 = 1$.

5. The function

$$f(z) = f(x+iy) = \begin{cases} \frac{xy^2(x+yi)}{x^2+y^4}, & \text{for } z \neq 0 \\ 0, & \text{for } z = 0 \end{cases}$$

   is differentiable at $z_0 = 0$.

False: *Take* $h = \Delta x + i\Delta y = \alpha t + i(\beta t)$, *where* α *and* β *are nonzero real constants and* t *is a real parameter. Then*

$$\lim_{h\mapsto 0} \frac{f(0+h)-f(0)}{h} = \lim_{t\mapsto 0} \frac{(\alpha t)(\beta^2 t^2)(\alpha t + \beta t i) - 0}{(\alpha^2 t^2 + \beta^4 t^4)(\alpha t + \beta t i)} = \lim_{t\mapsto 0} \frac{(\alpha t)(\beta^2)}{\alpha^2 + \beta^4 t^2} = 0.$$

Along the curve $x = y^2$, *we have*

$$\begin{aligned} \lim_{h\mapsto 0} \tfrac{f(0+h)-f(0)}{h} &= \lim_{(\Delta x,\Delta y)\mapsto(0,0)} \tfrac{\Delta x(\Delta y)^2(\Delta x + i\Delta y)}{((\Delta x)^2+(\Delta y)^4)(\Delta x+i\Delta y)} \\ &= \lim_{\Delta y\mapsto 0} \tfrac{(\Delta y)^2(\Delta y)^2}{((\Delta y)^4+(\Delta y)^4)} \\ &= \tfrac{1}{2} \neq 0. \end{aligned}$$

Q2. (10 Points) (Suggested time: 10 Minutes) Describe the locus in the complex plane consisting of the points (*if any*) satisfying the following equations:

1. $|z - 3 + 4i| \geq |z + 5 + 6i|$

   The equation

   $$|z - (3 - 4i)| = |z - (-5 - 6i)|$$

   represents the set of points in the xy*-plane equidistant from the points* $P(3,-4)$ *and* $Q(-5,-6)$, *i.e. the points on the line passing through the midpoint of the line segment* $\overline{PQ}$ *and perpendicular to it. The inequality represents then* **the points on and to the left of the line** $y = -(4x+9)$.

2. $|z - 1 + i| + |z + 1 - i| = 2.$

   If this equation had z_0 *as a solution, then we would have*

   $$\begin{aligned} (|z_0 - (1-i)| + |z_0 + (1-i)|)^2 &= (2)^2 \\ |z_0 - (1-i)|^2 + |z_0 + (1-i)|^2 + 2\,|z_0 - (1-i)|\,|z_0 + (1-i)| &= 4 \\ 2(|z_0|^2 + |1-i|^2) + 2\,|z_0 - (1-i)|\,|z + (1-i)| &= 4 \\ |z_0|^2 + 2\,|z_0 - (1-i)|\,|z_0 + (1-i)| &= 0, \end{aligned}$$

   which is impossible. So there are **no points** *satisfying* $|z - 1 + i| + |z + 1 - i| = 2$ *(Compare Ahlfors, Problem 4/11).*

Q3. (20 points) (Suggested time: 20 Minutes) Give a *counterexample* to **two** of the following **false** statements:

1. If $f(z) : \mathbb{C} \to \mathbb{C}$ is differentiable at z_0, then it's analytic at z_0.

   Counterexample:
$$f(z) = |z|^2 = x^2 + y^2 \text{ at } z_0 = 0.$$
   $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial v}{\partial y} = 0 = \frac{\partial v}{\partial x}$. *So* $f(z)$ *is not differentiable at* $z_0 \neq 0$. *At* $z_0 = 0$ *we have*
$$f'(0) = \lim_{h \mapsto 0} \frac{|h+0|^2 - 0}{h} = \lim_{h \mapsto 0} \overline{h} = 0,$$
   hence $f(z)$ *is differentiable on* $E = \{0\}$ *and clearly not analytic at* $z_0 = 0$.

2. If $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ is such that $u(x,y)$ and $v(x,y)$ satisfy the Cauchy-Riemann equations at $z = z_0$, then $f(z)$ is differentiable at z_0.

