Galois Connection in Fuzzy Binary Relations, Applications for Discovering Association Rules and Decision Making

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Abstract. Galois connection in crisp binary relations has proved to be useful for several applications in computer science. Unfortunately, data is not always presented as a crisp binary relation but may be composed of fuzzy values, thus forming a fuzzy binary relation. This paper aims at defining the notion of fuzzy galois connection corresponding to a fuzzy binary relation in two steps: firstly by defining the term fuzzy maximal rectangle and secondly, by extending the galois lattice structure to fuzzy binary relations. Applications concerning discovery of fuzzy association rules and decision making are also presented.
Keywords: *Galois connection, fuzzy relation, learning, decision making, association rules.*

1 Introduction

Galois lattice structure and galois connection have shown their usefulness for several applications in computer science. Several papers have been published in different applied journals using the galois lattice structure of a crisp binary relation for learning, classification, information retrieval, reasoning, finding additional information, object oriented programming, database organization and automatic entity extraction [1, 2, 4].

Unfortunately, data is not always presented as a crisp binary relation but may be composed of fuzzy values, thus forming a fuzzy binary relation.

In their previous attempts at fuzzy binary relation decompostion, the authors defined and extended the notion of difunctionality to fuzzy binary relations [7]. This paper aims at defining the notion of fuzzy galois connection corresponding to a fuzzy binary relation in two steps: firstly by defining the term fuzzy maximal rectangle and secondly, by extending the galois lattice structure to fuzzy binary relations.

This paper is organized as follows: In Section 2, the fundamental operations and properties of fuzzy sets are recalled. In Section 3, the mathematical definitions and properties of a classical galois lattice structure are recalled. In Section 4,the notion of a fuzzy maximal rectangle is defined; then a galois lattice structure is mathematically extended to fuzzy binary relations and fuzzy galois connection is defined. In Section 5, applications of fuzzy galois connection to the discovery of association rules and decision making using classification are given.

2 Mathematical Background

Here we review some definitions and results that will be needed in the sequel. For details we refer to [6, 7]

2.1 Fuzzy Sets

Let U be a set, called the universe of discourse. An element of U is denoted by lowercase letters.

A fuzzy set is defined as a collection of elements $x \in U$ which includes a degree of membership for each of its elements. The membership degree for each element lies within the range $[0, 1]$. It may also be expressed by a membership function.

Example 1. If $U = \{p, q, r\}$ then $\widetilde{X} = \{p/0.5, q/0.1, y/0.9\}$ is an example of a fuzzy set. The degrees of membership of p, q and r in $\widetilde{X}$ are $0.5, 0.1$ and 0.9 respectively. The fuzzy values $0.5, 0.1$ and 0.9 determine the strength of membership of a particular element. Here, r with a membership degree of 0.9 has a strong membership of $\widetilde{X}$.

Crisp sets can be defined as special cases of fuzzy sets with the membership degrees restricted to the values of 0 and 1 only.

Example 2. If $U = \{a, b, c\}$ then $Y = \{a, b\}$ is an example of a crisp set. The elements a and b are called members of the set Y. Y can also be written as a fuzzy set with membership degrees restricted to the values of 0 and 1 only. Thus $Y = \{a/1, b/1, c/0\}$ is another way of expressing the set Y.

2.2 Basic Set-Theoretic operations on Fuzzy Sets

The fundamental set-theoretic operations on fuzzy sets which are of concern to us are inclusion, union, intersection.

A fuzzy set $\widetilde{A}$ is said to be included in another fuzzy set $\widetilde{B}$ if $\forall x \in \widetilde{A}\ \exists x \in \widetilde{B} \mid \mu_{\widetilde{A}}(x) \leq \mu_{\widetilde{B}}(x)$.

The membership function $\mu_{\widetilde{C}}(x)$ of the intersection $\widetilde{C} = \widetilde{A} \cap \widetilde{B}$ is defined by $\mu_{\widetilde{C}}(x) = min\{\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(x)\}$.

The membership function $\mu_{\widetilde{D}}(x)$ of the union $\widetilde{D} = \widetilde{A} \cup \widetilde{B}$ is defined by $\mu_{\widetilde{D}}(x) = max\{\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(x)\}$.

