Curvature collineations of some static spherically symmetric space–times

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Curvature collineations of some static spherically symmetric space–times are derived and compared with isometries and Ricci collineations for corresponding space–times. © *1996 American Institute of Physics.* [S0022-2488(96)04106-0]

I. INTRODUCTION

Over the past few years there has been much interest in the classification of solutions of the Einstein field equations in terms of their isometries. These isometries are given by Killing vectors (KVs), along which the Lie derivative of the metric tensor is zero, admitted by the space–time. Each independent KV gives rise to a conservation law for the spacetime. The classification by Petrov[1,2] was incomplete in that it did not provide a list of metrics for a given set of isometries, though a complete list of isometries was available. In an extensive study of spherically symmetric spacetimes, Takeno[3] used the curvature invariants of such space–times to classify them according to their isometries and these invariants. Following a different approach Qadir *et al.*[4] obtained a classification of such space–times by their isometries and provided a complete list of distinct space–time metrics. It appears that Takeno missed some metrics (for example, nonstatic spacetime like the Einstein universe but with the role of t and r inter changed, so that the isometry group is $SO(1,3)XR$ instead of $SO(4)XR$).

Though the classification of space–times in terms of their isometries is important, Katzin, Davies, and Lavine[5,6] argue that the symmetries of the matter field would be given by Ricci collineations (RCs), along which the Lie derivative of the Ricci tensor is zero. A complete classification of spherically symmetric, static metrics in terms of RCs has been obtained[7,8] and is being extended to the nonstatic cases.[9]

Katzin *et al.* also argue that the symmetries of the Riemann tensor, called curvature collineations (CCs), would also provide insights into general relativity. Though they give a theorem on connection between RCs and CCs, no explicit attempt to classifying spacetimes according to their CCs has been given. Keeping this point in mind and the complexity of the system of CC equations, we consider some specific spacetimes to obtain their CCs using some special methods. It is hoped that this would enable one to extend these methods to obtain a classifications of general space–times according to their CCs.

In the next section we give the set of coupled quadratic CC equations and their form for spherically symmetric static space–times. In the third section we solve this set of equations for various specific cases. A summary and conclusion is given in the last section.

II. CC EQUATIONS

A CC, ξ, satisfies the equation

$$\underset{\xi}{\mathscr{L}}R=0, \tag{1}$$

where R is the Riemann Christoffel curvature tensor. In a torsion free space, in a coordinate basis, this equation reduces to the set of partial differential equations (PDEs)

$$R^{\alpha}{}_{bcd,f}\xi^{f}+R^{\alpha}{}_{fcd}\xi^{f}{}_{,b}+R^{\alpha}{}_{bfd}\xi^{f}{}_{,c}+R^{\alpha}{}_{bcf}\xi^{f}{}_{,d}-f^{f}{}_{bcd}\xi^{\alpha}{}_{,f}=0. \tag{2}$$

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In a given 4-dimensional space–time there are actually 256 coupled PDEs to be solved for four unknown functions of four variables. However, for spherically symmetric static space–times in which at the most only six independent components (3 $R^0{}_{i0i}$, where $i=1,...,3$, 2 $R^1{}_{\alpha 1\alpha}$, where $\alpha=1$, 2, and 1 $R^2{}_{323}$) of the Riemann tensor can survive, the system of CC equations reduces to 22 sets of coupled CC equations consisting of *54* PDEs to be solved only. These coupled CC equations (without requiring summation over repeated indices) are given by

$$R^i{}_{1i1}\xi^1{}_{,0}+R^\alpha{}_{0\alpha 0}\xi^0{}_{,1}=0, \quad (i,\alpha)=(0,1),(2,2),(3,3), \tag{3}$$

$$R^i{}_{2i2}\xi^2{}_{,0}+R^\alpha{}_{0\alpha 0}\xi^0{}_{,2}=0, \quad (i,\alpha)=(0,2),(1,1),(3,3), \tag{4}$$

