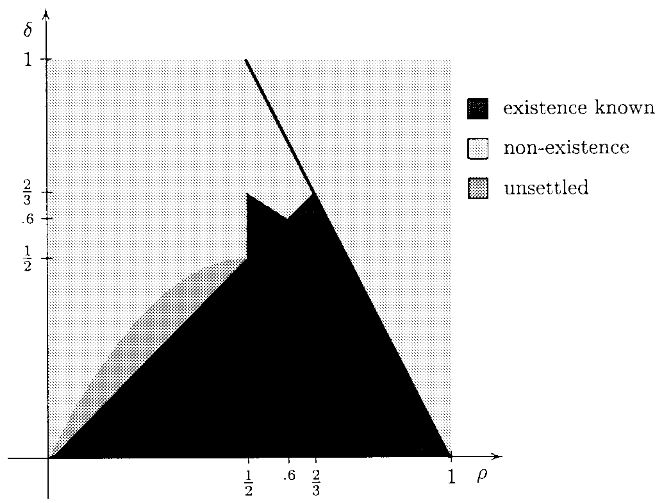

Fig. 2. Y: the domain of unrestricted codes.

where $\mu(\delta)$ is the best upper bound on the rate of an unrestricted code as a function of δ.

III. Conclusion and Open Problems

We have geometrically characterized the domain of linear and unrestricted binary codes in the (ρ, δ) plane. For $\delta > 1/2$ it might be worth shelling the domain according to the size of the code $M \leq 6, 7, \cdots$. A similar study for q-ary codes for $q > 2$ would also be of interest.

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New Single Asymmetric Error-Correcting Codes

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***Abstract*—New single asymmetric error-correcting codes are proposed. These codes are better than existing codes when the code length n is greater than 10, except for $n = 12$ and $n = 15$. In many cases one can construct a code C containing at least $\lceil 2^n/n \rceil$ codewords. It is known that a code with $|C| \geq \lceil 2^n/(n+1) \rceil$ can be easily obtained. It should be noted that the proposed codes for $n = 12$ and $n = 15$ are also the best known codes that can be explicitly constructed, since the best of the existing codes for these values of n are based on combinatorial arguments. Useful partitions of binary vectors are also presented.**

***Index Terms*— Asymmetric error-correcting codes, constant-weight codes, lower bounds, partitions.**

I. Introduction

In this correspondence new codes of asymmetric distance 2, capable of correcting a single asymmetric error, are presented. The asymmetric distance between two binary vectors, x and y, of length n is defined by

$$\Delta(x, y) = \max\{N(x, y), N(y, x)\}$$

where $N(x, y) = |\{i\colon x_i = 1 \text{ and } y_i = 0\}|$ and the minimum asymmetric distance of a code C is defined by

$$\Delta(C) = \min\{\Delta(x, y)\colon x, y \in C; x \neq y\}.$$

The function Δ is used to measure both the asymmetric distance between two binary vectors and to measure the asymmetric distance between a set of vectors (or a code). One may use the notation $\Delta(\{x, y\})$ instead of $\Delta(x, y)$ for consistency; however, for simplicity the latter form is used here.

The Hamming distance between two vectors can be defined as $D(x, y) = |\{i\colon x_i \neq y_i\}|$.

The theory and construction of asymmetric codes have been studied since the 1950's, and several code construction procedures and bounds have been published [1]–[6]. For example, it is known that a single asymmetric error-correcting code with $|C| \geq \lceil 2^n/(n+1) \rceil$ can be obtained by the group code [1]. Construction procedures which produce slightly better codes and upper bounds for these codes can be found in the literature, for example, in [3]–[6]. Additionally, bounds and tables for constant-weight codes have been reported [7], [8]; these are sometimes useful in constructing partitions as well as asymmetric error-correcting codes.

The main idea of the proposed construction method is to form the code from the Cartesian product of two sets of smaller codes, say

$$C = A_1 \times B_1 \; \cup \; A_2 \times B_2 \; \cup \; A_3 \times B_3 \cdots.$$

The choice of some simple properties about A's and B's guarantees the generation of a code with asymmetric distance 2, as explained in Section II.