2.3 Cartesian Product

The cartesian product of a crisp set with a fuzzy set is an important notion as it will be used to define the notion of a fuzzy maximal rectangle. Since a crisp set can be defined as a fuzzy set with membership values equal to 0 or 1 only, so, the cartesian product $\widetilde{R} = A \times \widetilde{B}$ is defined by

$$\widetilde{R} = \{(x, y)/min(\mu_A(x), \mu_{\widetilde{B}}(y)) \mid \forall x \in A, y \in \widetilde{B}\} \tag{1}$$

Example 3. Consider $A = \{a/1, b/0\}$ and $\widetilde{B} = \{p/0.1, q/0.9\}$. So $\widetilde{R} = \{(a, p)/0.1, (b, p)/0, (a, q)/0.9, (b, q)/0\}$. The resulting relation $\widetilde{R}$ is a fuzzy binary relation.

3 Classical Galois Lattice

In this section, we start by presenting some formal properties of rectangular relations. Along all this section and the following ones, J stands for any set of indices.

Definition 1. *Let R be a binary relation defined on a set E. A rectangle $A \times B$ is a cartesian product of two sets (A, B) such that $A \times B \subseteq R$. A is the domain of the rectangle (A, B) and B is its range.*

Definition 2. *Let (A, B) be a rectangle of a given relation R defined on a set E. The rectangle (A, B) is said to be maximal if whenever $A \times B \subseteq A' \times B' \subseteq R$, then $A = A'$ and $B = B'$.*

In[5], the authors showed that, using an order relation ($\ll$), the set of rectangles of a binary relation R can be organized under a complete and distributive lattice.

Definition 3. *Let R be a binary relation defined on a set E and T_r the set of rectangles of R ordered by the relation $\ll$. $(T_r, \ll)$ is a complete and distributive lattice with the smallest element $(\emptyset, E)$ and the biggest one $(E, \emptyset)$, and where the supremum P and the infinimum H of any set of rectangles of T_r are given respectively as follows :*

$$P_{j \in J}(A_j, B_j) = (\cup_{j \in J} A_j, \cap_{j \in J} B_j) \tag{2}$$

$$H_{j \in J}(A_j, B_j) = (\cap_{j \in J} A_j, \cup_{j \in J} B_j) \tag{3}$$

3.1 Galois Connection in Crisp Binary Relations

Let R be a binary relation defined on a set E. For two sets A and B such that $A, B \subseteq E$, we define the operators $f(A) = A^R$ and $h(B) = B^Q$ as follows:

$$f(A) = A^R = \{d \mid \forall g, g \in A \Rightarrow (g,d) \in R\} \tag{4}$$

$$h(B) = B^Q = \{g \mid \forall d, d \in B \Rightarrow (g,d) \in R\} \tag{5}$$

The operators ^{R} and ^{Q} define a galois connection[10] between the ordered sets $(A, \ll)$ and $(B, \ll)$ by satisfying the following conditions:

$$A_i \subseteq A_j \Rightarrow A_i^R \supseteq A_j^R \tag{6}$$

$$B_i \subseteq B_j \Rightarrow B_i^Q \supseteq B_j^Q \tag{7}$$

$$A_i \subseteq A_i^{RQ} \text{ and } B_i \subseteq B_i^{QR} \tag{8}$$

Proposition 1. *A pair (f, h) or $(^R, ^Q)$ of maps is called a galois connection if and only if*

$$A \subseteq B^Q \Leftrightarrow B \subseteq A^R \tag{9}$$

Proposition 2. *For every galois connection (f, h) or $(^R, ^Q)$*

$$f = fgf \text{ and } g = gfg \tag{10}$$

3.2 Sub-lattice of maximal rectangles

The set of maximal rectangles of a binary relation R, using the galois connection operators ^{R} and ^{Q} can be organized under a complete sub-lattice[5].

Definition 4. *Let R be a binary relation defined on a set E and $T_{r\max}$ the set of maximal rectangles of R ordered by the relation $\ll^{\max}$. Hence, $(T_{r\max}, \ll^{\max})$ is a complete sub-lattice where the supremum $P^{\max}$ and the infinimum $H^{\max}$ of any set of maximal rectangles of $T_{r\max}$ are given respectively as follows:*

$$P^{\max}_{j\in J}(A_j, B_j) = (\cup_{j\in J} A_j, \cap_{j\in J} B_j) \tag{11}$$

$$H^{\max}_{j\in J}(A_j, B_j) = (\cap_{j\in J} A_j, \cup_{j\in J} B_j) \tag{12}$$

4 Fuzzy Galois lattice

In this section, we start by introducing the notions of fuzzy rectangle and fuzzy maximal rectangle and we prove some formal properties; we also define a fuzzy galois connection by proving some of its conditions. Then we propose to organize the set of fuzzy rectangles under a complete and distributive lattice.