   Counterexample: $f(z) = f(x+iy) = \sqrt{|xy|}$ *at* $z_0 = 0$. *Then*
$$\frac{\partial u}{\partial x}_{(0,0)} = \lim_{h \mapsto 0} \frac{\sqrt{|(x+h)(0)| - 0}}{h} = 0 \text{ and } \frac{\partial u}{\partial y}_{(0,0)} = \lim_{h \mapsto 0} \frac{\sqrt{|(0)(0+h)| - 0}}{h} = 0.$$
   Clearly $\frac{\partial v}{\partial x} = 0 = \frac{\partial v}{\partial y}$. *Hence* $u(x,y)$ *and* $v(x,y)$ *satisfy the Cauchy-Riemann equations at* $z_0 = 0$. *However for* $h = \Delta x + i\Delta y = \Delta x + i\Delta x$ *we have*
$$\lim_{h \mapsto 0} \frac{f(0+h) - f(0)}{h} = \lim_{(\Delta x, \Delta y) \mapsto (0,0)} \frac{\sqrt{|\Delta x \Delta y| - 0}}{\Delta x + \Delta y i} = \lim_{\Delta x \mapsto 0} \frac{\sqrt{|\Delta x \Delta x|}}{\Delta x + \Delta x i} = \lim_{\Delta x \mapsto 0} \frac{|\Delta x|}{\Delta x(1+i)},$$
   which does not exist. Hence $f(z) = \sqrt{|xy|}$ *is not differentiable at* $z_0 = 0$.

3. The product of two convergent series of complex numbers is convergent.

   Counterexample: Consider the convergent alternating series $\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} b_n$. Then $(\sum_{n=0}^{\infty} a_n)(\sum_{n=0}^{\infty} b_n) = \sum_{n=0}^{\infty} c_n$, where
$$c_n = \sum_{j=0}^{n} a_j b_{n-j} = (-1)^n \sum_{j=0}^{n} \frac{1}{\sqrt{(j+1)(n+1-j)}}.$$
   The function $f_n(x) = \frac{1}{\sqrt{(x+1)(n+1-x)}}$ has its minimum on the interval $[0, n]$ as $f(\frac{n}{2}) = \frac{2}{n+2}$, hence for n odd we have
$$c_n = (-1)^n \sum_{j=0}^{n} \frac{1}{\sqrt{(j+1)(n+1-j)}} \leq (-1)^n (n+1) \frac{2}{n+2},$$
   i.e. $\lim_{n \mapsto \infty} c_n \neq 0$ and consequently $\sum_{n=0}^{\infty} c_n$ diverges.

Q4. (60 points) (Suggested time: 40 Minutes) Prove **four** of the following statements:

1. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions converging uniformly to f on a region $E \subseteq \mathbb{C}$, then f is continuous on E.

   Proof: *(Consult Ahlfors: Page 36).*

2. Every (twice continuously differentiable) harmonic function $u(x, y) : \mathbb{R}^2 \to \mathbb{R}$ has a harmonic conjugate.

   Proof: *Define*
$$v(x, y) = \int_0^y \frac{\partial u}{\partial x}(x, t)dt + \varphi(x).$$
   Then
$$\begin{aligned}
\tfrac{\partial v}{\partial x}(x, y) &= \tfrac{\partial}{\partial x}(\textstyle\int_0^y \frac{\partial u}{\partial x}(x, t)dt) + \varphi'(x) \\
&= \textstyle\int_0^y \frac{\partial^2 u}{\partial x^2}(x, t)dt + \varphi'(x) && \textit{(by Leibniz's Rule)} \\
&= -\textstyle\int_0^y \frac{\partial^2 u}{\partial y^2}(x, t)dt + \varphi'(x) && \textit{(Since } u(x,y) \textit{ is harmonic)} \\
&= -\tfrac{\partial u}{\partial y}(x, y) + \tfrac{\partial u}{\partial y}(x, 0) + \varphi'(x).
\end{aligned}$$
   Since $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, *we conclude that* $\varphi'(x) = -\frac{\partial u}{\partial y}(x, 0)$ *and consequently* $\varphi(x) = -\int_0^x \frac{\partial u}{\partial y}(s, 0)ds$. *So*
$$v(x, y) = \int_0^y \frac{\partial u}{\partial x}(x, t)dt - \int_0^x \frac{\partial u}{\partial y}(s, 0)ds.$$
   It's easy to check that $v(x, y)$ is harmonic and that $u(x, y)$, $v(x, y)$ satisfy the Cauchy-Riemann equations.

3. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of continuous functions on $G \subseteq \mathbb{C}$ with $|f_k(z)| \leq M_k$ for all $z \in G$ and all $k \in \mathbb{N}$. If $\sum_{k=1}^{\infty} M_k$ converges, then $\sum_{k=1}^{\infty} f_k$ converges *absolutely* and *uniformly* on G.