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TABLE I
NEW SINGLE ASYMMETRIC ERROR-CORRECTING CODES

n	proposed codes	existing codes
2	2	2^a
3	2	2^a
4	4	4^a
5	6	6^a
6	12	12^b
7	16	18^c
8	28	36^c
9	52	62^c
10	104	108^c
11	180*	174^c
12	336	340^d
13	652*	624^d
14	1204*	1139^d
15	2188	2216^d
16	4232*	4168^d
17	7968*	7688^d
18	14624*	13951^d
19	28032*	26265^d
20	53856*	49940^a
21	101576*	95326^a
22	195700*	182362^a

[a] Code by Varshamov [1]

[b] Code by Kim and Freiman [2]

[c] Code by Delsarte and Piret [3]

[d] Code by Zhang and Xia [4]

* Proposed code improving the existing code

For an easy reference, Table I lists the new codes and the best known codes given in [1]–[4].

II. CONSTRUCTION METHOD

Before describing the construction method, we give the following definition.

Definition 1: Let A be the set of all the 2^p binary vectors of length p and let $A_1, A_2, \cdots, A_{p'}$ be a partition of A, i.e., $A_i \cap A_j = \phi$ and $\cup A_i = A$, such that $\Delta(A_i) \geq 2$ for $1 \leq i \leq p'$.

Let B be the set of the 2^{q-1} even-weight binary vectors of length q and $B_1, B_2, \cdots, B_{q'}$ be a partition of B such that $\Delta(B_j) \geq 2$ for $1 \leq j \leq q'$.

Let C be the code obtained by the Cartesian product of $A_i \times B_i$, i.e.,

$$C = A_1 \times B_1 \cup A_2 \times B_2 \cup A_3 \times B_3 \cdots. \quad (1)$$

When $p' \neq q'$, some A_i's or B_i's will be empty; in particular, $A_i \times B_i$ is empty for $i > \min(p', q')$. Obviously, the code C has $\Sigma_{i=1}^{m} |A_i| * |B_i|$ codewords where $m = \min(p', q')$. Several authors [7]–[10] have used the partitioning construction to design codes. We employ the same approach to construct asymmetric error-correcting codes, also our procedure to construct some partitions used this method.

Theorem 1: The code C, obtained in (1), of length $n = p + q$ is a single asymmetric error-correcting code.

Proof: Let $x, y \in C$ and $x \neq y$. Let $x = x'x''$ and $y = y'y''$ where $x' \in A_i$, $x'' \in B_i$, $y' \in A_j$, and $y'' \in B_j$.

Case 1: $i = j$: either $x' \neq y' \Rightarrow \Delta(x', y') \geq 2 \Rightarrow \Delta(x, y) \geq 2$ or $x'' \neq y'' \Rightarrow \Delta(x'', y'') \geq 2 \Rightarrow \Delta(x, y) \geq 2$.

Case 2: $i \neq j$: Here we have $D(x', y') \geq 1$ since $x' \neq y'$, and $D(x'', y'') \geq 2$ since $x'' \neq y''$ and x'' and y'' are both even. Therefore, $D(x, y) \geq 3 \Rightarrow \Delta(x, y) \geq 2$. □

Example 1: To construct a single asymmetric error-correcting code with $n = 6$, let $p = 2$ and $q = 4$. Then $A = \{00, 01, 10, 11\}$ can be partitioned into $A_1 = \{00, 11\}, A_2 = \{01\}$ and $A_3 = \{10\}$. And $B = \{0000, 0011, 0101, \cdots, 1111\}$ can be partitioned into $B_1 = \{0000, 0011, 1100, 1111\}$, $B_2 = \{0101, 1010\}$, $B_3 = \{0110, 1001\}$.

