International Journal of Systems Science
volume ??, number ??, Month?? 200?, pages 1–8

Taylor & Francis
Taylor & Francis Group

Model reduction of bilinear systems described by input–output difference equation

S. A. AL-BAIYAT

A class of single input single output bilinear systems described by their input–output difference equation is considered. A simple expression for the Volterra kernels of the system is derived in terms of the coefficients of difference equation. An algorithm, based on the singular value decomposition of a generalized Hankel matrix, is also developed. The algorithm is then used to find a reduced-order bilinear state-space model. The Hankel approach will be extensively studied under different data length cases and different orders of the state-space models. A numerical example is presented to illustrate the effectiveness of the proposed algorithm.

1. Introduction

Many control systems encountered in practice may not be adequately described by first-principle models. Modelling of such processes is often achieved by using a parametric external model based on input–output data (Diaz and Desrochers 1988, Haber and
1 Unbehaven 1990). The most well known of these models are the Volterra series expansion, the autoregressive moving average (ARMA) and neural networks (Mohler 1991, Levin and Narendra 1995). In particular, the Volterra series have proven to be valuable tools and are used extensively in modelling of nonlinear systems (Brockett 1976, D'Alessandro *et al*. 1974). It expands the impulse response model of a linear system by representing nonlinearity as a part of higher order impulses termed kernels.

In spite of their success in characterizing the input–output relationship, the external models alone are not suitable for the analysis and design of control systems. On the other hand, analysis and control applications, such as stability, controllability, observability and feedback design, are well understood in the state-space set up (Atassi and Khalil 2001, Serrani *et al*. 2001). One of the objectives of the present paper is to bridge the gap between the two modelling approaches by finding a reduced-order state-space model for the given input–output model for bilinear systems.

Bilinear systems are considered as a subclass of nonlinear systems under the assumption of linearity in the state or in the control, but not jointly. Interest in studying bilinear systems has grown over the years, mainly because such systems are general enough to model several important processes in engineering, economics, biology, ecology, etc. (Mohler 1991), and at the same time they are specific enough to support a rich mathematical structure (D'Alessandro *et al*. 1974). Moreover, bilinear systems can be used to approximate quite general nonlinear systems (Brockett 1976).

The approximation of higher-order complex systems to lower-order models have attracted the attention of many researchers during the past two decades (Moor 1981, Pernebo and Silverman 1982, Muscato 2000). Various model reduction schemes have been proposed in the literature. Early methods were concentrated on the retention of dominant poles in the reduced-order model, as in aggregation methods, or the matching of the several moments of the original systems, as in Pade approximation methods. However, one approach by Moor has dramatically changed the status of model reduction. This approach is the balanced realization. Balanced model reduction of linear dynamic systems proved to be a very efficient scheme for the approximation of large-scale systems. In this approach, strongly controllable and strongly observable states in a balanced representation are retained in the reduced-order model as the dominant part of the original higher-order system.

Received 5 August 2002. Revised 4 March 2004. Accepted 10 June 2004.

Electrical Engineering Department, King Fahd University of Petroleum & Minerals, Dhahran 31261, Saudi Arabia.
e-mail: sbaiyat@kfupm.edu.sa

International Journal of Systems Science ISSN 0020–7721 print/ISSN 1464–5319 online © 200? Taylor & Francis Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/00207720410001734237

Balancing theory has the advantage of being computationally simple using standard matrix software. This simplicity has formalized a balancing approach in several directions. The stability issue was addressed by Pernebo and Silverman (1982) in which it was shown that the reduced-order system retains its stability, balancing and minimality properties. The issue of error between the full- and reduced-order systems was addressed by many researchers. Two popular performance norms, H_∞ and H_2, were used to quantify the closeness between the full- and reduced-order models. Enns (1984) and Wang *et al.* (1999) and many others have used the H_∞ norm. The H_2 norm was used by Spanos *et al.* (1992) and Diab *et al.* (1998), just to name a few.

An alternative equivalent formulation of the model reduction problem is based on the singular value decomposition of the Hankel matrix. This approach is very important if the available information is the input–output representation or the input–output data of the system. Many algorithms based on the Hankel matrix and which compute realizations of linear systems exist in the literature. The most widely known is that of Ho and Kalman (1965). Bettayeb (1981) has used this algorithm to find a reduced-order model equivalent to the reduced-order model resulting from Moor's balanced reduction. The Hankel matrix was also used by Kung (1978) to find a reduced-order realization. Sreeram and Agathoklis (1991) used a weighted impulse response Gramian to find a reduced-order model for single input single output discrete time systems. The approach was developed further to the multi-input multi-output by Ang *et al.* (1995). Xiao *et al.* (1997) have extended the work of Ang *et al.* to two-dimensional systems.

The success of the application of a balanced model reduction scheme to several practical systems motivated many researchers to generalize the balancing concept to more general dynamic systems. State-space balanced representation and balanced model reduction of bilinear systems have been treated by Hsu *et al.* (1983) and Al-Baiyat *et al.* (1994). Zhang and Lam (2002) have proposed an H_2 model reduction method for continuous time bilinear systems. Moreover, an explicit H_2 norm error bound was also given in that approach. Recently Zhang *et al.* (2003) have investigated the stability issue of reduced-order discrete time bilinear systems and under some reachability or observability condition have shown that the reduced-order system is stable. As mentioned above, the literature is quite limited for the model reduction problem of bilinear systems and much more is needed to generalize many of the properties of the model reduction for linear systems to bilinear systems.