   Proof: *(Compare Ahlfors: Page 37). Let*
$$S_n(z) = f_1(z) + ... + f_n(z).$$
   Then for $n > m$ we have for all $z \in G$:
$$|S_n(z) - S_m(z)| = |f_{m+1}(z) + ... + f_n(z)| \leq \sum_{m+1}^{n} |f_k(z)| \leq \sum_{m+1}^{n} M_k.$$
   By assumption $\sum_{k=1}^{\infty} M_k$ *is convergent and so it's Cauchy. Hence* $\{S_n(z)\}_{n=1}^{\infty}$ *is Cauchy and consequently convergent to* $S(z) := \lim_{n\mapsto\infty} S_n(z) =$. *Given* $\in > 0$, *there exists* $n_0 \in N$, *such that* $\sum_{n+1}^{\infty} M_k < \in$ *for* $n \geq n_0$. *Hence for all* $z \in G$ *we have for all* $n \geq n_0$:
$$|S(z) - S_n(z)| = \left|\sum_{k=1}^{\infty} f_k(z) - \sum_{k=1}^{n} f_k(z)\right| = \left|\sum_{k=n+1}^{\infty} f_k(z)\right| \leq \sum_{k=n+1}^{\infty} M_k < \in,$$
   i.e. $\{S_n(z)\}_{n=1}^{\infty}$ *(and consequently* $\sum_{k=1}^{\infty} f_k(z)$*) converges absolutely and uniformly to* $S(z)$ *on* G.

4. A sequence $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{C}$ converges to z_0 in $\mathbb{C} \Leftrightarrow d(z_n, z_0) \mapsto 0$ (where $d(z_n, z_0)$ denotes the *chordal distance* between z_n and z_0 in the extended complex plane $\mathbb{C}_\infty$).

   Proof: ($\Rightarrow$) *Assume that* $\{z_n\}_{n=1}^{\infty}$ *in* C *converges to* z_0 *in* C*. Then given* $\in > 0$ *there exists some* $n_0 \in N$ *such that*

$$|z_n - z_0| < \frac{\in}{2} \text{ for } n \geq n_0.$$

   Hence for $n \geq n_0$*, we have*

$$|d(z_n, z_0) - 0| = d(z_n, z_0) = \frac{2\,|z_n - z_0|}{\sqrt{1 + |z_n|^2}\sqrt{1 + |z_0|^2}} \leq 2\,|z_n - z_0| < 2\frac{\in}{2} = \in,$$

   i.e. $d(z_n, z_0) \mapsto 0$.

   ($\Leftarrow$) *Assume that* $d(z_n, z_0) \mapsto 0$*. Then clearly* $\{z_n\}_{n=1}^{\infty}$ *is bounded, i.e. there exists some* $M > 0$*,such that* $|z_n| \leq M$ *for all* $n \in N$*. Since* $d(z_n, z_0) \mapsto 0$*, given arbitrary* $\in > 0$ *there exists a natural number* n_0*, such that for* $n \geq n_0$ *we have*

$$d(z_n, z_0) = |d(z_n, z_0) - 0| < \frac{2 \in}{\sqrt{1 + M^2}\sqrt{1 + |z_0|^2}}.$$

   Hence for $n \geq n_0$ *we get*

$$\begin{aligned} |z_n - z_0| &= \frac{\sqrt{1 + |z_n|^2}\sqrt{1 + |z_0|^2}}{2} d(z_n, z_0) \\ &< \frac{\sqrt{1 + M^2}\sqrt{1 + |z_0|^2}}{2} \frac{2 \in}{\sqrt{1 + M^2}\sqrt{1 + |z_0|^2}} = \in, \end{aligned}$$

   i.e. $\{z_n\}_{n=1}^{\infty}$ *converges to* z_0.

5. The series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges on $\{z \in \mathbb{C} : |z| \leq 1,\, z \neq 1\}$.

Proof: *The radius of convergence is*

$$R = \frac{1}{\lim_{n\mapsto\infty} \sup(\frac{1}{n})^{\frac{1}{n}}} = \frac{1}{\lim_{n\mapsto\infty} n^{\frac{1}{n}}} = 1,$$

hence the converges absolutely for $|z| < 1$.