$$R^i{}_{3i3}\xi^3{}_{,0}+R^\alpha{}_{0\alpha 0}\xi^0{}_{,3}=0, \quad (i,\alpha)=(0,3),(1,1),(2,2), \tag{5}$$

$$R^i{}_{1i1,f}\xi^f+2R^i{}_{1i1}\xi^1{}_{,1}=0, \quad i=0,2,3 \text{ and } f=1 \text{ or } 2, \tag{6}$$

$$R^i{}_{1i1}\xi^1{}_{,2}+R^\alpha{}_{2\alpha 2}\xi^2{}_{,1}=0, \quad (i,\alpha)=(0,0),(2,1),(3,3), \tag{7}$$

$$R^i{}_{1i1}\xi^1{}_{,3}+R^\alpha{}_{3\alpha 3}\xi^3{}_{,1}=0, \quad (i,\alpha)=(0,0),(3,1),(2,2), \tag{8}$$

$$R^i{}_{2i2}\xi^2{}_{,3}+R^\alpha{}_{3\alpha 3}\xi^3{}_{,2}=0, \quad (i,\alpha)=(0,0),(1,1),(3,2), \tag{9}$$

$$R^i{}_{2i2,f}\xi^f+2R^i{}_{2i2}\xi^2{}_{,2}=0, \quad i=0,1,3 \text{ and } f=1 \text{ or } 2, \tag{10}$$

$$R^i{}_{3i3,f}\xi^f+2R^i{}_{3i3}\xi^3{}_{,3}=0, \quad i=0,1,2 \text{ and } f=1 \text{ or } 2, \tag{11}$$

$$R^i{}_{0i0,f}\xi^f+2R^i{}_{0i0}\xi^0{}_{,0}=0, \quad i=1,2,3 \text{ and } f=1 \text{ or } 2, \tag{12}$$

$$(R^0{}_{i0i}-R^\alpha{}_{i\alpha i})\xi^0{}_{,\alpha}=0, \quad (i,\alpha)=(1,3),(2,3),(1,2),(3,2),(2,1),(3,1), \tag{13}$$

$$(R^0{}_{i0i}-R^\alpha{}_{i\alpha i})\xi^\alpha{}_{,0}=0, \quad (i,\alpha)=(1,3),(2,3),(1,2),(3,2),(2,1),(3,1), \tag{14}$$

$$(R^\alpha{}_{i\alpha i}-R^\beta{}_{i\beta i})\xi^\alpha{}_{,\beta}=0, \quad (i,\alpha,\beta)=(0,1,2),(3,1,2),(0,1,3),(2,1,3), \tag{15}$$

$$(R^\alpha{}_{i\alpha i}-R^\beta{}_{i\beta i})\xi^\beta{}_{,\alpha}=0, \quad (i,\alpha,\beta)=(0,1,2),(3,1,2),(0,3,2),(1,3,2), \tag{16}$$

$$(R^\alpha{}_{i\alpha i}-R^\beta{}_{i\beta i})\xi^\beta{}_{,\alpha}=0, \quad (i,\alpha,\beta)=(0,1,3),(2,1,3),(0,2,3),(1,2,3). \tag{17}$$

III. SOLUTION OF THE CC EQUATIONS

We solve the CC equations for Minkowski, De Sitter (anti-De Sitter), Einstein (anti-Einstein), Schwarzschild, and Reissner–Nordstrom metrics along with three Bertotti–Robinson-like metrics.[10] However, since the problem of solving CC equations is trivial in Minkowski space–time, we do not solve this case explicitly and only give results. The CC equations in the De Sitter and anti-De Sitter metrics reduce to the Killing equations. We therefore only quote results without giving details for these two cases. We present the complete procedure for solving the CC equations for the Einstein metric. As the same methods apply for the anti-Einstein, Schwarzschild, Reissner–Nordstrom, and Bertotti–Robinson-like metrics, we again only quote the results for them.

For the Einstein metric

$$ds^2=dt^2-\frac{dr^2}{1-r^2/R^2}-r^2(d\vartheta^2+\sin^2\vartheta\, d\phi^2), \tag{18}$$