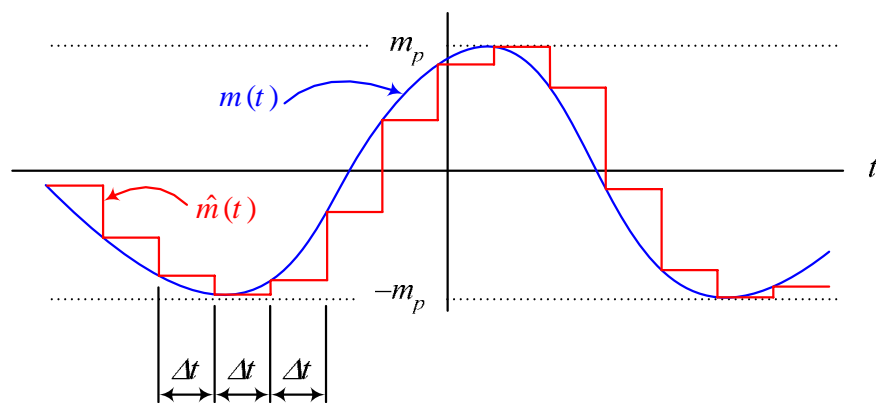


Wideband FM and PM

For the FM signal shown below, the value of k_f may not satisfy the condition $k_f \ll 1$, and therefore the approximation used in narrowband FM may not be applicable. For FM signals

$$g_{FM}(t) = A \cdot \cos \left[\omega_c t + k_f \int_{-\infty}^t m(\alpha) d\alpha \right].$$

that do not satisfy $k_f \ll 1$, computing the bandwidth is more difficult than for narrowband signals. To compute the bandwidth of wideband FM, we will approximate the signal $g_{FM}(t)$ by another signal that results from modulating a sampled version of the message signal. That is, instead of using the signal $m(t)$ as the message signal for modulating the carrier, we will use an approximation $\hat{m}(t)$ that is obtained by sampling $m(t)$ as shown in the following figure.



The value of Δt that we will use is the maximum allowable value that will insure that the $m(t)$ can be reconstructed from $\hat{m}(t)$ without loss of information. This maximum value for Δt is obtained using the Nyquist sampling theorem, which states that for a signal $m(t)$ with a bandwidth of B_m (Hz), the minimum sampling frequency is $2B_m$. Therefore, the maximum for Δt is given by $\Delta t = 1/2B_m$. Of course a smaller value for Δt will be better since it gives a better approximation for $m(t)$ but is unnecessary.

So, to find the approximate bandwidth of FM signal, let us assume that the original message signal $m(t)$ is bounded in amplitude by the two values m_p and $-m_p$. Therefore,

$$-m_p \leq m(t) \leq m_p.$$

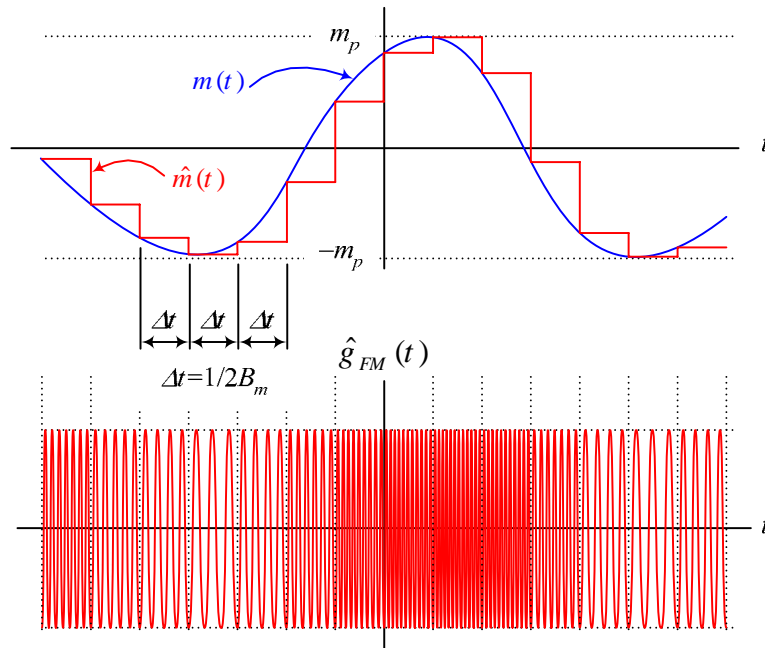
Then, we can straight forward say that

$$-m_p \leq \hat{m}(t) \leq m_p.$$

Now, the approximate signal $\hat{g}_{FM}(t)$ to $g_{FM}(t)$ is obtained by FM modulating $\hat{m}(t)$ as

$$\hat{g}_{FM}(t) = A \cdot \cos \left[\omega_c t + k_f \int_{-\infty}^t \hat{m}(\alpha) d\alpha \right]$$

Since $\hat{m}(t)$ is constant over periods of $\Delta t = 1/2B_m$, the instantaneous frequency of $\hat{g}_{FM}(t)$ will be constant over periods of $\Delta t = 1/2B_m$. The signal $\hat{g}_{FM}(t)$ will look like as follows.



