

Single Side Band Suppressed Carrier (SSBSC or SSB for short)

In QAM, we have seen that you can modulate two signals at the same frequency using sine and cosine carriers and demodulate them without having interference. However, still each signal would require twice the bandwidth of the original message signal. In fact, for real time-domain signals, the parts of the spectrum of that signal that fall in the positive region and negative region frequency carry exactly the same information and therefore, need not both to be transmitted. This type of amplitude modulation is called Single Side Band (SSB) modulation. Since the carrier is usually not transmitted in this type of modulation, it is called SSB-suppressed carrier (SSBSC). For short, we will call it SSB modulation.

To understand the process of transmitting only one of the side bands of a signal, we will start discussing the process in frequency-domain and then try to find the time-domain representation of the SSB signal. So, our aim in the next few pages is to write an expression of $g_{USB}(t)$ and $g_{LSB}(t)$, which are the time-domain signal representations of the upper- and lower-side band signals, in terms of the original message $m(t)$.

Consider the baseband message signal $m(t)$ with the frequency spectrum $M(\omega)$ shown in part (A) of the figure in the next page. Assuming that the signal $m(t)$ is a real signal, the magnitude of its spectrum is an even function and the phase of its spectrum is an odd function (so, the information contained in the part of the spectrum with positive frequency is exactly the same as the information contained in the part with negative frequency). The spectrum $M(\omega)$ can be split into two parts called $M_+(\omega)$ and $M_-(\omega)$ as shown in parts (B) and (C). $M_+(\omega)$ is the part of $M(\omega)$ with positive frequency and $M_-(\omega)$ is the part with negative frequency. It is clear that

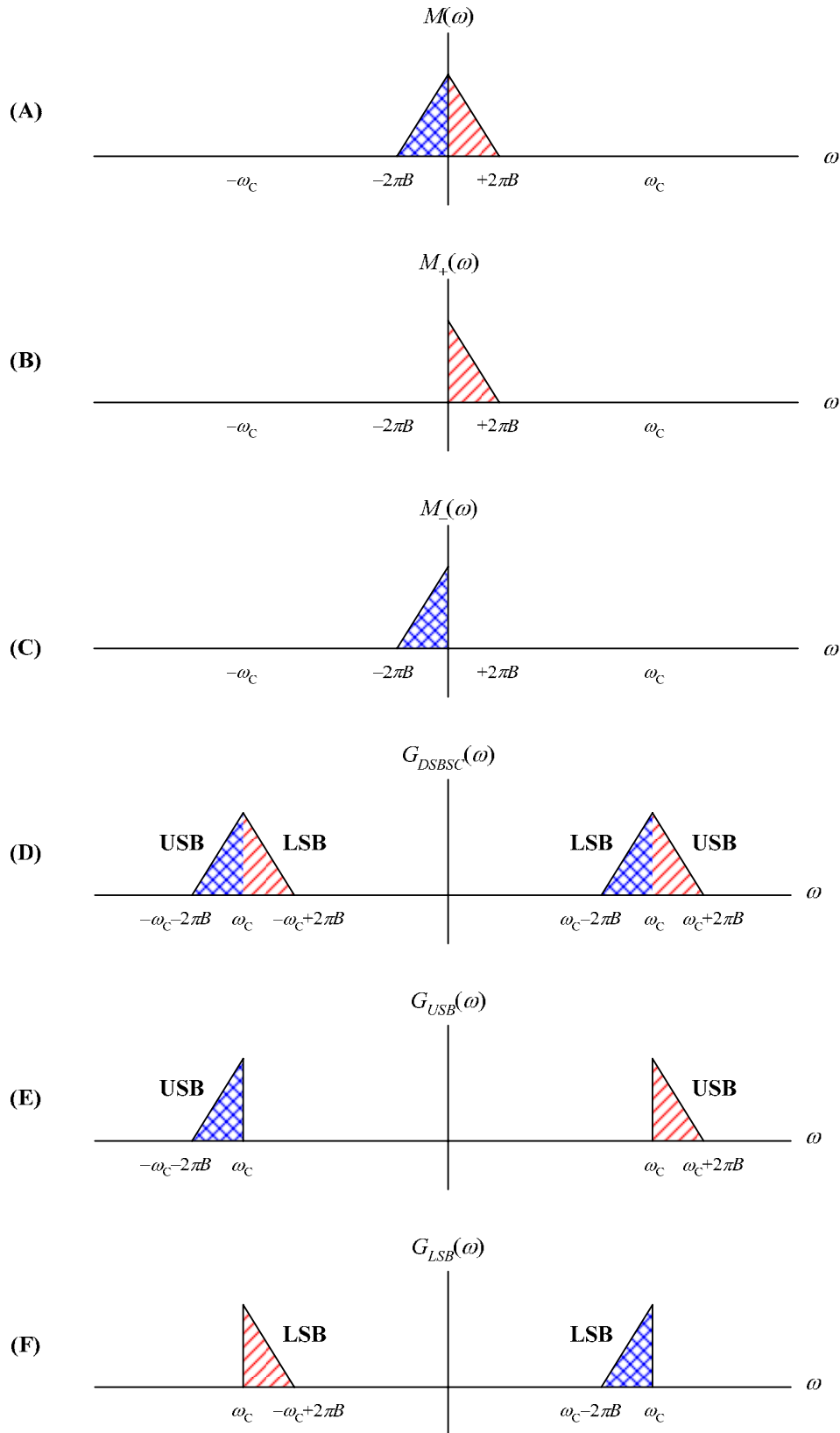
$$M(\omega) = M_+(\omega) + M_-(\omega).$$