Definition 5. *Let $\widetilde{R}$ be a fuzzy binary relation defined from E to $\widetilde{F}$. A fuzzy rectangle of $\widetilde{R}$ is a couple of two sets $(A, \widetilde{B})$ such that $A \times \widetilde{B} \subseteq \widetilde{R}$. i.e., $\mu_{A\times\widetilde{B}}(x, y) \leq \mu_{\widetilde{R}}(x, y)$, where $x \in E$ and $y \in \widetilde{F}$.*

Definition 6. *Let $\widetilde{R}$ be a fuzzy binary relation defined from E to $\widetilde{F}$. The relation $A \times \widetilde{B}$, such that $A \subseteq E$ and $\widetilde{B} \subseteq \widetilde{F}$ is called fuzzy rectangular relation associated with the rectangle $(A, \widetilde{B})$ of $\widetilde{R}$. A is the domain of this relation and $\widetilde{B}$ is its range.*

Remark 1. The correspondence between the fuzzy rectangles $(A_i, \widetilde{B}_i)$ and the associated fuzzy rectangular relations $(A_i \times \widetilde{B}_i)$ is a bijective one, except when $A_i = \emptyset$ or $\widetilde{B}_i = \emptyset$. For instance, the fuzzy rectangles $(\emptyset, \widetilde{B}_1)$ and $(A_1, \emptyset)$ are both associated with the null rectangular relation $\emptyset$. This is the main reason for which we have made the distinction between fuzzy rectangles and fuzzy rectangular relations in the sense that the concept of a fuzzy rectangle allows us to obtain a lattice structure.

Definition 7. *A fuzzy maximal rectangle, or fuzzy concept is defined as the cartesian product $A \times \widetilde{B} \subseteq \widetilde{R}$, where A is a crisp set and $\widetilde{B}$ is a fuzzy one, whenever $A \times \widetilde{B} \subseteq A' \times \widetilde{B}' \subseteq R$, then $A = A'$ and $B = \widetilde{B}'$.*

Proposition 3. *Let $\{A_j, B_j\}$ (with $j \in J$) be a set of fuzzy rectangular relations of a fuzzy binary relation $\widetilde{R}$. The relation $(\cup_{j\in J}A_j) \times (\cap_{j\in J}\widetilde{B}_j)$ is a rectangular relation of $\widetilde{R}$. Therefore $(\cup_{j\in J}A_j, \cap_{j\in J}\widetilde{B}_j)$ is a rectangle of $\widetilde{R}$.*

Proof. We have to prove that $\forall(a,b) \in (\cup_{j\in J}A_j) \times (\cap_{j\in J}\widetilde{B}_j) \Rightarrow (a,b) \in \widetilde{R}$

$$\forall(a,b) \in (\cup_{j\in J}A_j) \times (\cap_{j\in J}\widetilde{B}_j) \Rightarrow a \in \cup_{j\in J}A_j \Rightarrow \exists k \in J \mid a \in A_k \tag{13}$$
$$\forall(a,b) \in (\cup_{j\in J}A_j) \times (\cap_{j\in J}\widetilde{B}_j) \Rightarrow \mu_{\cap_{j\in J}\widetilde{B}_j}(b) = \min_{j\in J}(\mu_{\widetilde{B}_j}(b))$$

Hence, and particularly for k we have

$$\mu_{\cap_{j\in J}\widetilde{B}_j}(b) \leq \mu_{\widetilde{B}_k}(b) \tag{14}$$
$$(13)\,\text{and}\,(14) \Rightarrow \mu_{(\cup_{j\in J}A_j)\times(\cap_{j\in J}\widetilde{B}_j)}(a,b) \leq \mu_{(A_k,B_k)}(a,b) \leq \mu_{\widetilde{R}}(a,b)$$

Since $(A_k, \widetilde{B}_k)$ is a fuzzy rectangle of $\widetilde{R} \Rightarrow (\cup_{j\in J}A_j) \times (\cap_{j\in J}\widetilde{B}_j)$ is a fuzzy rectangle of $\widetilde{R}$.

Proposition 4. *Let $\{A_j, \widetilde{B}_j\}$ (with $j \in J$) be a set of fuzzy rectangular relations of a fuzzy binary relation $\widetilde{R}$. The relation $(\cap_{j\in J}A_j) \times (\cup_{j\in J}\widetilde{B}_j)$ is a rectangular relation of $\widetilde{R}$. Therefore $(\cap_{j\in J}A_j, \cup_{j\in J}\widetilde{B}_j)$ is a fuzzy rectangle of $\widetilde{R}$.*

Proof. Conversely follows from the proof of the previous proposition.