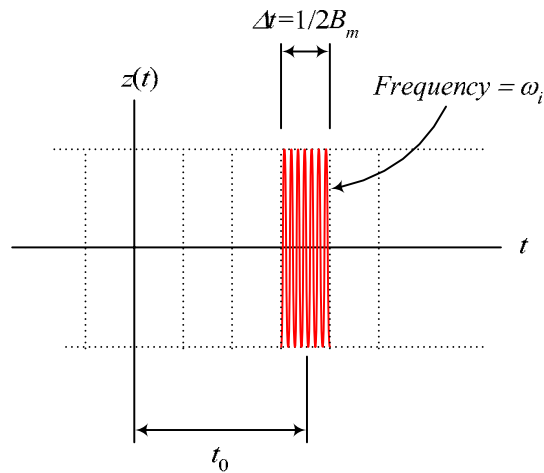
Since $-m_p \leq \hat{m}(t) \leq m_p$, the instantaneous frequency of $\hat{g}_{FM}(t)$ (and also $g_{FM}(t)$) will be in the range

$$\omega_c - k_f m_p \leq \omega_i(t) \leq \omega_c + k_f m_p .$$

This means that the instantaneous frequency changes over a range of $\Delta\omega = k_f \cdot m_p$ (this can also be written as $\Delta f = k_f \cdot m_p / 2\pi$) on each side around the carrier frequency ω_c .

So, to approximate the bandwidth of the original FM signal $g_{FM}(t)$, we will compute the approximate bandwidth of the approximation signal $\hat{g}_{FM}(t)$ by finding its frequency spectrum. Since $\hat{g}_{FM}(t)$ is composed of blocks of sinusoids with different frequencies that are in the range of frequencies of $\omega_c - k_f m_p \leq \omega_i(t) \leq \omega_c + k_f m_p$, we can find the spectrum of each of these blocks independently and then add these spectrums to get the overall spectrum of $\hat{g}_{FM}(t)$.

Consider the part of $\hat{g}_{FM}(t)$ that shown below



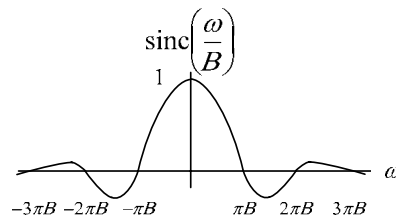
The signal $z(t)$ is given by

$$z(t) = A \cdot \text{rect} [2B_m (t - t_0)] \cdot \cos(\omega_i t),$$

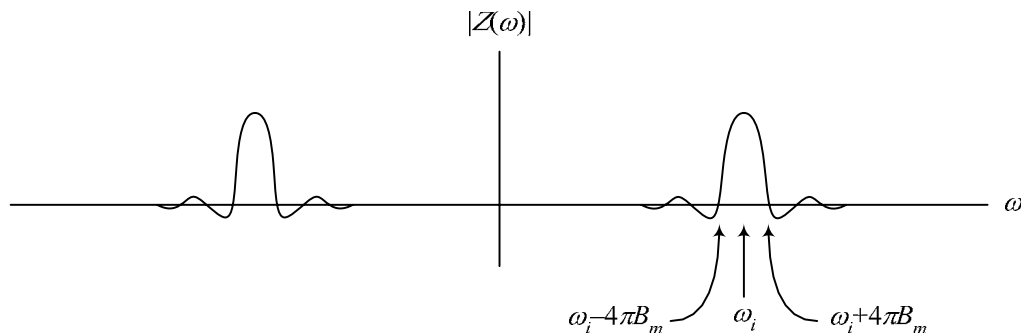
where $\omega_c - k_f m_p \leq \omega_i(t) \leq \omega_c + k_f m_p$. The Fourier transform of $z(t)$ is

$$Z(\omega) = \frac{A\pi}{4B_m} \left[\text{sinc} \left(\frac{\omega - \omega_i}{4B_m} \right) \cdot e^{-j(\omega - \omega_i)t_0} + \text{sinc} \left(\frac{\omega + \omega_i}{4B_m} \right) \cdot e^{-j(\omega + \omega_i)t_0} \right].$$