We obtain a code C of length 6 where

$$C = A_1 \times B_1 \cup A_2 \times B_2 \cup A_3 \times B_3$$

having $2 * 4 + 1 * 2 + 1 * 2 = 12$ codewords as follows:

00	0000	
00	0011	
00	1100	
00	1111	$A_1 \times B_1$
11	0000	
11	0011	
11	1100	
11	1111	
01	0101	
01	1010	$A_2 \times B_2$
10	0110	
10	1001	$A_3 \times B_3$

III. PARTITIONING

In order to maximize the size of the code C of length n, appropriate values of p and q, satisfying $n = p + q$, must be chosen. Once p and q are chosen, "good" A and B partitions should be obtained.

The norm of a partition $P = \{P_1, P_2, \cdots, P_m\}$ is defined as sum of squares, i.e. $\Sigma_{i=1}^{m} |P_i|^2$, as in [7]. In almost all cases, partitions with better norms produce better codes.

In general, we may not be able to obtain the best partition (in fact, it may not exist) but as a rule of thumb a good partition should have as few classes as possible and classes' size should be maximized. After the selection of the A and B partitions, the code is formed by taking the Cartesian product of the largest class of A with the largest class of B, then the second largest with the second largest, and so on. Without loss of generality, any partition $P = \{A_1, A_2, \cdots, A_m\}$ is assumed to satisfy $|A_j| \geq |A_{j+1}|$ for all j.

A Partitions

Most of the A partitions given in Table II are obtained using the Abelian group partitioning given in [1] and [5]. In some cases partitions that are better than group partitions can be obtained. For example, for $p = 5$, Table III gives a better partition. The Appendix explains a procedure which can be used to obtain better A partitions in many other cases, for example, for $p = 6, 10$, and 11.

B Partitions

The B partitions shown in Table IV are obtained from the partitions of the constant-weight vectors into classes with Hamming distance 4 (see [7]). For example, the entries for $q = 4$ which are 4, 2, and 2 are obtained as follows.

First, the vectors of weight 0 are partitioned into one class, namely, $\{0000\}$; the vectors of weight 2 are partitioned into three classes, $\{0011, 1100\}, \{1001, 0110\}$, and $\{1010, 0101\}$; and the vectors of weight 4 have one class which is $\{1111\}$. The eight even-weight vectors of length 4 can then be partitioned into three classes of size

TABLE II
A PARTITIONS

p	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	Remark
1	1	1											[1] Z_2
2	2	1	1										[1] Z_3
3	2	2	2	2									[1] Z_4
4	4	3	3	3	3								[1] Z_5
5	6	6	6	6	4	4							see Table III
6	12	10	10	8	8	8	8						Appendix A
7	16	16	16	16	16	16	16	16					[1] Z_8
8	32	28	28	28	28	28	28	28	28				[5] $Z_3 \times Z_3$
9	52	52	51	51	51	51	51	51	51	51			[1] Z_{10}
10	104	102	102	102	102	90	88	84	84	84	82		Appendix A
11	180	180	176	172	172	168	168	168	168	168	164	164	Appendix A

TABLE III
THE SIX A PARTITIONS FOR $p = 5$

A_1	A_2	A_3	A_4	A_5	A_6
00001	00010	00100	10000	01000	00000
11000	01100	10010	01010	10100	11100
00110	10001	01001	00101	00011	01011
10011	11010	10101	10110	10111	11111
01101	00111	01110	11001		
11110	11101	11011	01111		

4, 2, and 2, respectively, as follows:

$$\{0000, 0011, 1100, 1111\}, \{1001, 0110\}, \text{ and } \{1010, 0101\}$$

where each partition is of asymmetric distance 2. The constant-weight partitions of different weights are listed in [7] for binary vectors of length up to 14. Partitions of larger even-weight vectors can be obtained using the procedure given in [7], and partitions of different even weights can be assembled (as given in the above example) to obtain partitions of all even-weight vectors of the required length.

