A Pansystems Approach to Sorting Based on Binary Relations

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Abstract

In computer science, there are many sorting algorithms. But all the computer algorithms are done through a certain steps of operations. What these algorithms are really done is not so clear, so it usually is difficult for people to figure out the thought of the algorithm and further improve it. Set theory and binary relations have order relations which have a close relationship with sorting in computer science. But traditional research mainly focuses on the pure mathematics operations on set theory. Pansystems strengthen the research work in set theory by combining mathematics with its philosophy background. In this paper, using pansystems PMT-combination [1] method, we re-inspect the reflexive relation and equivalence relation in set theory and find out the common characteristic of them. Then using pansystems body-expansion, we expand the traditional partial order to a new kind of order-relation. Furthermore, we point out the relationship between pansystems body-shadow relation, partial order, extended partial order and semi-partial order. The new research work on order relations in set theory further be used to interpret the sorting operation in computer science.

Keywords: *Pansystems, Set Theory, Sorting, Binary Relations, Equivalence Relation*

1. Introduction

Set theory is based on definition of set and the explanations of it, which is based on the fundamental relation: *membership*. All other relations such as equals, subset are derived from the basic membership relation. Pansystems has done a lot of research on set theory and all the relations in set theory can be classified into two basic relations in pansystems' term: general whole-part relation and general body-shadow relation. Many operations and theorems are built on these two basic relations. In this paper, based on the binary relations in set theory, we provide their pansystems' extensions.

2. Binary relations

Binary relation is an important concept in set theory. Set theory is the basis of modern science. Its concept and application has infiltrated into many branches of mathematics, physics and other natural sciences. A set cannot contain two or more identical elements and the order in which the elements of a set are listed is irrelevant. The definition of set decides that it cannot represent the internal structure of these objects.

Although set itself cannot represent internal structure such as order and relations between objects, it has the potential to represent it. Another related concept of set -- sequence can be used to represent the orders and relations between objects.

Definition 2.1 Sequence pair (a,b) represents a sequence of two elements a,b in a set. [2]

Based on the sequence pair, we can define the relations on set.

Definition 2.2 Given non-empty set A,B, define $A\times B=\{(a,b)\,|\,a\in A,b\in B\}$, $A\times B$ is the set of all sequences (a,b).

Because relations are set, the operations which can be done on set can also be applied to relations. But besides the usual set operations, relations have its unique operations.

Definition 2.3 The reverse operation of a relation

R is defined as: $R^{-1} = \{(x, y) \mid (y, x) \in R\}$ 。

For example:

$R = \{(a,b),(b,c)\}$, then $R^{-1} = \{(b,a),(c,b)\}$

Obviously, $(R^{-1})^{-1} = R$.

The definition of relations can be extended to relations between n sets.

Definition 2.4 $A_1 \times A_2 \times \cdots \times A_n$ is the set of all sequences $(a_1, a_2, \cdots, a_n)$, in which $a_1 \in A_1, a_2 \in A_2, \cdots, a_n \in A_n$,

if $A_i = A, i = 1, 2, \cdots, n$, we write $A_1 \times A_2 \times \cdots \times A_n$ as A^n ,called the n -ary relations of A . When $n = 2$, we call A^2 the binary relations on A .

3. Pansystems theory

Pansystems is a framework whose pursuit is universal considerations of philosophy, mathematics and technology (PMT-combination). [1] It pursues the relative unity of flexible balance of the universality, exactness and concrete operability or relative feasibility. It is a transfield multilayer network-like and Encyclopaedia-connecting research with the emphasis on pansystems melting the philosophy, mathematics, technology, post-modern systems thought and certain aesthetic principles into a unified entity. This newborn direction is now enrolled as one of new theories of modern system science, one of methodologies of world mathematicians.