Proposition 5. *The following relation $\widetilde{\ll}$ defined on $\widetilde{R}$ is a partial order relation: $(A_1, \widetilde{B}_1) \widetilde{\ll} (A_2, \widetilde{B}_2) \Leftrightarrow A_1 \subseteq A_2$ and $\widetilde{B}_2 \subseteq \widetilde{B}_1$, where $(A_1, \widetilde{B}_1)$, $(A_2, \widetilde{B}_2) \in \widetilde{R}$.*

Proof. According to the definition of a partial order relation, we have to prove that $\widetilde{\ll}$ is reflexive, antisymmetrical and transitive.

– Reflexivity

  $\forall A_1 \subseteq E$, $\widetilde{B}_1 \subseteq \widetilde{F}$ we have $A_1 \subseteq A_1$ and $\widetilde{B}_1 \subseteq \widetilde{B}_1$ (by reflexivity of $\subseteq$). Hence, $(A_1, \widetilde{B}_1) \widetilde{\ll} (A_1, \widetilde{B}_1)$.

– Antisymmetry

  $(A_1, \widetilde{B}_1) \widetilde{\ll} (A_2, \widetilde{B}_2)$ and $(A_2, \widetilde{B}_2) \widetilde{\ll} (A_1, \widetilde{B}_1)$
  $\Leftrightarrow A_1 \subseteq A_2$ and $\widetilde{B}_2 \subseteq \widetilde{B}_1$ and $A_2 \subseteq A_1$ and $\widetilde{B}_1 \subseteq \widetilde{B}_2$
  $\Rightarrow A_1 = A_2$ and $\widetilde{B}_1 = \widetilde{B}_2$
  $\Rightarrow (A_1, \widetilde{B}_1) = (A_2, \widetilde{B}_2)$.

– Transitivity

  $(A_1, \widetilde{B}_1) \widetilde{\ll} (A_2, \widetilde{B}_2)$ and $(A_2, \widetilde{B}_2) \widetilde{\ll} (A_3, \widetilde{B}_3)$.
  $\Leftrightarrow A_1 \subseteq A_2$ and $\widetilde{B}_2 \subseteq \widetilde{B}_1$ and $A_2 \subseteq A_3$ and $\widetilde{B}_3 \subseteq \widetilde{B}_2$
  $\Rightarrow A_1 \subseteq A_3$ and $\widetilde{B}_3 \subseteq \widetilde{B}_1$
  $\Leftrightarrow (A_1, \widetilde{B}_1) \ll (A_3, \widetilde{B}_3)$.

Theorem 1. *Let $\widetilde{R}$ be a fuzzy binary relation defined from E to $\widetilde{F}$ and T_r the set of fuzzy rectangles of $\widetilde{R}$ ordered by the relation $\widetilde{\ll}$. $(T_r, \widetilde{\ll})$ is a complete and distributive lattice with a smallest element $(\emptyset, \widetilde{F})$ and a biggest one $(E, \emptyset)$, and where:*

$$P_{j\in J}(A_j, \widetilde{B}_j) = (\cup_{j\in J}A_j, \cap_{j\in J}\widetilde{B}_j) \tag{15}$$
$$H_{j\in J}(A_j, \widetilde{B}_j) = (\cap_{j\in J}A_j, \cup_{j\in J}\widetilde{B}_j) \tag{16}$$

Proof. First, let us show that any set of fuzzy rectangles of $\widetilde{R}$ has a smallest superior boundary and a biggest inferior one which are both fuzzy rectangles of $\widetilde{R}$. Then, we show that P is distributive relatively to H and conversely.

- Smallest Superior Boundary

  $\forall j \in J$, $(A_j, \widetilde{B}_j) \,\widetilde{\lll}\, (C, \widetilde{D})$
  $\Leftrightarrow \forall j \in J$, $(A_j \subseteq C)$ and $(\widetilde{D} \subseteq \widetilde{B}_j)$
  $\Leftrightarrow \cup_{j\in J} A_j \subseteq C$ and $\widetilde{D} \subseteq \cap_{j\in J} \widetilde{B}_j$
  $\Leftrightarrow (\cup_{j\in J} A_j, \cap_{j\in J} \widetilde{B}_j) \,\widetilde{\lll}\, (C, \widetilde{D})$
  The rectangle $(\cup_{j\in J} A_j, \cap_{j\in J} \widetilde{B}_j)$ is a fuzzy rectangle of T_r.
  Infact, according to proposition 3, $P_{j\in J}(A_j, \widetilde{B}_j) = (\cup_{j\in J} A_j, \cap_{j\in J} \widetilde{B}_j)$, is the smallest superior boundary.