It was shown in [8] that when $q = 2^i$, or $q = 3 \times 2^i$ for $i \geq 1$, the even-weight vectors can be partitioned into $q - 1$ classes. For example, when $q = 4$, we obtain a partition with the following *three* sizes: 4, 2, 2. We note that the procedure given in [8] also works when $q = 5 \times 2^i$, for $i \geq 1$; this is because, as given in Table IV, the even-weight vectors of length 10 can be partitioned into *nine* classes, and consequently, when $q = 5 \times 2^i$, it can be partitioned into $5 \times 2^i - 1$ classes.

Theorem 2: Let $q = r \times 2^i$ where $1 \leq r \leq 6$ and $i \geq 1$. One can construct a code C of length $n = 2q - 2$ containing at least $\lceil 2^n/n \rceil$ codewords.

Proof: Let $n = 2q - 2$ and let $p = q - 2$. The set A of all the $2^p = 2^{q-2}$ binary vectors can be partitioned into $A_1, A_2, \cdots, A_{q-1}$, e.g., using any Abelian group of size $q - 1$. Since $q = r \times 2^i$ where $1 \leq r \leq 6$ and $i \geq 1$, as described above, the set B of the 2^{q-1} binary even-weight vectors of length q can be partitioned into $q - 1$ classes, viz., $B_1, B_2, \cdots, B_{q-1}$.

Therefore,

$$C = A_1 \times B_1 \ \cup \ A_2 \times \cdots B_2 \ \cup \ A_{q-1} \times B_{q-1}$$

of size

$$|C| = \sum_{i=1}^{q-1} |A_i| * |B_i|.$$

Since it is assumed that $|A_j| \geq |A_{j+1}|$ and $|B_j| \geq |B_{j+1}|$, the size of $|C|$ is minimized when

$$|A_i| = \frac{2^{q-2}}{q-1} \quad \text{and} \quad |B_i| = \frac{2^{q-1}}{q-1}, \qquad \text{for } 1 \leq i \leq q-1.$$

TABLE IV
B PARTITIONS

q	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}
1	1										
2	2										
3	2	1	1								
4	4	2	2								
5	4	3	3	3	3						
6	8	6	6	6	6						
7	12	11	10	10	9	8	4				
8	24	22	20	20	18	16	8				
9	36	35	35	35	33	32	32	13	5		
10	72	70	70	70	62	60	54	40	14		
11	125	124	118	117	110	101	100	94	79	46	10
12	248	246	234	234	224	198	192	176	136	94	66

Therefore,

$$\begin{aligned} |C| &\geq \sum_{i=1}^{q-1} \frac{2^{q-2}}{q-1} \times \frac{2^{q-1}}{q-1} \\ &= \frac{2^{2q-3}}{q-1} = \frac{2^{2q-2}}{2q-2} = \frac{2^n}{n}. \end{aligned}$$

Hence, $|C| \geq \lceil 2^n/n \rceil$.

IV. CONCLUDING REMARKS

The proposed codes improve the existing lower bounds for all codes of length $n > 10$ (except $n = 12$ and $n = 15$). Moreover, the proposed codes for $n = 12$ and $n = 15$ are the best known codes that can be explicitly constructed, since the codes given in [4] are based on combinatorial arguments. It has been shown that, in many cases, the proposed codes C contains at least $\lceil 2^n/n \rceil$ codewords. Although we present here codes only up to $n = 22$, the construction procedures can be applied to larger word sizes to obtain code lengths larger than the best existing codes. This contention has not been proved here, but from the trend we observe in Table I, as well as by using numerical verification of the statement for larger values of n (for $n \leq 40$), we suspect it might be true even for $n > 40$. In addition, whenever we use a value of q which can be partitioned into $q - 1$ classes, we can obtain $|C| > 2^n/n$ not only for $n = 2q - 2$ but also for other values of n in this vicinity where "good" p partitions exist. For example, we can get 15 partitions of even-weight vectors when $q = 16$, and so we can obtain codes with size more than $\lceil 2^n/n \rceil$ not only when $n = 30$, but for $n = 28$ and 29 as well.

Obviously, better A and B partitions will yield better codes. A similar technique can also be used to construct codes capable of correcting more than one asymmetric error.