The concept of system and generalized system is widely used in modern systems science, physical science and sociology, and the concept of transformation, change and generalized transformation, in fact can be found everywhere. The concept of symmetry and generalized symmetry (or pansymmetry) is all along the core of methods in mathematics, physics and cybernetics. And in the modern methodology, the concepts of generalized system, generalized transformation and pansymmetry are widely contained in a series of concepts or relations such as: form and content, affirmation and negation of negation, qualitative change and quantitative change, possibility and reality, etc.

The generalized system can be considered as a unified entity of certain given things set and its certain panstructure set, where panstructure may be considered as a generalization and extension of a series of concepts such as relation, relation of relations, dynamic relations, relation with parameter, mathematical structure, etc. The concept of generalized system emphasizes the compositions and relations of things. [4]

Pansystems provides a new view of traditional mathematics theories. Pansystems summaries all kinds of relations and abstracts them into two basic pansystems relations: the Whole-Part relation and the Body-Shadow relation.

Mapping and functions in set theory is also called transformation, they are in fact special binary relations. If $f \subset A \times B$ is a binary relation on A and B , for each $x \in A$, there exists only one $y \in B$ and makes $(x, y) \in f$, then f is called a function. Written : $f : A \to B$ or $y = f(x)$, and also: $f(A) = \{f(x) \mid x \in A\}$, we call A domain, and body in pansystems term. B is called range, $f(A)$ is the image or shadow of A in pansystems term. If $D \subset A$, then $f(D) = \{f(x) \mid x \in D\}$ is called the miniature of A .

Pansystems body-shadow relation is a very widely used concept. Many mathematics relations such as mapping, function can be viewed as body-shadow relations.

Example: 3.1 Given set $A = \{s1, s2, s3, s4, s5\}$ represents a set of apples.

Set $B = \{$ red, green, yellow $\}$ to represent the apples' color in the set. Then f : color of apples will map each apple to a unique element in B .

The relation between apples and theirs colors is a body-shadow relation. If we want to know which apple has the specific color in B , we should look it up in the mapping function f. This kind of process which from shadow back to body is called embody. More specifically, if we define f as follows:

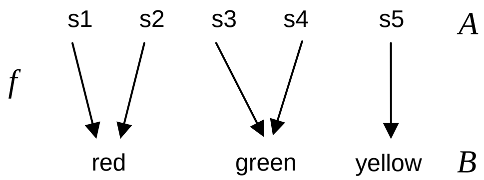

Figure 3.1 body-shadow relations of apples

If we use apples $D=\{s1,s3,s5\}$ to represent all the apples in the set, then $f(D)$ can be viewed as the miniature of the all the apples, they represent the apples which are in different categories of colors. From the miniature, we can get an overview of the status of the apples. But reversely, from the representative elements, we can expand it to a group of apples which have the same color. s1 in set D represent $s1,s2$ in set A, and s3 in set D represent $s3,s4$ in set A, s5 represent s5 in set A.

Pansystems body expansion can be used in many fields, especially in places where we should inspect things through different scales and different levels. The expansion can be done through relation operations and hence can be implemented by computers automatically. Pansystems relation expansion can make computer operations simulate the ability of changing scales and levels when processing certain problems.

4. Pansystems extension on binary relations

Pansystems has a lot of extensions on set theory. We will give the pansystems extension on binary relations as follows. [6 Pansystems view of the world]

Definition 4.1 If $f\subset A\times A$, $g\subset A\times A$, we define the composition of relations as: $f\circ g=\{(x,y)\mid \exists t\in A,(x,t)\in f,$ $(t,y)\in g\}\subset A\times A$.

Definition 4.2 If $f\subset A\times A$, we define the reverse relations as $f^{-1}=\{(x,y)\mid (y,x)\in f\}$

Definition 4.3 Given non-empty set A, we define the diagonal relation as $I=I(A)=\{(x,x)\mid x\in A\}$.