Both $M_+(\omega)$ and $M_-(\omega)$ can be represented in terms of $M(\omega)$ by noticing that each is a step function that multiplies $M(\omega)$. That is

$$\begin{aligned} M_+(\omega) &= M(\omega)u(\omega) \\ M_-(\omega) &= M(\omega)u(-\omega) \end{aligned} \tag{1}$$

Since the step function $u(\omega)$ and the sign function $\text{sgn}(\omega)$ are related by a scaling and a shifting up/down relationship given by

$$\begin{aligned} \text{sgn}(\omega) &= 2u(\omega) - 1 \\ u(\omega) &= \frac{1}{2} + \frac{1}{2}\text{sgn}(\omega), \\ u(-\omega) &= \frac{1}{2} - \frac{1}{2}\text{sgn}(\omega) \end{aligned} \tag{2}$$



We can substitute for the step functions in the two equations labeled (q) with the $\text{sgn}(\omega)$ function as shown above in (2). This gives

$$\begin{aligned}
 M_+(\omega) &= M(\omega)u(\omega) = M(\omega) \left[\frac{1}{2} + \frac{1}{2} \text{sgn}(\omega) \right] \\
 &= \frac{1}{2} M(\omega) + \frac{1}{2} \underbrace{M(\omega) \text{sgn}(\omega)}_{jM_h(\omega)} \\
 M_-(\omega) &= M(\omega)u(-\omega) = M(\omega) \left[\frac{1}{2} - \frac{1}{2} \text{sgn}(\omega) \right] \\
 &= \frac{1}{2} M(\omega) - \frac{1}{2} \underbrace{M(\omega) \text{sgn}(\omega)}_{jM_h(\omega)}
 \end{aligned} \tag{3}$$

Let us define the term $M(\omega) \cdot \text{sgn}(\omega)$ to be $jM_h(\omega)$, where the subscript h stands for ‘‘Hilbert’’. Therefore,

$$M_h(\omega) = -jM(\omega) \text{sgn}(\omega). \tag{4}$$

So, we can write $M_+(\omega)$ and $M_-(\omega)$ as

$$\begin{aligned}
 M_+(\omega) &= \frac{1}{2} M(\omega) + \frac{j}{2} M_h(\omega) \\
 M_-(\omega) &= \frac{1}{2} M(\omega) - \frac{j}{2} M_h(\omega)
 \end{aligned} \tag{5}$$

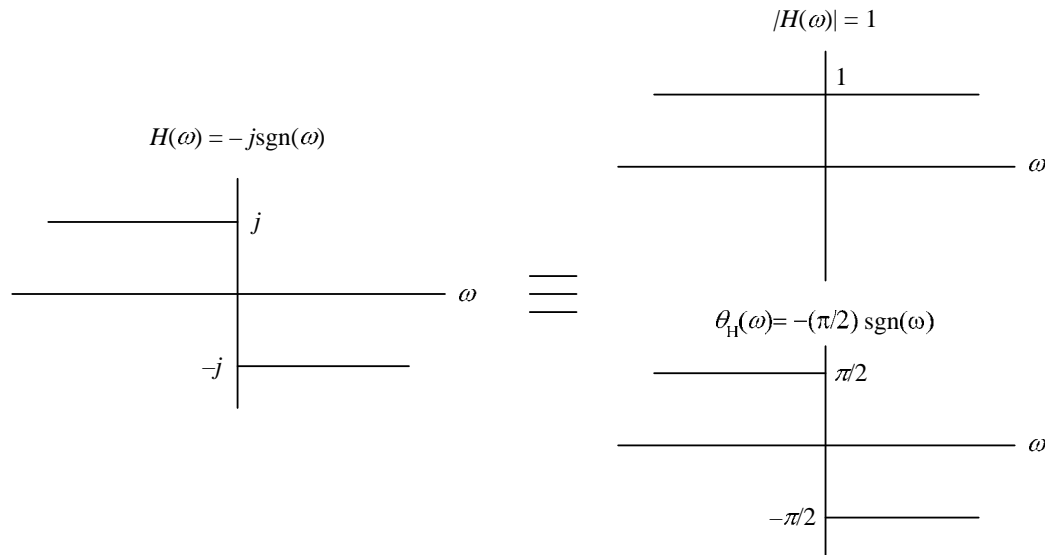
If we define $m_+(t)$ and $m_-(t)$ to be the inverse Fourier transforms of $M_+(\omega)$ and $M_-(\omega)$, respectively, and $m_h(t)$ to be the inverse Fourier transform of $M_h(\omega)$, we get

$$\begin{aligned}
 m_+(t) &= \frac{1}{2} m(t) + \frac{1}{2} j m_h(t) \\
 m_-(t) &= \frac{1}{2} m(t) - \frac{1}{2} j m_h(t)
 \end{aligned} \tag{6}$$

