Trigonometric Fourier Series

Let $g(t)$ be one of the following:

1. a periodic signal with period T_0,
2. any signal in which we are interested in the time interval of $t_1 \le t \le t_1 + T_0$,

The signal $g(t)$ in the interval $t_1 \le t \le t_1 + T_0$ can be represented by the **trigonometric Fourier series** in terms of a sum of the following sinusoids:

$$\{1, \cos(\omega_0 t), \cos(2\omega_0 t), \cos(3\omega_0 t), \ldots\ldots, \sin(\omega_0 t), \sin(2\omega_0 t), \sin(3\omega_0 t), \ldots\ldots\},$$

or the **compact Fourier series** sinusoids:

$$\{1, \cos(\omega_0 t+\theta_1), \cos(2\omega_0 t+\theta_2), \cos(3\omega_0 t+\theta_3), \ldots\ldots\}, \qquad (1)$$

where the frequency ω_0 and T_0 are related by $T_0 = 2\pi / \omega_0$.

The basis for this representation is that the different sinusoids shown above are "orthogonal". Two signals $a(t)$ and $b(t)$ are orthogonal over a period T of time if

$$\int_T a(t)\cdot b(t)dt = 0.$$

This is true for all of the sinusoids given above in (1) over a period $T_0 = 2\pi / \omega_0$.

The representation of $g(t)$ is given by

$$g(t) = a_0 + a_1\cos(\omega_0 t) + a_2\cos(2\omega_0 t) + a_3\cos(3\omega_0 t) + \ldots + b_1\sin(\omega_0 t) + b_2\sin(2\omega_0 t) + b_3\sin(3\omega_0 t) + \ldots \qquad t_1 \le t \le t_1 + T_0$$

which can be written as

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t) \qquad t_1 \le t \le t_1 + T_0$$

or using the compact sinusoids as

$$g(t) = C_0 + \sum_{n=1}^{\infty} C_n\cos(n\omega_0 t + \theta_n) \qquad t_1 \le t \le t_1 + T_0$$

The coefficients a_0, a_n, and b_n shown above can be evaluated using

$$a_0 = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} g(t)dt,$$

The coefficient a_0 represents the average (or the DC value) of the function. So, for functions that have zero DC, the coefficient a_0 will be zero.

$$a_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} g(t)\cos(n\omega_0 t)dt \qquad n = 1, 2, 3, \ldots$$

$$b_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} g(t)\sin(n\omega_0 t)dt \qquad n = 1, 2, 3, \ldots$$

A trigonometric identity in the form of

$$a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t) = C_n\cos(n\omega_0 t + \theta_n),$$

exits, where the relation between the coefficients a_n and b_n and the coefficients C_n and θ_n is given by

$$C_n = \sqrt{a_n^2 + b_n^2},$$

and

$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$$

such that $C_0 = a_0$.

Comments:

1. All cosine functions $\cos(n\omega_0 t)$ are even functions (they are symmetric about the y–axis). Since an odd function can never be represented in terms of even functions, a_n of the Fourier series for an odd function are always zero for all values of *n*.
2. All sine functions $\sin(n\omega_0 t)$ are odd functions (they are anti–symmetric about the y–axis or symmetric about the origin). Since an even function can never be represented in terms of odd functions, b_n of the Fourier series for an even function are always zero for all values of *n*.
3. For a periodic function that is not even and not odd, at least some of the coefficients a_n and some of the coefficients b_n will be non zero.
4. The frequency ω_0 is called the fundamental frequency of the periodic signal $f(t)$ and the multiple of this frequency $n\omega_0$ is called the n^{th} harmonic of this fundamental frequency. The fundamental frequency represents the lowest frequency component contained in $f(t)$. Two signals: a sine wave with frequency ω_0 and a square wave with frequency ω_0 will sound similar when played using a speaker except that the square wave will also contain higher harmonics.

5. The Fourier series of part of a non–periodic signals is similar to the Fourier series of a periodic signal that is obtained by repeating that part of non–periodic signal to the right and to the left.

Exponential Fourier Series

We also can represent the function $g(t)$ in terms of complex exponentials as

$$g(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} = D_0 + \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} D_n e^{jn\omega_0 t}$$

where

$$D_n = \frac{1}{T_0}\int_{T_0} g(t)e^{-jn\omega_0 t}dt \, .$$

Therefore, D_n is related to C_n and θ_n as

$$|D_n| = |D_{-n}| = \frac{1}{2}C_n, \qquad \angle D_n = -\angle D_{-n} = \theta_n \, .$$

Examples:

1. Find the Fourier series coefficients a_n and b_n for

   a) the aperiodic signal $g(t) = |t|$, $-0.5 \le t \le 1.5$,

   Solution:

   Although this signal is non–periodic, we can still find its Fourier series expansion between the two points $t = -0.5$ to 1.5 as follows.

   We will consider T_0 to be $T_0 = 1.5 - (-0.5) = 2$ sec ➔ $\omega_0 = 2\pi/T_0 = \pi$ rad/s

$$\begin{aligned}
a_0 &= \frac{1}{2}\int_{-0.5}^{1.5} |t|\,dt \\
&= \frac{1}{2}\int_{-0.5}^{0} -t \cdot dt + \frac{1}{2}\int_{0}^{1.5} t \cdot dt \\
&= -\frac{1}{4}t^2\Big|_{-0.5}^{0} + \frac{1}{4}t^2\Big|_{0}^{1.5} \\
&= -\frac{1}{4}(0-0.25) + \frac{1}{4}(2.25-0) \\
&= 0.625
\end{aligned}$$

$$a_n = \frac{1}{2}\int_{-0.5}^{1.5} |t| \cos(n\omega_0 t)dt$$

$$= \frac{1}{2}\int_{-0.5}^{0} -t \cdot \cos(n\omega_0 t)dt + \frac{1}{2}\int_{0}^{1.5} t \cdot \cos(n\omega_0 t)dt$$

Now we will have to integrate using the integration by parts method. That is

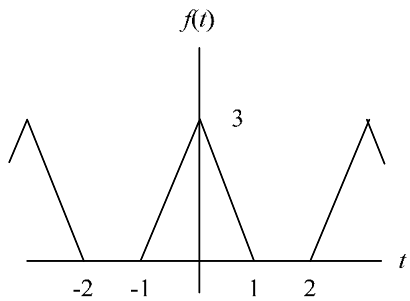
$$\int_T udv = uv\big|_T - \int_T vdu$$

So, if we let $u = t$ and $dv = \cos(n\omega_0 t)\cdot dt$,
we will get, $du = dt$ and $v = (1/n\omega_0)\sin(n\omega_0 t)$

$$a_n = -\frac{1}{2}\left(\left[\frac{t}{n\pi}\sin(n\pi t)\right]_{-0.5}^{0} - \int_{-0.5}^{0}\frac{1}{n\pi}\sin(n\pi t)dt\right) + \frac{1}{2}\left(\left[\frac{t}{n\pi}\sin(n\pi t)\right]_{0}^{1.5} - \int_{0}^{1.5}\frac{1}{n\pi}\sin(n\pi t)dt\right)$$

Now completing the remaining integrations and evaluating the coefficients a_n becomes straight forward. Computing the coefficients a_n is performed in exactly the same manner.

b) the periodic signal $f(t)$ shown below



c) the periodic signal $h(t)$ shown below