For a given value of n, appropriate values of p and q must be chosen in order to maximize the code length. The values of p and q that produce codes for $n \leq 22$ are listed in Table V. We note that, in general, $p < q \leq p+5$. Furthermore, we note that, for the values of n discussed in the correspondence, q is even, and p is the largest integer less than q; the only exceptions occur when $n = 15$ and $n = 19$.

Finally, it was *noticed* that in many cases the single asymmetric error-correcting codes satisfy $\lfloor 2^n/n \rfloor \leq |C| \leq \lfloor 2^n/(n-1) \rfloor$ for $2 \leq n \leq 22$. For $n = 11, 12$, the lower bound is not satisfied as shown in Table V. The codes have to be slightly improved for $n = 11$ and 12 to satisfy the lower bound, e.g., for $n = 12$ one needs to get a code with 341 codewords instead of 340 codewords. The existing upper bounds are more than $\lfloor 2^n/(n-1) \rfloor$ for $n \geq 11$, so the observation about the upper bound may not be true for $n \geq 11$ but it is plausible.

APPENDIX
IMPROVED A PARTITIONS

The improved A partitions—which are partitions of binary vectors such that the minimum asymmetric distance between elements in any class is at least 2—given here are obtained using a procedure very similar to the method given in Section II for constructing asymmetric error-correcting codes. To obtain partitions of all binary vectors of length p, we start with two numbers s and t, where $s = \lfloor (p-1)/2 \rfloor$, and $t = \lceil (p+1)/2 \rceil$. Note that $p = s + t$. The goal is to produce $p+1$ partitions which are better than the group partitions. (It may, however, be possible to find an A partition, which is more useful for constructing asymmetric error-correcting codes, than a $p+1$ partition.)

Then, we employ in different distinct combinations all partitions of vectors of length s, and all partitions of the odd- as well as the even-weight vectors of length t to produce the desired partitions of length p. We know that it is possible to get $s+1$ partitions of the vectors of length s, and t partitions of all odd- (or even-) weight vectors of length t [7]; the former is the same as the A partitions, and the latter are similar to the B partitions. Thus it is always possible to obtain $2t$ partitions of binary vectors of length p. When p is odd, $2t = p+1$, but when p is even, $2t = p+2$. The proposed procedure produces better A partitions (than the group partitions) for all odd values of p, but only for certain even values of p, as detailed below.

When $p \bmod 4 = 0$, we get more partitions using our procedure, so we prefer the $p+1$ partitions obtained using the group method. For odd values of p, i.e., when $p \bmod 4 = 1$ or 3, we can always get $p+1$ partitions, and in many cases we obtain better partitions than the group partitions. However, we observe that our procedure produces a flat partition (that is, all partitions having nearly equal elements), as does the group method, when $p = 2^i - 1$. Finally, when $p \bmod 4 = 2$, we get $p+1$ partitions which are better than those obtained using the group method whenever $t = r \times 2^i$, where $1 \leq r \leq 6$; this is because, as described in Section III, we can get $t-1$ partitions of the even-weight vectors of length t in these cases. Therefore, for all odd values of p, and for quite a few cases when p is even, we can obtain A partitions which are at least as good as (and in most cases better than) those obtained using the group method.

As an example, the A partition for $p = 6$ of sizes 12, 10, 10, 8, 8, 8, and 8 is illustrated.

We get $s = \lfloor 6 - 1/2 \rfloor = 2$ and $t = \lceil 6 + 1/2 \rceil = 4$. Recall that one can partition all binary vectors of length 2, $S = \{00, 01, 10, 11\}$, into $S_1 = \{00, 11\}$, $S_2 = \{01\}$, $S_3 = \{10\}$. All the even-weight binary vectors of length 4, $T = \{0000, 0011, 0101, \cdots, 1111\}$, can be partitioned into $T_1 = \{0000, 0011, 1100, 1111\}$, $T_2 = \{0101, 1010\}$, $T_3 = \{0110, 1001\}$. And the eight odd-weight vectors $T' = \{0001, 0010, 0100, \cdots, 1110\}$ can be partitioned into four classes $T'_1 = \{0001, 1110\}$, $T'_2 = \{0010, 1101\}$, $T'_3 = \{0100, 1011\}$, and $T'_4 = \{1000, 0111\}$. Now we can obtain the following seven A partitions of all the 2^6 binary vectors:

$$A_1 = S_1 \times T_1 \ \cup\ S_2 \times T_2 \ \cup\ S_3 \times T_3 \text{ of size } 12$$
$$A_2 = S_1 \times T_2 \ \cup\ S_2 \times T_3 \ \cup\ S_3 \times T_1 \text{ of size } 10$$
$$A_3 = S_1 \times T_3 \ \cup\ S_2 \times T_1 \ \cup\ S_3 \times T_2 \text{ of size } 10$$
$$A_4 = S_1 \times T'_1 \ \cup\ S_2 \times T'_2 \ \cup\ S_3 \times T'_3 \text{ of size } 8$$
$$A_5 = S_1 \times T'_2 \ \cup\ S_2 \times T'_3 \ \cup\ S_3 \times T'_4 \text{ of size } 8$$
$$A_6 = S_1 \times T'_3 \ \cup\ S_2 \times T'_4 \ \cup\ S_3 \times T'_1 \text{ of size } 8$$
$$A_7 = S_1 \times T'_4 \ \cup\ S_2 \times T'_1 \ \cup\ S_3 \times T'_2 \text{ of size } 8.$$

Notice that $A_1 \cup A_2 \cup \cdots \cup A_7$ contain all the 64 binary vectors of length 6, $A_i \cap A_j = \phi$ when $i \neq j$, and $\Delta(A_i) \geq 2$ for $1 \leq i \leq 7$.

The sizes of $A_1, A_2, \cdots, A_7$ are 12, 10, 10, 8, 8, 8, and 8, respectively, as given in Table II. This can be contrasted with the flat partition of the 2^6 binary vectors: 10, 9, 9, 9, 9, 9, and 9 obtained using the group method, where Z_7, the only Abelian group is used.

This procedure may be deemed as a generalized version of the code construction procedure proposed in Section II. Clearly, each A_i in the above example is obtained in the same way C is obtained, only using different combinations of S, T, and T' partitions. Theorem 1 shows that the code C obtained using our procedure produces a set of binary vectors satisfying the minimum asymmetric distance of 2. Therefore, we can offer arguments similar to those made in the proof of Theorem 1 to show that each of the A_i's in the above procedure produces partitions satisfying the minimum asymmetric distance property. We also note that each of the A_i's employs a unique combination of S, T, and T' partitions, and that $\Sigma\ A_i = 2^p$. Thus it can be concluded that the proposed partitioning method produces valid partitions of the 2^p binary vectors.

We note that it is not necessary to always match the even partitions with rotated versions of the other partitions. For instance, if we have

TABLE V
CARDINALITY OF THE ASYMMETRIC CODES
LENGTH n VERSUS. $\lfloor 2^n/n \rfloor$ AND $\lfloor 2^n/n-1 \rfloor$

n	$\lfloor \frac{2^n}{n} \rfloor$	existing code	p	q	proposed code	$\lfloor \frac{2^n}{n-1} \rfloor$	upper bound [6]
2	2	2^a	0	2	2	4	2
3	2	2^a	1	2	2	4	2
4	4	4^a	0	4	4	5	4
5	6	6^a	1	4	6	8	6
6	10	12^b	2	4	12	12	12
7	18	18^c	3	4	16	21	18
8	32	36^c	2	6	28	36	36
9	56	62^c	3	6	52	64	62
10	102	108^c	4	6	104	113	117
11	186	174^c	5	6	180*	204	210
12	341	340^d	4	8	336	372	410
13	630	624^d	5	8	652*	682	786
14	1170	1139^d	6	8	1204*	1260	1500
15	2184	2216^d	6	9	2188	2340	2828
16	4096	4168^d	6	10	4232*	4369	5430
17	7710	7688^d	7	10	7968*	8192	10379
18	14563	13951^d	8	10	14624*	15420	19898
19	27594	26265^d	7	12	28032*	29127	38008
20	52428	49940^a	8	12	53856*	55188	73174
21	99864	95326^a	9	12	101576*	104857	140798
22	190650	182326^a	10	12	195700*	199728	271953