Now, let us look carefully at the $M_h(\omega)$, which is defined in (4). This signal is a transformed version of $M(\omega)$. The transformation is known as the Hilbert transform. Based on the shape of $\text{sgn}(\omega)$, we see that this transform simply flips the positive part of the spectrum of $M(\omega)$ and multiplies the whole thing by j . If we define the transfer function of this transform to be $H(\omega)$, where

$$H(\omega) = -j \text{sgn}(\omega), \tag{7}$$

we see that $H(\omega)$ can be represented graphically using any of the two equivalent forms shown below.



So, this transform can be thought of simply as a transform that produces input signal shifted by an angle of $\pi/2$ to the right (shifted by $-\pi/2$).

*** As an exercise, apply this transform to the Fourier transform of a cosine function $\cos(\omega_c t)$. You should get a $\sin(\omega_c t)$. If you apply it to $\sin(\omega_c t)$, you will get $-\cos(\omega_c t)$.

To get the representation of the Hilbert transform in time-domain (i.e., to find $m_h(t)$ in terms of $m(t)$), from the table of Fourier transforms we notice the FT pair

$$\text{sgn}(t) \Leftrightarrow \frac{2}{j\omega}$$

Using the symmetry between time and frequency property given in the table of FT properties, we get

$$\frac{2}{jt} \Leftrightarrow 2\pi \text{sgn}(-\omega),$$

or

$$\frac{1}{\pi t} \Leftrightarrow -j \text{sgn}(\omega)$$

Now, since

$$M_h(\omega) = -j \text{sgn}(\omega) \cdot M(\omega),$$

we see that $m_h(t)$ is the convolution in time-domain of the inverse Fourier transforms of two functions $-j\text{sgn}(\omega)$ and $M(\omega)$, which gives

$$\begin{aligned} m_h(t) &= \frac{1}{\pi} * m(t) \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{m(\alpha)}{t - \alpha} d\alpha \end{aligned} \quad (8)$$

This is the Hilbert transform of a signal in time-domain.

Now we are ready to represent the SSB signals in terms of the message signals. First, we notice that the USB and LSB signals in frequency-domain shown in parts (E) and (F) of the above figure can be represented in terms of shifted versions of $M_+(\omega)$ and $M_-(\omega)$ as

$$\begin{aligned} G_{USB}(\omega) &= M_+(\omega - \omega_c) + M_-(\omega + \omega_c) \\ G_{LSB}(\omega) &= M_+(\omega + \omega_c) + M_-(\omega - \omega_c) \end{aligned} \quad (9)$$

So,

$$\begin{aligned} g_{USB}(t) &= m_+(t)e^{j\omega_c t} + m_-(t)e^{-j\omega_c t} \\ g_{LSB}(t) &= m_+(t)e^{-j\omega_c t} + m_-(t)e^{j\omega_c t} \end{aligned} \quad (10)$$

Since,

$$\begin{aligned} m_+(t) &= \frac{1}{2}m(t) + \frac{1}{2}jm_h(t) \\ m_-(t) &= \frac{1}{2}m(t) - \frac{1}{2}jm_h(t) \end{aligned} \quad (11)$$

Substituting this (11) in (10), we get

$$\begin{aligned} g_{USB}(t) &= \frac{1}{2}m(t)e^{j\omega_c t} + \frac{1}{2}jm_h(t)e^{j\omega_c t} + \frac{1}{2}m(t)e^{-j\omega_c t} - \frac{1}{2}jm_h(t)e^{-j\omega_c t} \\ &= \boxed{m(t)\cos(\omega_c t) - m_h(t)\sin(\omega_c t)} \\ g_{LSB}(t) &= \frac{1}{2}m(t)e^{j\omega_c t} - \frac{1}{2}jm_h(t)e^{j\omega_c t} + \frac{1}{2}m(t)e^{-j\omega_c t} + \frac{1}{2}jm_h(t)e^{-j\omega_c t} \\ &= \boxed{m(t)\cos(\omega_c t) + m_h(t)\sin(\omega_c t)} \end{aligned} \quad (12)$$

where $m_h(t)$ is given by (8).

These are the representations of the upper- and lower-side band signals in terms of message signal.