Minimum Spanning Tree

- What is a Minimum Spanning Tree.
- Constructing Minimum Spanning Trees.
- What is a Minimum-Cost Spanning Tree.
- Applications of Minimum Cost Spanning Trees.
- Prim’s Algorithm.
  - Example.
  - Implementation.
- Kruskal’s algorithm.
  - Example.
  - Implementation.
- Review Questions.

What is a Minimum Spanning Tree.

- Let G = (V, E) be a simple, connected, undirected graph that is not edge-weighted.

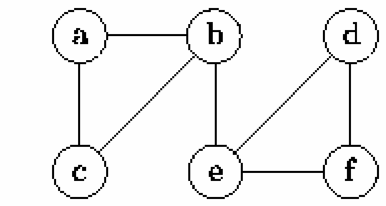
- A spanning tree of G is a free tree (i.e., a tree with no root) with | V | - 1 edges that connects all the vertices of the graph.

- Thus a minimum spanning tree for G is a graph, T = (V', E') with the following properties:
  - V' = V
  - T is connected
  - T is acyclic.

- A spanning tree is called a tree because every acyclic undirected graph can be viewed as a general, unordered tree. Because the edges are undirected, any vertex may be chosen to serve as the root of the tree.

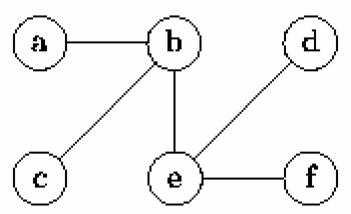
Constructing Minimum Spanning Trees

- Any traversal of a connected, undirected graph visits all the vertices in that graph. The set of edges which are traversed during a traversal forms a spanning tree.



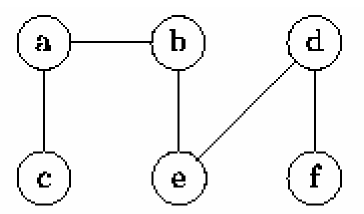
(a) Graph G

- For example, Fig:(b) shows the spanning tree obtained from a breadth-first traversal starting at vertex b.



(b) Breadth-first spanning tree of G rooted at b

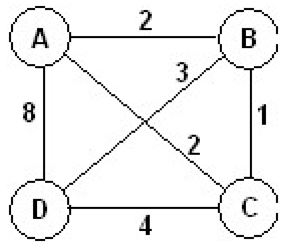
- Similarly, Fig:(c) shows the spanning tree obtained from a depth-first traversal starting at vertex c.



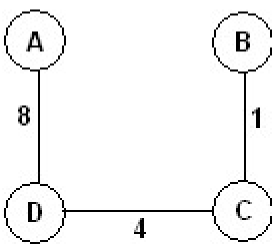
(c) Depth-first spanning tree of G rooted at c

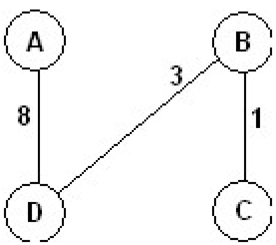
What is a Minimum-Cost Spanning Tree

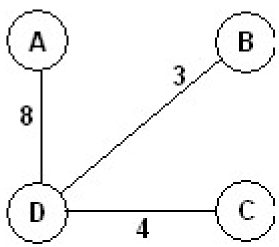
- For an edge-weighted , connected, undirected graph, G, the total cost of G is the sum of the weights on all its edges.
- A minimum-cost spanning tree for G is a minimum spanning tree of G that has the least total cost.
- Example: The graph

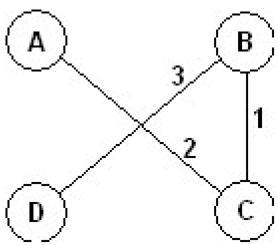


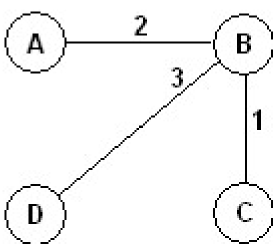
Has 16 spanning trees. Some are:



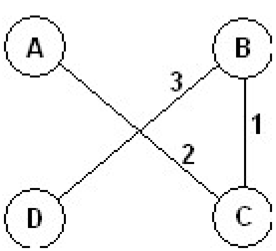


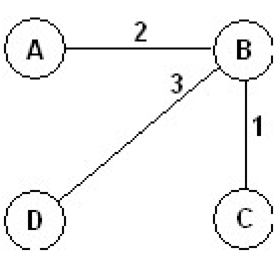






The graph has two minimum-cost spanning trees, each with a cost of 6:





Applications of Minimum-Cost Spanning Trees

Minimum-cost spanning trees have many applications. Some are:

- Building cable networks that join n locations with minimum cost.
- Building a road network that joins n cities with minimum cost.
- Obtaining an independent set of circuit equations for an electrical network.
- In pattern recognition minimal spanning trees can be used to find noisy pixels.