(a, b, c, d, and * are the same as in Table I.)

four partitions in each of S, T, and T', then the following combination would give A partitions better than those obtained using the simple rotation strategy used in the previous example:

$$
\begin{aligned}
A_1 &= S_1 \times T_1 \ \cup\ S_2 \times T_2 \ \cup\ S_3 \times T_3 \ \cup\ S_4 \times T_4\\
A_2 &= S_1 \times T_2 \ \cup\ S_2 \times T_1 \ \cup\ S_3 \times T_4 \ \cup\ S_4 \times T_3\\
A_3 &= S_1 \times T_3 \ \cup\ S_2 \times T_4 \ \cup\ S_3 \times T_1 \ \cup\ S_4 \times T_2\\
A_4 &= S_1 \times T_4 \ \cup\ S_2 \times T_3 \ \cup\ S_3 \times T_2 \ \cup\ S_4 \times T_1\\
A_5 &= S_1 \times T_1' \ \cup\ S_2 \times T_2' \ \cup\ S_3 \times T_3' \ \cup\ S_4 \times T_4'\\
A_6 &= S_1 \times T_2' \ \cup\ S_2 \times T_1' \ \cup\ S_3 \times T_4' \ \cup\ S_4 \times T_3'\\
A_7 &= S_1 \times T_3' \ \cup\ S_2 \times T_4' \ \cup\ S_3 \times T_1' \ \cup\ S_4 \times T_2'\\
A_8 &= S_1 \times T_4' \ \cup\ S_2 \times T_3' \ \cup\ S_3 \times T_2' \ \cup\ S_4 \times T_1'.
\end{aligned}
$$

We observe that the "best" strategy depends on the lengths and distributions of S, T, and T'. It is clear that the efficiency of the resulting single asymmetric error-correcting code depends on our ability to find "good" partitions, which suggests that more research in this direction is in order.

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New Constant Weight Codes from Linear Permutation Groups

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***Abstract*—New constant weight codes are found by considering certain linear permutation groups. A code is obtained as a collection of orbits of words under such a group. This leads to a difficult optimization problem, where a stochastic search heuristic, tabu search, is used to find good solutions in a feasible amount of time. Nearly 40 new codes of length at most 28 are presented.**

***Index Terms*—Combinatorial optimization, constant weight codes, permutation groups, tabu search.**

I. Introduction

The aim of this correspondence is to construct new constant weight codes with the help of an optimization heuristic, tabu search. These new codes give improved lower bounds on $A(n, d, w)$, the maximum number of binary words of length n, minimum distance d, and constant weight w.

Several recent papers have discussed this problem and a variety of optimization methods have been applied [4], [5], [10]. Unfortunately, a search without limitations on the structure of the code does not work well if we are searching for a large code; in such cases, we can predefine a structure (automorphism group) of the code to facilitate the search.

The approach of searching for t-designs with predefined automorphisms was considered by Kramer and Mesner in [11]. For constant weight codes, a similar approach was taken by Brouwer [1], Brouwer *et al.* [2], and Kibler [9]. We have developed this approach further and carried out a computer-aided search for new codes. We have managed to improve 37 codes with $n \leq 28$ and $4 \leq d \leq 12$ in the tables of [2].

The automorphism groups used in this correspondence are primarily linear permutation groups. The codes obtained are invariant under such groups, with the occasional exception of a few words of a code. The main groups used in this work are affine groups (also in finite rings), projective special linear groups, and subgroups of these. These groups are discussed in Section II.

The problem of finding codes of maximal size invariant under a given permutation group is an instance of the problem of finding the largest cliques in a graph with weighted vertices, which is NP-complete [6]. That is why we use a stochastic search algorithm, tabu search, which we believe can here relatively well handle

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