- Biggest Inferior Boundary

  We have to prove that any set of fuzzy rectangles of T_r has a biggest inferior boundary.
  Let $(A_j, \widetilde{B}_j)$ be a fuzzy rectangle of T_r :
  $\forall j \in J$, $(C, \widetilde{D}) \,\widetilde{\lll}\, (A_j, \widetilde{B}_j)$
  $\Leftrightarrow \forall j \in J$, $(C \subseteq A_j)$ and $(\widetilde{B}_j \subseteq \widetilde{D})$
  $\Leftrightarrow C \subseteq \cap_{j\in J} A_j$ and $\cup_{j\in J} \widetilde{B}_j \subseteq \widetilde{D}$
  $\Leftrightarrow (C, \widetilde{D}) \,\widetilde{\lll}\, (\cap_{j\in J} A_j, \cup_{j\in J} \widetilde{B}_j)$

- The rectangle $(\cap_{j\in J} A_j, \cup_{j\in J} \widetilde{B}_j)$ is a fuzzy rectangle of T_r.
  Infact, according to proposition 4, $H_{j\in J}(A_j, \widetilde{B}_j) = (\cap_{j\in J} A_j, \cup_{j\in J} \widetilde{B}_j)$, is the biggest inferior boundary.

- By noting that $P_{j\in\{1,2\}}(A_j, \widetilde{B}_j) = (A_1 \cup A_2, \widetilde{B}_1 \cap \widetilde{B}_2)$ and that $H_{j\in\{1,2\}}(A_j, \widetilde{B}_j) = (A_1 \cap A_2, \widetilde{B}_1 \cup \widetilde{B}_2)$ we could easily show that P is distributive relatively to H and conversely by using the correspondent properties and the duality of the operators $\cup$ and $\cap$.

- According to the definition of $\widetilde{\lll}$, we could easily show that $(\emptyset, \widetilde{F})$ is the smallest element of $(T_r, \widetilde{\lll})$ and that $(E, \emptyset)$ is the biggest one.

4.1 Galois Connection in Fuzzy Binary Relations

Let $\widetilde{R}$ be a fuzzy binary relation defined from E to $\widetilde{F}$. For two sets A and $\widetilde{B}$ such that $A \subseteq E$ and $\widetilde{B} \subseteq \widetilde{F}$ we define the operators $\widetilde{f}(A) = A^{\widetilde{R}}$ and $\widetilde{h}(\widetilde{B}) = B^{\widetilde{Q}}$ as follows:

$$\widetilde{f}(A) = A^{\widetilde{R}} = \{d/\alpha \mid \forall g, g \in A,\, \alpha = \min \mu_{\widetilde{R}}(g,d)\} \tag{17}$$

$$\widetilde{h}(\widetilde{B}) = \widetilde{B}^{\widetilde{Q}} = \{g \mid \forall d,\, d \in \widetilde{B}, \Rightarrow \mu_{\widetilde{R}}(g,d) \geq \mu_{\widetilde{B}}(d)\} \tag{18}$$

Proposition 6. *The operators* $^{\widetilde{R}}$ *and* $^{\widetilde{Q}}$ *form a fuzzy galois connection on the ordered sets* $(A, \widetilde{\lll})$ *and* $(B, \widetilde{\lll})$ *by satisfying the following conditions:*

$$A_i \subseteq A_j \Rightarrow A_i^{\widetilde{R}} \supseteq A_j^{\widetilde{R}} \tag{19}$$

$$\widetilde{B}_i \subseteq \widetilde{B}_j \Rightarrow \widetilde{B}_i^{\widetilde{Q}} \supseteq \widetilde{B}_j^{\widetilde{Q}} \tag{20}$$

$$A \subseteq A^{\widetilde{R}\widetilde{Q}} \text{ and } \widetilde{B} \subseteq \widetilde{B}^{\widetilde{Q}\widetilde{R}} \tag{21}$$

Proof. Let $A, A_i,\, A_j \in E$ and $\widetilde{B}, \widetilde{B}_i,\, \widetilde{B}_j \in \widetilde{F}$.