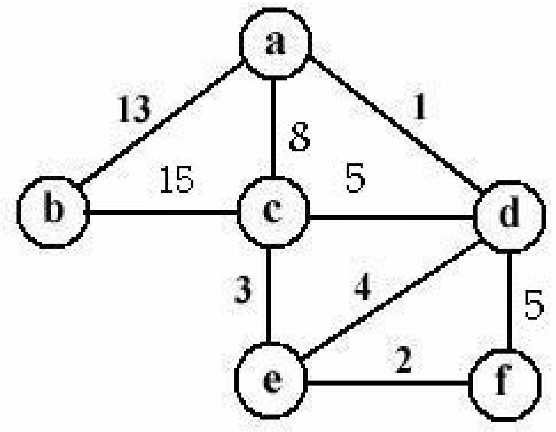
Prim's Algorithm

- Prim's algorithm finds a minimum cost spanning tree by selecting edges from the graph one-by-one as follows:
- It starts with a tree, T, consisting of the starting vertex, x.
- Then, it adds the shortest edge emanating from x that connects T to the rest of the graph.
- It then moves to the added vertex and repeats the process.

```
Consider a graph G=(V, E);
Let T be a tree consisting of only the starting vertex x;
while (T has fewer than IVI vertices)
{
        find a smallest edge connecting T to G-T;
        add it to T;
}
```

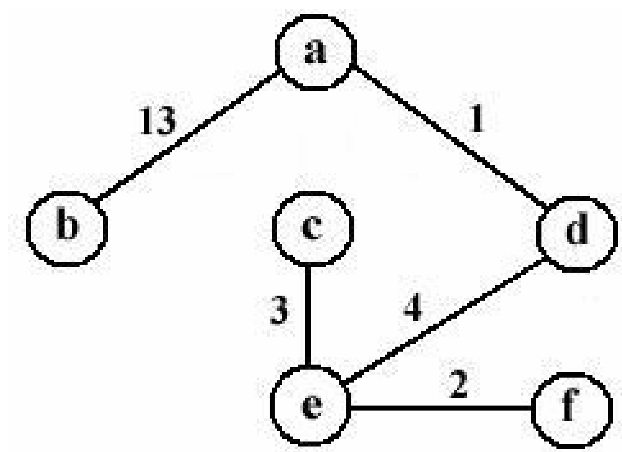
Example

Trace Prim's algorithm starting at vertex a:



Pass:	initially	1	2	3	4	5	6	weight	vl
Active vertex:		a	d	e	f	c	b		
a	0							0	-
b	∞	13	13	13	13	13		13	a
c	∞	8	5	3	3			3	e
d	∞	1						1	a
e	∞	∞	4					4	d
f	∞	∞	5	2				2	e

The resulting minimum-cost spanning tree is:



Implementation of Prim's Algorithm.

- Prims algorithn can be implememted similar to the Dijskra's algorithm as shown below:

```
public static Graph primsAlgorithm(Graph g, Vertex start){
   int n = g.getNumberOfVertices();
   Entry table[] = new Entry[n];
   for(int v = 0; v < n; v++)
      table[v] = new Entry();

   table[g.getIndex(start)].distance = 0;
   PriorityQueue queue = new BinaryHeap(g.getNumberOfEdges());
   queue.enqueue(new Association(new Integer(0), start));
   while(!queue.isEmpty()) {
      Association association = (Association)queue.dequeueMin();
      Vertex v1 = (Vertex) association.getValue();
      int n1 = g.getIndex(v1);
      if(!table[n1].known){
         table[n1].known = true;
         Iterator p = v1.getEmanatingEdges();
         while (p.hasNext()){
            Edge edge = (Edge) p.next();
            Vertex v2 = edge.getMate(v1);
            int n2 = g.getIndex(v2);
            Integer weight = (Integer) edge.getWeight();
            int d = weight.intValue();
```