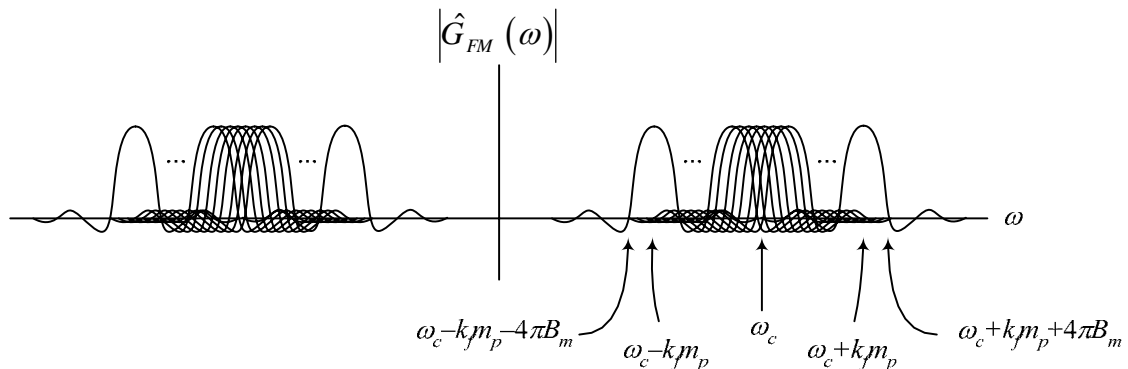
Remembering that the sinc function looks like the following



Sketching the magnitude of $Z(\omega)$ (i.e., magnitude spectrum of $z(t)$) will give the following (since the complex exponentials have a magnitude of one and the two components shifted to the left and right almost do not interfere with each other)



Adding the spectrum of the different signals like $z(t)$ given above will give us the spectrum of the approximation signal $\hat{g}_{FM}(t)$



If we assume that the sidebands (the small humps at the two edges of a sinc function) have negligible power, and knowing that $\Delta\omega = k_f \cdot m_p$, we see that the bandwidth of an FM signal is approximately equal to

$$\begin{aligned}
 BW_{FM} &\approx 2k_f \cdot m_p + 8\pi B_m \quad (\text{rad/s}) \\
 &\approx 2 \cdot \Delta\omega + 8\pi B_m \quad (\text{rad/s})
 \end{aligned}$$

Using the fact that $\Delta f = k_f \cdot m_p / 2\pi$, the bandwidth in Hz becomes

$$\begin{aligned}
 BW_{FM} &\approx \frac{2k_f \cdot m_p}{2\pi} + 4B_m \quad (\text{Hz}) \\
 &\approx 2 \cdot \Delta f + 4B_m \quad (\text{Hz}) \\
 &\approx 2[\Delta f + 2B_m] \quad (\text{Hz})
 \end{aligned}$$

In practice, this bandwidth is higher than the actual bandwidth of FM signals. Consider for example narrowband FM. Using this formula for the bandwidth, we see that the bandwidth is twice the actual bandwidth. In fact, a more accurate relationship is known as CARSON's Rule, which is given by

$$\boxed{
 \begin{aligned}
 BW_{FM} &\approx 2 \cdot \Delta f + 2B_m \quad (\text{Hz}) \approx 2[\Delta f + B_m] \quad (\text{Hz}) \\
 &\approx 2[\Delta\omega + 2\pi B_m] \quad (\text{rad/s})
 \end{aligned}
 }$$

where

$$\boxed{B_m = \text{Bandwidth of the Message Signal } m(t) \text{ in Hz,}}$$

and

$$\boxed{\Delta\omega = k_f \cdot m_p \quad \rightarrow \quad \Delta f = k_f \cdot m_p / 2\pi.}$$

We will define a quantity β such that

$$\beta = \frac{\Delta\omega}{B_m} = \frac{k_f m_p}{B_m}$$

This is known as the MODULATION INDEX of the FM signal.

Exercise: A sinusoidal message signal $m(t)$ with a frequency of 5000 Hz and amplitude of 5 V is used to generate an FM signal with amplitude $A = 10$ V using a modulator with $k_f = 2\pi 3000$ and a carrier frequency of 5 MHz. Find the following:

- a) The frequency deviation and the range of instantaneous frequency (find both in rad/s and in Hz) for the FM signal.
- b) The modulation index of the FM signal.
- c) The Bandwidth of the FM signal (both in rad/s and Hz).