- Then

$$A_i^{\widetilde{R}} = \{d/\alpha_i \mid \forall g, g \in A_i,\, \alpha_i = \min \mu_{\widetilde{R}}(g,d)\}$$
$$A_j^{\widetilde{R}} = \{d/\alpha_j \mid \forall g, g \in A_j,\, \alpha_j = \min \mu_{\widetilde{R}}(g,d)\}$$

  If $A_i \subseteq A_j \Rightarrow \alpha_i \geq \alpha_j$. Hence $A_i^{\widetilde{R}} \supseteq A_j^{\widetilde{R}}$ This proves (19).

– For (20), if $\widetilde{B}_i \subseteq \widetilde{B}_j \Rightarrow \forall d \in \widetilde{B}_i$, $\mu_{\widetilde{B}_j}(d) \geq \mu_{\widetilde{B}_i}(d)$. Then

$$\widetilde{B}_i^{\widetilde{Q}} = \{g_i \mid \forall d,\, d \in \widetilde{B}_i, \Rightarrow \mu_{\widetilde{R}}(g_i, d) \geq \mu_{\widetilde{B}_i}(d)\}$$
$$\widetilde{B}_j^{\widetilde{Q}} = \{g_j \mid \forall d,\, d \in \widetilde{B}_j, \Rightarrow \mu_{\widetilde{R}}(g_j, d) \geq \mu_{\widetilde{B}_j}(d)\}$$

If $g_j \in B_j^{\widetilde{Q}} \Rightarrow \mu_{\widetilde{R}}(g_j, d) \geq \mu_{\widetilde{B}_j}(d) \geq \mu_{\widetilde{B}_i}(d) \Rightarrow g_j \in B_i^{\widetilde{Q}}$. Hence $B_i^{\widetilde{Q}} \supseteq B_j^{\widetilde{Q}}$. This proves (20).

– For (21)

$$A^{\widetilde{R}} = \{d/\alpha \mid \forall g, g \in A,\, \alpha = \min \mu_{\widetilde{R}}(g, d)\}$$
$$A^{\widetilde{R}\widetilde{Q}} = \{g \mid \forall d,\, d \in A^{\widetilde{R}}, \Rightarrow \mu_{\widetilde{R}}(g, d) \geq \min \mu_{\widetilde{R}}(g, d)\}$$

Obviously, if $g \in A \Rightarrow g \in A^{\widetilde{R}\widetilde{Q}} \Rightarrow A \subseteq A^{\widetilde{R}\widetilde{Q}}$. This proves the first part of (21).

– Similarly, let $d \in \widetilde{B}$ with $\mu_{\widetilde{B}}(d)$. Then

$$\widetilde{B}^{\widetilde{Q}} = \{g \mid \forall d,\, d \in \widetilde{B}, \Rightarrow \mu_{\widetilde{R}}(g, d) \geq \mu_{\widetilde{B}}(d)\}$$
$$\widetilde{B}^{\widetilde{Q}\widetilde{R}} = \{d/\alpha \mid \forall g, g \in \widetilde{B}^{\widetilde{R}},\, \alpha = \min \mu_{\widetilde{R}}(g, d)\}$$

$\Rightarrow \alpha \geq \mu_{\widetilde{B}}(d) \Rightarrow \widetilde{B} \subseteq \widetilde{B}^{\widetilde{Q}\widetilde{R}}$. This proves the other part of (21).

Proposition 7. *A pair* $(^{\widetilde{R}}, ^{\widetilde{Q}})$ *of maps is a fuzzy galois connection if and only if* $A \subseteq \widetilde{B}^{\widetilde{R}} \Leftrightarrow \widetilde{B} \subseteq A^{\widetilde{Q}}$.

Proof. $A \subseteq \widetilde{B}^{\widetilde{R}}$ by (19) $\Rightarrow A^{\widetilde{Q}} \supseteq \widetilde{B}^{\widetilde{Q}\widetilde{R}}$ and by (21) $\Rightarrow A^{\widetilde{Q}} \supseteq \widetilde{B}$.

This proves that $A \subseteq \widetilde{B}^{\widetilde{R}} \Rightarrow \widetilde{B} \subseteq A^{\widetilde{Q}}$ The other direction follows symmetrically.

Proposition 8. *For a fuzzy galois connection* $(^{\widetilde{R}}, ^{\widetilde{Q}})$, $A^{\widetilde{R}} = A^{\widetilde{R}\widetilde{Q}\widetilde{R}}$ *and* $\widetilde{B}^{\widetilde{Q}} = B^{\widetilde{Q}\widetilde{R}\widetilde{Q}}$.