Implementation of Prim's Algorithm Cont'd

```
            if(!table[n2].known && table[n2].distance > d){
               table[n2].distance = d; table[n2].predecessor = v1;
               queue.enqueue(new Association(new Integer(d), v2));
            }
         }
      }
   }
   GraphAsLists result = new GraphAsLists(false);
   Iterator it = g.getVertices();
   while (it.hasNext()){
      Vertex v = (Vertex) it.next();
      result.addVertex(v.getLabel());
   }
   it = g.getVertices();
   while (it.hasNext()){
      Vertex v = (Vertex) it.next();
      if (v != start){
         int index = g.getIndex(v);
         String from = v.getLabel();
         String to = table[index].predecessor.getLabel();
         result.addEdge(from, to, new Integer(table[index].distance));
      }
   }
   return result;
}
```

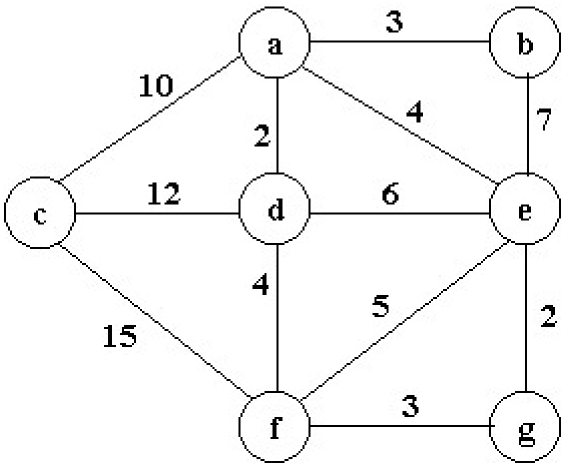
Kruskal's Algorithm.

- Kruskal’s algorithm also finds the minimum cost spanning tree of a graph by adding edges one-by-one.

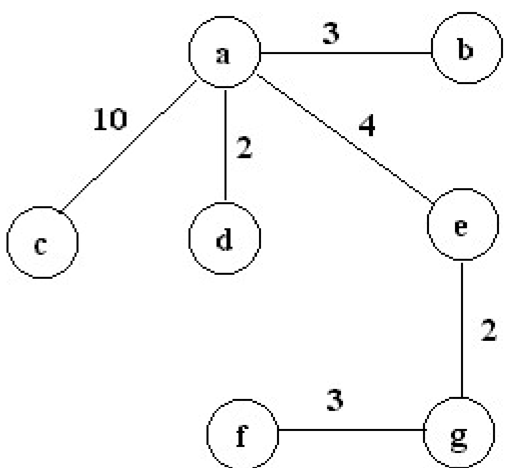
```
enqueue edges of G in a queue in increasing order of cost.
T = ϕ ;
while(queue is not empty){
  dequeue an edge e;
  if(e does not create a cycle with edges in T)
     add e to T;
}
return T;
```

Example for Kruskal's Algorithm.

Trace Kruskal's algorithm in finding a minimum-cost spanning tree for the undirected, weighted graph given below:



edge	ad	eg	ab	fg	ae	df	ef	de	be	ac	cd	cf
weight	2	2	3	3	4	4	5	6	7	10	12	15
insertion status	√	√	√	√	√	x	x	x	x	√	x	x
insertion order	1	2	3	4	5					6		



The minimum cost is: 24

Implementation of Kruskal's Algorithm

```
public static Graph kruskalsAlgorithm(Graph g){
   Graph result = new GraphAsLists(false);
   Iterator it = g.getVertices();
   while (it.hasNext()){
      Vertex v = (Vertex)it.next();
      result.addVertex(v.getLabel());
   }
   PriorityQueue queue = new BinaryHeap(g.getNumberOfEdges());
   it = g.getEdges();
   while(it.hasNext()){
      Edge e = (Edge) it.next();
      if (e.getWeight()==null)
         throw new IllegalArgumentException("Graph is not weighted");
      queue.enqueue(e);
   }

   while (!queue.isEmpty()){
      Edge e = (Edge) queue.dequeueMin();
      String from = e.getFromVertex().getLabel();
      String to = e.getToVertex().getLabel();
      if (!result.isReachable(from, to))
         result.addEdge(from,to,e.getWeight());
   }
   return result;
}
```

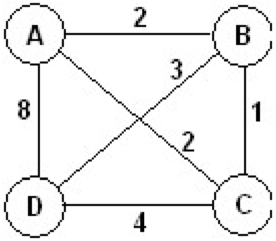
adds an edge only, if it does not create a cycle