Definition 4.4 Relations which contains diagonal relation I has the properties of reflexive. Written as: $R[A]=\{f\mid I\subset f\},\ f\subset A^2\}$

Definition 4.5 Given set A and its binary relation f, if $f=f^{-1}$, then f is a symmetry relation. Written as: $S[A]=\{f\mid f=f^{-1}\}$

Definition 4.6 Given set A and its binary relation f, if $f\cap f^{-1}\subset I$, then f is a anti-symmetry relation. Written as: $S_a[A]=\{f\mid f\cap f^{-1}\subset I\}$

Definition 4.7 Given set A and its binary relation f, if $f^2\subset f$, then f is a transitive relation. Written as: $T[A]=\{f\mid f^2\subset f\}$

Definition 4.8 Given set A and its binary relation f, if f satisfies reflexive and symmetry, f is a tolerance relation.

Written as: $E_s[A]=R[A]\cap S[A]$.

Definition 4.9 Given set A and its binary relation f, if f satisfies reflexive, symmetry and transitive, f is a equivalence relation. Written as: $E[A]=R[A]\cap S[A]\cap T[A]$.

Definition 4.10 Given set A and its binary relation f, if f satisfies reflexive, anti-symmetry and transitive, f is a partial relation. Written as: $L[A]=R[A]\cap S_a[A]\cap T[A]$.

Definition 4.11 Given set A and its binary relation f, if f satisfies reflexive and transitive, f is a semi-partial order relation. Written as: $L_s[A]=R[A]\cap T[A]$.

Definition 4.12 Given set A and its binary relation f, we use $C[A]=\{f\mid f\cup f^{-1}=A^2\}$ represents all the complete relations on set A.

Definition 4.13 Given set A and its binary relation f, if f satisfies complete order and partial order, f is a complete order relation. Written as: $L_c[A]=C[A]\cap L[A]$.

Definition 4.14 Given set A and its binary relation f, we use

$T_q[A]=\{f \mid f_1^{(2)}, f_1 \circ f_2, f_2 \circ f_1 \subset f,$

$f_1 = f - f^{-1}, f_2 = f \cap f^{-1}\}$ represents all the semi-transitive relations on set A.

Definition 4.15 Given set A and its binary relation f, we use $U[A]=\{f \mid f^t \cap f^{-t} \subset I\}$ represents all the relations which are uni-direction.

Definition 4.16 Given set A and its binary relation f, if f satisfies reflexive and semi-transitive, f is a quasi-partial-order relation. Written as: $L_q[A]=R[A] \cap T_q[A]$.

In the above definitions,

$X \in \{S_a, T, T_q, L, L_s, L_q, U, L_c\}$ can be used as the mathematics model of general order relations.

$Y \in \{R, S, E_s, E\}$ can be used as the general identical mathematics model.

$\bar{Y}=\{f \mid A^2 - f \in Y\}$ can be used as the differentiate mathematics model.

The above definitions are pansystems' extension on binary relations.

Besides the above extensions, pansystems also have a range of pansystems equivalence relation operators which can convert ordinary relation to equivalences.

There are 22 Pansystems equivalence operators in sum, $\delta_i, i=0,1,\cdots,21$. They can be divided into 3 categories, δ_0, $\delta_1 \sim \delta_{10}$, $\delta_{11} \sim \delta_{21}$.

The pansystems equivalence relation operators δ_i is defined as follows:

$\delta_0(g)=\max\{f \mid f \subseteq g, f$ is an equivalence relation$\}$

$\delta_k(g)=[\delta_k(g)]^t, k=1,2,\cdots,10,$ t represents the transitive closure.

$\delta_{k+11}(g)=\delta_0(\varepsilon_k(g)), k=0,1,\cdots,10,$ in which $\varepsilon_3(g)=\delta_3(g),$ ε is tolerance relation operator, $\varepsilon_i, i=1,2,\cdots,10,$ They are defined as follows:

$\varepsilon_0(g)=\max\{f \mid f \subseteq g, f$ is a tolerance relation$\}$

$\varepsilon_1(g)=g \cup g^{-1} \cup I(g),$

$\varepsilon_2(g)=\varepsilon_1(g \cap g^{-1}),$

$\varepsilon_3(g)=\varepsilon_1(g^t \cap g^{-t}),$ t represents the transitive closure, g^{-t} represents the reverse relation of g^t.