Proof. With $\widetilde{B} = A^{\widetilde{R}}$ by (21)

$$\Rightarrow A^{\widetilde{R}} \subseteq A^{\widetilde{R}\widetilde{Q}\widetilde{R}} \tag{22}$$

and from $A \subseteq A^{\widetilde{R}\widetilde{Q}}$ by (19)

$$\Rightarrow A^{\widetilde{R}} \supseteq A^{\widetilde{R}\widetilde{Q}\widetilde{R}} \tag{23}$$

(22) and (23) $\Rightarrow A^{\widetilde{R}} = A^{\widetilde{R}\widetilde{Q}\widetilde{R}}$. The other part can be proved similarly.

4.2 Fuzzy Galois sub-Lattice of fuzzy maximal rectangles

We can remark that $A \times A^{\widetilde{R}}$ is the biggest relation of the form $A \times X \subseteq \widetilde{R}$ and that $B^{\widetilde{Q}} \times B$ is the biggest relation of the form $X \times B \subseteq \widetilde{R}$. In other words, $^{\widetilde{R}}$ computes the maximal range for a domain A and $^{\widetilde{Q}}$ computes the maximal domain for a range $\widetilde{B}$.

Similarly to the classical case, we propose to organize the set of fuzzy maximal rectangles under a complete sub-lattice.

Theorem 2. *Let* $\widetilde{R}$ *be a fuzzy binary relation defined from* E *to* $\widetilde{F}$ *and* $T_{r\max}$ *be the set of fuzzy maximal rectangles of* $\widetilde{R}$ *ordered by the relation* $\widetilde{\ll}^{\max}$. $(T_{r\max}, \widetilde{\ll}^{\max})$ *is a complete sub-lattice where the supremum* (P^{max}) *and the infinimum* (H^{max}) *of any set of maximal rectangles of* $T_{r\max}$ *are given respectively by :*

$$P_{j\in J}^{max}(A_j, \widetilde{B}_j) = ((\cup_{j\in J} A_j)^{\widetilde{R}\widetilde{Q}}, \cap_{j\in J} \widetilde{B}_j) \tag{24}$$
$$H_{j\in J}^{\max}(A_j, \widetilde{B}_j) = (\cap_{j\in J} A_j, (\cup_{j\in J} \widetilde{B}_j)^{\widetilde{Q}\widetilde{R}}) \tag{25}$$

Proof. This theorem can be proved in a similar manner as theorem 1.

Example 4. Let us consider the fuzzy binary relation $\widetilde{R}$ given in the following table:

	p_1	p_2	p_3	p_4	Class
o_1	0.5	1	0.7	0.5	C1
o_2	0.6	0.7	1	0.5	C2
o_3	1	0.9	1	0.1	C3
o_4	1	0.9	0.9	0.1	C3

Table 1: The fuzzy binary relation $\widetilde{R}$

The fuzzy concepts in $\widetilde{R}$ are given below labelled FC_0 to FC_9.

Label	Fuzzy Concept
FC_0	$\emptyset \times \{p_1/1.0, p_2/1.0, p_3/1.0, p_4/0.5\}$
FC_1	$\{o_1\} \times \{p_1/0.5, p_2/1.0, p_3/0.7, p_4/0.5\}$
FC_2	$\{o_2\} \times \{p_1/0.6, p_2/0.7, p_3/1.0, p_4/0.5\}$
FC_3	$\{o_3\} \times \{p_1/1.0, p_2/0.9, p_3/1.0, p_4/0.1\}$
FC_4	$\{o_1, o_2\} \times \{p_1/0.5, p_2/0.7, p_3/0.7, p_4/0.5\}$
FC_5	$\{o_2, o_3\} \times \{p_1/0.6, p_2/0.7, p_3/1.0, p_4/0.1\}$
FC_6	$\{o_3, o_4\} \times \{p_1/1.0, p_2/0.9, p_3/0.9, p_4/0.1\}$
FC_7	$\{o_1, o_3, o_4\} \times \{p_1/0.5, p_2/0.9, p_3/0.7, p_4/0.1\}$
FC_8	$\{o_2, o_3, o_4\} \times \{p_1/0.6, p_2/0.7, p_3/0.9, p_4/0.1\}$
FC_9	$\{o_1, o_2, o_3, o_4\} \times \{p_1/0.5, p_2/0.7, p_3/0.7, p_4/0.1\}$