Implementation of Kruskal's Algorithm – Cont'd

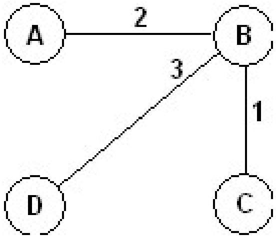
```
public abstract class AbstractGraph implements Graph {
   public boolean isReachable(String from, String to){
      Vertex fromVertex = getVertex(from);
      Vertex toVertex = getVertex(to);
      if (fromVertex == null || toVertex==null)
         throw new IllegalArgumentException("Vertex not in the graph");
      PathVisitor visitor = new PathVisitor(toVertex);
      this.preorderDepthFirstTraversal(visitor, fromVertex);
      return visitor.isReached();
   }

   private class PathVisitor implements Visitor {
      boolean reached = false;
      Vertex target;
      PathVisitor(Vertex t){target = t;}

      public void visit(Object obj){
         Vertex v = (Vertex) obj;
         if (v.equals(target)) reached = true;
      }
      public boolean isDone(){return reached;}
      boolean isReached(){return reached;}
   }
}
```

Prim’s and Kruskal’s Algorithms

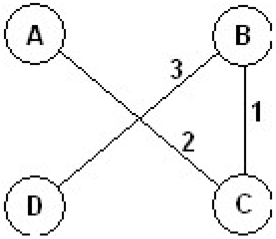
Note: It is not necessary that Prim's and Kruskal's algorithm generate the same minimum-cost spanning tree.
For example for the graph:



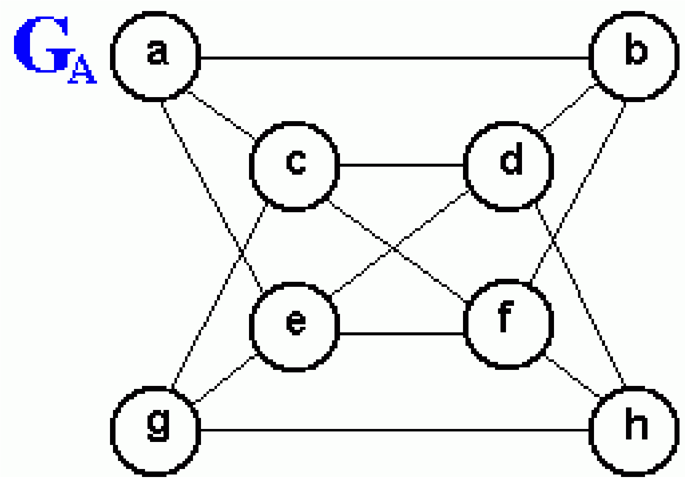
Kruskal's algorithm (that imposes an ordering on edges with equal weights) results in the following minimum cost spanning tree:

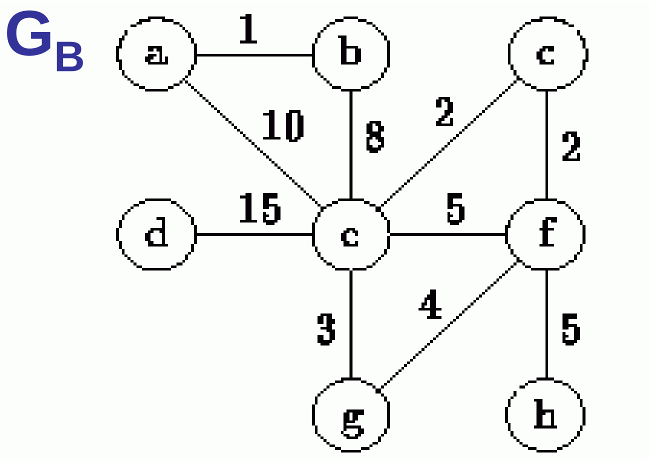


The same tree is generated by Prim's algorithm if the start vertex is any of: A, B, or D; however if the start vertex is C the minimum cost spanning tree is:



Review Questions





1. **Find the breadth-first spanning tree and depth-first spanning tree of the graph GA shown above.**
2. **For the graph GB shown above, trace the execution of Prim's algorithm as it finds the minimum-cost spanning tree of the graph starting from vertex a.**
3. **Repeat question 2 above using Kruskal's algorithm.**