$\varepsilon_4(g)=\varepsilon_1(g \circ g^{-1}),$

$\varepsilon_5(g)=\varepsilon_1(g^{-1} \circ g),$

$\varepsilon_{k+5}(g)=\varepsilon_k(g^-), k=1,2,\cdots,5.$ [6,7]

Reflexive and equivalence are both binary relations on set theory. They all concern the relations between different objects in given set. But undeniable, set theory originates from reality. Using PMT-combination method, pansystems have inspected set theory and by combining the recognition with reality, pansystems provide a new point of view with related set theory.

For any given set, we have deleted all other attributions of these objects and remained only some specific attributions. All the objects in a set in fact have the same attributions which the set is defined by; the set of red apples for example is referred to the apples which have the color red. Any element in the set has the specific attributions: apple and red color. But in practice, after we divide the world into different sets, we usually use sets alone to do the operations. The related attributions are usually neglected. The relation between attributions and their related set is a body-shadow relation in pansystems term. All the relations and their operations can be interpreted and expanded by combining with the set's body-shadow relation.

Example 4.1: Given a set of balls which have different colors, the details are as follows: a1, a2, a3 are red; a4, a5 are green, a6 is blue. As shown in Figure 4.1

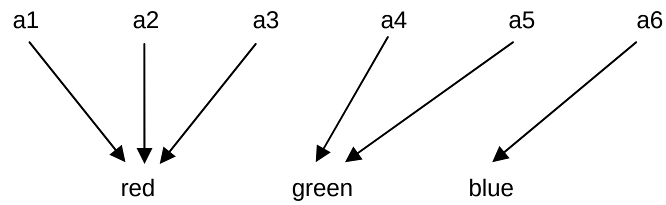

Figure 4.1 Information about balls

If we write the body-shadow relation as f, $R = f \circ f^{-1}$ will get a binary relation on set $A = \{a1, a2, a3, a4, a5, a6\}$. R is an equivalence relation on A. Under the scale of color, the set can be divided into three equivalence classes: {red}, {green} and {blue}. In the binary relation R, every element has a relation with itself, this is called reflexive. In this example, reflexive means every element's color is the same with itself. The elements in the equivalence classes, {red}= { $a1, a2, a3$ } for example, they can be classified into the same equivalence class means that under the scale color, they are equal. Equivalence relation and equivalence classes in fact mean the objects equals under certain circumstance.

In some sense, reflexive relation is a kind of special equivalence. It is the equivalence relation under any circumstance, because each object equals to itself in every place. This kind of equivalence is absolute, while the equivalence relation is conditional, different objects may equal under certain circumstance.

Given that reflexive relations and equivalence relations are both concerned with the equal relation between objects, they can be exchanged in certain places and expand the traditional relation operation.

5. General order and sorting

The reflexive relation and the equivalence is conforms in some sense. Reflexive relation is the relation on certain point, so we can replace the point's reflexive relation with an equivalence relation and hence expand the original binary relation.

In definition 2.9, if we substitute I with equivalence relation $E[A)$ on A, the original anti-symmetry relation can be extended as: $S_a^*[A) = \{ f \cap f^{-1} \in E[A)\}$.

Furthermore, we can expand other binary relations concerning with anti-symmetry relations. The typical relation is partial order relation:
Based on definition 2.13, we can get $L^*[A) = R[A) \cap S_a^*[a) \cap T[A)$, this relation is in fact a subset of semi-partial order which is defined in definition 2.14.

