Ricci collineations of static spherically symmetric spacetimes

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The Ricci collineations of static spherically symmetric spacetimes are classified and their relationship with isometries is discussed. A general theorem about this relationship is stated and its extension to all spherically symmetric spacetimes is discussed.

I. INTRODUCTION

Whereas the geometrical nature of a spacetime is encoded into the metric tensor, by virtue of the Einstein field equations, the physics is given more explicitly by the Ricci tensor. As such, while the *Killing vectors* (KVs), or isometries, provide information of the symmetries inherent in the spacetime, the symmetries of the matter–energy field are provided by the *Ricci collineations* (RCs), vector fields along which the Ricci tensor is invariant under Lie transport.[1] It seems intuitively clear that the matter–energy field may possess *more* symmetries than the underlying spacetime but never *less*. However, it is not immediately obvious from geometrical considerations that this should be true. It is, therefore, of interest to investigate this relationship between RCs and KVs. In this article we limit the investigation to static, spherically symmetric space–times.

KVs are vector fields along which the metric tensor is invariant under Lie transport. They satisfy the Killing equations, which can be written in component form as

$$(K_{ab}) \qquad g_{ab,c}k^c+g_{ac}k^c{}_{,b}+g_{bc}k^c{}_{,a}=0, \quad (a,b,c=0,1,2,3), \tag{1}$$

where g_{ab} are the components of the metric tensor, k^c are the components of the KV, and the "$,a$" denotes differentiation with respect to x^a. The corresponding RC equations are

$$(C_{ab}) \qquad R_{ab,c}C^c+R_{ac}C^c{}_{,b}+R_{bc}C^c{}_{,a}=0, \tag{2}$$

where R_{ab} are the components of the Ricci tensor and C^c are the components of the RC. Notice the formal similarity between the two equations. However, there is a fundamental difference. The metric tensor must be nonsingular and hence, in any valid coordinate system, its determinant must be nonzero and noninfinite. While the Ricci tensor must also have a noninfinite determinant, it *can* be zero. This fact makes the two systems of equations fundamentally different. Also, the latter system of equations can be regarded as a highly nonlinear set of *third order* partial differential equations for the metric tensor.

For Einstein spaces[2] of the type $R_{ab}=\Lambda\, g_{ab}$ with nonzero constant, Λ, the RCs will simply be the KVs. However, for arbitrary spacetimes, and even the more general Einstein spaces given by $R_{ab}=\mathcal{R}(x^c)g_{ab}$, there is no reason, *a priori*, why an RC should be a KV or *vice versa*. In this article we investigate the question of which of the two will be a more specialized symmetry for spherically symmetric static spacetimes and when the two will be identical. On physical grounds, since the matter–energy field is "controlled" by the space–time, we would expect all KVs to be RCs.

The metric for spherically symmetric static spacetimes can generally be written in the form[3]

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