Table 2: The fuzzy concepts in $\widetilde{R}$

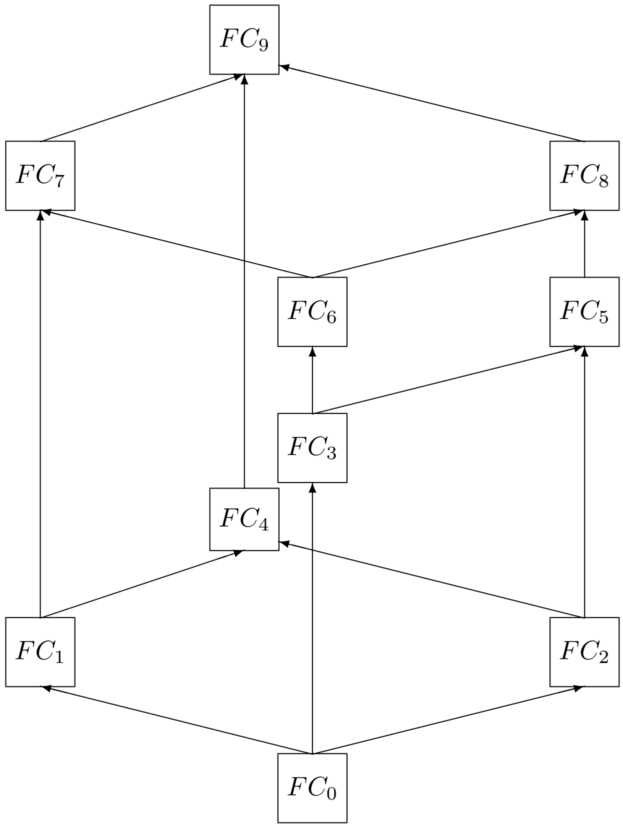

Figure 1: Fuzzy galois sub-lattice of $\widetilde{R}$

5 Applications

5.1 Fuzzy maximal rectangles and Decision making

The rectangular relation (or rectangle) of a crisp binary relation was widely used in many fields specially in learning and decision making[8, 9]. For instance, in the I.P.R[8] method, the rectangular decomposition[2] is applied to a binary relation in order to generate a set of optimal rectangles and each one of them generates a decision rule. Similarly to the crisp case, the fuzzy rectangle can be used in a decision making problem. Let $\widetilde{R}$ be a fuzzy relation defined from E to $\widetilde{F}$, $\widetilde{R}$ can be viewed as a set of objects (i.e., each row represents one object) and each object is described by a set of properties and a class label. Table 1 illustrates such a relation. Let o_x be a novel object for which we try to decide its eventual class given all the relation's objects. So, o_x can be approximated by the fuzzy concept or the maximal rectangle containing it, and from this concept we can decide the class of o_x.

Example 5. Let $o_x = \{p_1/1, p_2/0.9, p_3/0.9, p_4/0.1\}$. The concept corresponding to o_x is represented in table 2 and the chosen class is C3.

	p_1	p_2	p_3	p_4	Class
o_3	1	0.9	1	0.1	C3
o_4	1	0.9	0.9	0.1	C3

Table 3: The concept approximating o_x

5.2 Discovery of fuzzy association rules

Knowledge discovery, or Data mining, is the discovery of previously unknown, potentially useful and hidden knowledge in databases. During the past years, boolean association rules mining has received considerable attention. These associations describe dependencies among items in large databases. In this context, many efficient algorithms have been proposed in the literature. The most noticeable are Apriori, Manilla's algorithm, Partition and DIC. All these algorithms are based on the Apriori mining method [3]

Fuzzy Maximal Rectangles can be used to generate association rules among objects. Hidden and potentially useful information can be discovered among objects.

Example 6. Let us consider the fuzzy binary relation given in table 1. Suppose that the user is interested in discovering the rules, with a support at least equal to 2, where the property p_1 appears in the consequence. In the fuzzy sub-lattice, we look for the smallest fuzzy maximal rectangle containing p_1, which is $\{o_3, o_4\} \times \{p_1/1.0, p_2/0.9, p_3/0.9, p_4/0.1\}$. Hence, the associated fuzzy rule is:

If $p_2(0.9), p_3(0.9), p_4(0.1)$ then $p_1(1)$.

6 Conclusion

In this paper, we have proposed a fuzzy galois connection for fuzzy relations which satisfies all the conditions defined for fuzzy galois connections. We also proved that, similarly to the classical case, the set of fuzzy (maximal) rectangles can be organized under a complete and distributive lattice (complete sub-lattice). We believe that the introduced formal properties can be successfully applied to some practical fields of computer science, such as, Knowledge extraction (fuzzy association rules), Information retrieval and fuzzy reasoning, and Vertical and horizontal additional information, and so on.

Acknowledgment: We thank King Fahd University of Petroleum and Minerals and the General Direction of Scientific Research of Tunisia (DGRST) for partially financing this work.

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