For $z = 1$, $\sum_{n=1}^{\infty} \frac{z^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ *the divergent harmonic series.*

For $z \neq 1$ *and* $|z| = 1$ *consider* $A_n(z) := 1 + z + ... + z^n = \frac{1-z^{n+1}}{1-z}$. *Then*

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{z^k}{k} = z + \frac{z^2}{2} + ... + \frac{z^n}{n} \\ &= \frac{A_1 - 1}{1} + \frac{A_2 - A_1}{2} + ... + \frac{A_n - A_{n-1}}{n} \\ &= -1 + (1 - \frac{1}{2})A_1 + (\frac{1}{2} - \frac{1}{3})A_2 + ... + (\frac{1}{n-1} - \frac{1}{n})A_{n-1} + \frac{1}{n}A_n. \end{aligned}$$

For $z \neq 1$ *and* $|z| = 1$ *we have* $|A_n(z)| = \left|\frac{1-z^{n+1}}{1-z}\right| \leq \frac{1+|z|^{n+1}}{|1-z|} = \frac{2}{|1-z|}$, *hence for* $m > n \geq 1$ *we get*

$$\begin{aligned} |S_m - S_n| &= \left| -\frac{1}{n+1}A_n + (\frac{1}{n+1} - \frac{1}{n+2})A_{n+1} + ... + (\frac{1}{m-1} - \frac{1}{m})A_{m-1} + \frac{1}{m}A_m \right| \\ &= \left| (\frac{1}{n} - \frac{1}{n+1})A_n + (\frac{1}{n+1} - \frac{1}{n+2})A_{n+1} + ... + (\frac{1}{m-1} - \frac{1}{m})A_{m-1} + \frac{1}{m}A_m - \frac{1}{n}A_n \right| \\ &\leq \frac{2}{|1-z|}[(\frac{1}{n} - \frac{1}{n+1}) + (\frac{1}{n+1} - \frac{1}{n+2}) + ... + (\frac{1}{m-1} - \frac{1}{m}) + \frac{1}{m} + \frac{1}{n}] \\ &= \frac{4}{n\,|1-z|}. \end{aligned}$$

So $\lim_{n,m\mapsto\infty} |S_m(z) - S_n(z)| = \lim_{n\mapsto\infty} \frac{4}{n|1-z|} = 0$. *Consequently* $\{S_n(z)\}_{n=1}^{\infty}$ *is a convergent sequence for* $z \neq 1$ *and* $|z| = 1$.

6. Each rational function $R(z)$, which is real on the real line (the x-axis), is represented as a quotient of two polynomials with real coefficients.

Proof: *Let*

$$R(z) = \frac{P(z)}{Q(z)} = \frac{a_0 + a_1 z + ... + a_n z^n}{b_0 + b_1 z + ... + b_m z^m} = \frac{a_n \prod_{j=1}^{n}(z - z_j)}{b_m \prod_{k=1}^{m}(z - w_k)},$$

where $a_n \neq 0$, $b_m \neq 0$ *and we may assume that* $P(z)$ *and* $Q(z)$ *have no common factors, i.e.* $z_j \neq w_k$ *for all* $j = 1, ..., n$ *and* $k = 1, ..., m$. *By assumption the polynomial*

$$R(x)\,|Q(x)|^2 = P(x)\overline{Q(x)} = a_n \overline{b_m} \prod_{j=1}^{n}(x - z_j) \prod_{k=1}^{m}(x - \overline{w_k})$$

is real for real x. *Therefore it has real coefficients, in particular* $a_n \overline{b_m}$ *and* $c_0 := \frac{a_n}{b_m} = \frac{a_n \overline{b_m}}{|b_m|^2}$ *are real. Since nonreal roots of polynomial with real coefficients occur in pairs of conjugate zeros, this means for every nonreal* z_l *the set* $\{z_j, \overline{w_k} : j = 1, ..., n,\ k = 1, ..., m\}$ *contains* $\overline{z_l}$. *Since by assumption* $\{\overline{z_1}, ..., \overline{z_n}\} \cap \{\overline{w_1}, ..., \overline{w_m}\} = \varnothing$ *we conclude that* $z_l = \overline{z_j}$ *for some* $j = 1, ..., n$. *Hence the linear factors in* $P(z)$ *can be grouped in quadratic polynomials with real coefficients* $(z - z_l)(z - \overline{z_l}) = z^2 - 2\,\mathrm{Re}(z_l) + |z_l|^2$. *Similar arguments apply to zeros* $\overline{w_k}$. *So we can write*

$$R(z) = \frac{P(z)}{Q(z)} = \frac{P^*(z)}{Q^*(z)} = \frac{c_0 \prod_{j=1}^{n}(z - z_j)}{\prod_{k=1}^{m}(z - \overline{w_k})} = \frac{c_0 \prod_{j=1}^{n}(z - z_j)}{\prod_{k=1}^{m}(z - w_k)},$$

where both $P^*(z)$ *and* $Q^*(z)$ *are represented as products of linear and quadratic polynomials with real coefficients.*

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