Example 5.1 Suppose there are three kinds of flowers which have different colors: red, blue and yellow. There is a partial order relationship exists in their values. Red is more valuable than blue, and blue is more valuable than yellow. The detail information is shown as follows in Figure 5.1(c):

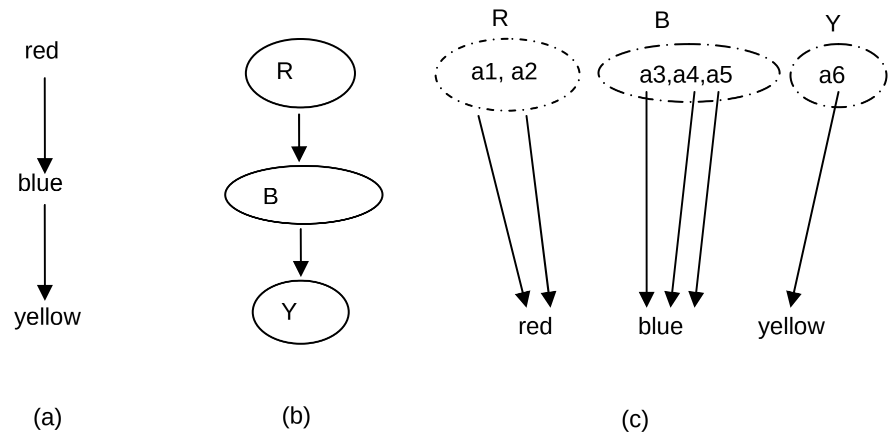

Figure 5.1 Expansion on partial order

In Figure 5.1 (a), the value of the three kinds of flowers forms a partial order. But in the partial order, there is only three points, and $f \cap f^{-1} = I$. But if we expand each point to specific flowers which have the related color, we will get an expanded relation. This is pansystems body expansion.

Suppose $R = \{a1, a2\}$, $B = \{a3, a4, a5\}$, $Y = \{a6\}$, we use the complete relation on R, B, Y to substitute the red, blue and yellow node respectively. We will get another binary relation in Figure 5.1 (b). Obviously, (a) satisfies partial order $L[C)$, $C = \{$ red, blue, yellow $\}$; (b) satisfies $L^*[A)$, $A = \{a1, a2, ..., a6\}$. (c) is the body-shadow relation between A and C . In Figure 5.1, we have expanded each point in (a) into an equivalence relation classes in (c), which is generated by a body-shadow relation. It also can be viewed that (b) is the combination of partial order in (a) and equivalence relation in (c).

In Figure 5.1 (b), if we write the relation between objects as $L^*[A)$,
then $L^*[A) = R[A) \cap S_a^*[a) \cap T[A)$,
$S_a^*[A) = \{ f \cap f^{-1} \in E[A)\}$, where
$E[A) = f \circ f^{-1}$, $f : A \to C$ is shown in Figure 5.1 (c).

Figure 5.1(b) is in fact a semi-partial order which is defined in definition 2.14. Obviously, (b) is in greater details than (a), and (b) is the result of body-expansion of (a).

Here in this example, from the body-shadow relations of objects, we can get an equivalence relation on the set. Using the equivalence relation, we can get a division of the set. Combing with the partial order exists in their shadows, we can get a semi-partial order between each object. Semi-partial order is in fact the combination of equivalence relation and partial order. The essence of semi-partial order is pansystems body-expansion. This kind of expansion can be found in many algorithms in computer science. Radix sort first form a semi-partial order then the partial order. [5] The B tree is in fact a body-expansion of basic binary tree. We also can consider to expand other data structures using the body-expansion method to get new data structures and algorithms.

6. Conclusion

In this paper, we introduced pansystems expansion on binary relations and then apply the body-expansion to real problems. By using pansystems PMT-combination method, we reveal the relation between body-shadow relation and set theory. Further, we disclose the common characteristic of reflexive relations and equivalence relations and based on this characteristic, we introduce the pansystem body-expansion operation on binary relations. The expansion further is used in the partial order relation and generates the semi-partial order. Through the pansystems extension on binary relations, we introduce semi-partial order and unite it with partial order on the philosophy level. This kind of extension laid the foundation for further researches, such as new data structures and new algorithms.

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