Formula Sheet for EE406

Energy of discrete time signal (DTS): $E = \sum_{n=-\infty}^{\infty} |x(n)|^2$

Average power of discrete time signal (DTS): $P = \lim_{n \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$

Convolution sum formula for LTI system

$$y(n) = x(n) * h(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

Convolution sum formula for causal LTI system

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{n} x(k)h(n-k)$$

Convolution sum formula for causal LTI system and the input is also causal

$$y(n) = \sum_{k=0}^{n} h(k)x(n-k) = \sum_{k=0}^{n} x(k)h(n-k)$$

Convolution sum formula for causal FIR system

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

Convolution sum formula for causal IIR system

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

Constant Coefficient Difference Equation (CCDE):

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

Direct Z-transform: $X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$

Summation of finite Geometric Series: $\sum_{k=N_1}^{N_2} a^k = \begin{cases} \dfrac{a^{N_1} - a^{N_2+1}}{1-a}, & a \neq 1 \\ N_2 - N_1 + 1, & a = 1 \end{cases}$

Summation of infinite Geometric Series: $\sum_{k=n}^{\infty} a^k = \dfrac{a^n}{1-a}, \quad |a| < 1$

Summation of finite Arithmetic Series: $\sum_{k=N_1}^{N_2} K = \dfrac{1}{2}\left[\left(N_2 + N_1\right)\left(N_2 - N_1 + 1\right)\right]$

The system function of CCDF: $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$

Partial Fraction Expansion (PFE): Distinct Poles: real or complex conjugate

$$\frac{X(z)}{z} = \frac{A_1}{(z - p_1)} + \frac{A_2}{(z - p_2)} + \cdots + \frac{A_N}{(z - p_N)}$$

$$A_k = \frac{X(z)}{z}(z - p_k)\Big|_{z = p_k}$$

Inverse z-transform for PFE: Distinct Poles: real

$$z^{-1}\left\{\frac{A_k}{\left(1 - p_k z^{-1}\right)}\right\} = \begin{cases} A_k \left(p_k\right)^n u(n), & causal \\ -A_k \left(p_k\right)^n u(-n-1), & noncausal \end{cases}$$

Inverse z-transform for PFE: Distinct Poles: complex conjugate (causal)

$$z^{-1}\left\{\frac{A_k}{\left(1 - p_k z^{-1}\right)} + \frac{A_k^*}{\left(1 - p_k^* z^{-1}\right)}\right\} = 2|A_k|\left(r_k\right)^n \cos\left(\beta_k n + \alpha_k\right) u(n)$$

$$A_k = |A_k| e^{j\alpha_k}$$

$$p_k = r_k e^{j\beta_k}$$

Partial Fraction Expansion (PFE): Multiple-order Poles

$$\frac{1}{(z-p_k)^m} = \frac{A_{1k}}{(z-p_k)} + \frac{A_{2k}}{(z-p_k)^2} + \cdots + \frac{A_{mk}}{(z-p_k)^m} = \sum_{s=1}^{m} \frac{A_{sk}}{(z-p_k)^s}$$

$$A_{sk} = \frac{1}{(m-s)!} \left\{ \frac{d^{m-s}}{dz^{m-s}} \left[\frac{X(z)}{z} (z-p_k)^m \right] \right\}_{z=p_k}, \quad s = 1, 2, \cdots, m$$

<u>Inverse z-transform for PFE: Double-order Poles: causal</u>

$$z^{-1} \left\{ \frac{p_k z^{-1}}{(1-p_k z^{-1})^2} \right\} = n (p_k)^n u(n)$$

<u>Fourier Series of Discrete-time Periodic Signals (DTFS)</u>

Analysis: $C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$

Synthesis: $x(n) = \sum_{k=0}^{N-1} C_k e^{j2\pi kn/N}$

<u>Fourier Transform of Discrete-time Periodic Signals (DTFT)</u>

Analysis: $X(\omega) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

Synthesis: $x(n) = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$

<u>The frequency response of a system</u> $H(\omega) = |H(\omega)| e^{j\theta(\omega)} = \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k}$

<u>The frequency response of a LTI system described by CCDE</u>

$$H(\omega) = H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

<u>The response of a relaxed LTI system to an input signal, *x(n)*, with a single frequency,</u> ω_k is

$y(n) = A e^{j\omega_k n} H(\omega_k)$

Where $x(n) = A e^{j\omega_k n}$

<u>The system response to a cosine input signal, *x(n)*, with a single frequency,</u> ω_k is

$$y(n) = A\left|H(\omega_k)\right|\cos\left(\omega_k n + \theta(\omega_k)\right)$$

Where $x(n) = A\cos\left(\omega_k n\right)$

The system response to a sine input signal, *x(n)*, with a single frequency, ω_k is

$$y(n) = A\left|H(\omega_k)\right|\sin\left(\omega_k n + \theta(\omega_k)\right)$$

Where $x(n) = A\sin\left(\omega_k n\right)$

Discrete Fourier Transform (DFT) of Discrete-time finite Signals of length *N*

Analysis: $X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \le k \le N-1$

Synthesis: $x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn}, \quad 0 \le n \le N-1$

Where $W_N = e^{-j(2\pi/N)}$

Matrix relation of DFT

$$\overline{X} = \overline{D}_N \overline{x}$$

$$\overline{x} = \overline{D}_N^{-1}\overline{X} = \frac{1}{N}\overline{D}_N^{*}\overline{X}$$

$$\overline{X} = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}, \ \overline{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \text{ and } \overline{D}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & & W_N^{(N-1)(N-1)} \end{bmatrix}$$

Time domain relation between DTFT, *x(n)*, and DFT, *y(n)*

$$y(n) = \sum_{m=-\infty}^{\infty} x(n+mN), \quad 0 \le n \le N-1$$

Circular convolution of two finite sequences

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n) x_2\left(\left(m-n\right)_N\right), \quad 0 \le m \le N-1$$