This paper considers a class of bilinear discrete-time systems described by a input–output difference equation and investigates the idea of finding a reduced state-space bilinear model. One of the main contributions is the finding of exact explicit expressions of impulse response kernels of bilinear systems in terms of coefficients of bilinear difference equations. The paper also provides an algorithm for finding a reduced-order state-space model of the given input–output model. In the algorithm, the system Hankel matrix is formed first and then singular value decomposition on this matrix is performed to identify a bilinear state-space model of any order. This last step is very robust as a consequence of the robust perturbation property of the singular value decomposition. In this work, the Hankel approach will be extensively studied under different data length cases and different orders of the state-space models.

It is well known in the literature that model reduction algorithms based on the Hankel matrix approach lend themselves to system identification (Diab *et al.* 1997, Xiao *et al.* 1997). Both concepts are based on singular value decomposition of the Hankel matrix. Moreover, most of the available algorithms are extensions to the work started by Kung (1978). The proposed algorithm in this work can also be used for the identification problem.

2.. Problem statement

Consider a single input single output discrete-time bilinear system described by the following input–output difference equation:

$$
\begin{aligned}
y(k) = \sum_{i=1}^{s} a_i y(k-i) + \sum_{i=1}^{s} b_i u(k-i) \\
+ \sum_{i=1}^{s} \sum_{j=1}^{i} \beta_{ij} y(k-i) u(k-j),
\end{aligned} \tag{1}
$$

where $u(k) \in \mathrm{R}$ and $y(k) \in \mathrm{R}$ are the input and output of the system, respectively.

These types of difference equations can be found either from available data or from mathematical modelling. In the former case, measurement data are fitted by nonlinear regression, for example to the above form of difference equations. In the later case, physical laws and relations lead, generally, to nonlinear or bilinear differential or difference equations. In the continuous time modelling case, a further step of discretization is done to obtain the difference equation.

Depending on the nature of the data and taking the system into consideration, the modelling leads to either stochastic difference equations ($u(k)$ above is a stochastic input) or frequently called time series (Priestley 1988), or to deterministic difference equations where the input is a deterministic input. The present work will concentrate on deterministic difference equations.

The first issue is to derive a simple expression for the Volterra kernels in terms of the coefficients of the bilinear difference equation.

Second, we address the issue of finding a reduced-order bilinear state-space model of the form:

$$x(k+1) = Ax(k) + Nx(k)u(k) + bu(k) \tag{2a}$$

$$y(k) = cx(k), \tag{2b}$$

where $x(k) \in \mathrm{R}^{\mathrm{r}}$ is the state vector, $u(k)$ is a scalar input, $y(k)$ is a scalar output, and $\{A, N, b, c\}$ are matrices of proper dimensions.

3. Preliminaries

This section reviews useful definitions and results related to the input–output representation and the realization theory of bilinear systems. The results will be needed in what follows and can be found in D'Alessandro *et al*. (1974) and Isidori (1973). We start by defining the reachability matrix for the nth-order system described by equation (2) as follows:

$$R_n = \begin{bmatrix} P_1 & P_2 & \dots & p_n \end{bmatrix}, \tag{3a}$$

where

$$P_1 = b \tag{3b}$$

$$P_i = [AP_{i-1} \quad NP_{i-1}] \qquad i \geq 2. \tag{3c}$$

The state space of system (2) is reachable if and only if

$$\operatorname{rank} R_n = n. \tag{4}$$

Similarly, the observability matrix for system (2) is defined as follows:

$$O_n = \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{bmatrix}, \tag{5a}$$

where

$$Q_1 = c, \quad Q_i = \begin{bmatrix} Q_{i-1}A \\ Q_{i-1}N \end{bmatrix} \qquad i \geq 2. \tag{5b}$$

The state space of system (2) is observable if and only if

$$\operatorname{rank} O_n = n. \tag{6}$$

An interesting external description for system (2) with zero initial conditions is given by the relation (Isidori 1973):

$$y(k) = \sum_{j=1}^{k} w_j u_j(k-j) \qquad k = 1, 2, 3, \dots, \tag{7}$$

where w_j is a $1 \times 2^{(j-1)}$ row vector, whose elements represent the discrete Volterra kernels, defined recursively from the matrices A, N, b, c of (2) as follows:

$$P_1 = b \tag{3b}$$

$$P_i = [AP_{i-1} \quad NP_{i-1}] \qquad i = 2, 3, 4, \dots \tag{3c}$$

$$w_j = cP_j \qquad j = 1, 2, 3, \dots. \tag{8}$$

The $2^{(j-1)} \times 1$ column vector $u_j(k-j)$ is defined recursively from the input sequence of system (2) as follows:

$$u_1(h) = u(h)$$

$$u_j(h) = \begin{bmatrix} u_{j-1}(h) \\ u_{j-1}(h)u(h+j-1) \end{bmatrix} \qquad j = 2, 3, 4, \dots. \tag{9}$$

Isidori used the infinite sequence $\{w_1, w_2, w_3, \dots, w_j, \dots\}$ to define a generalized Hankel matrix for the bilinear system (2). The generalized Hankel matrix is constructed as follows:

$$S_{1j} = w_j, \qquad j = 1, 2, 3, \dots. \tag{10}$$

S_{ij} $(i = 2, 3, \dots; j = 1, 2, \dots)$ is obtained from $S_{i-1,j+1}$ with this rule: form the partition

$$S_{i-1,j+1} = \begin{bmatrix} S^1_{i-1,j+1} & S^2_{i-1,j+1} \end{bmatrix} \tag{11}$$

assigning the same number of columns to both blocks on the right-hand side and put

$$S_{ij} = \begin{bmatrix} S^1_{i-1,j+1} \\ S^2_{i-1,j+1} \end{bmatrix}. \tag{12}$$

Then the infinite Hankel matrix is defined as

$$S = \begin{bmatrix} S_{11} & S_{12} & & \dots \\ S_{21} & S_{22} & & \dots \\ . & . & & \\ . & . & & \dots \\ . & . & & \end{bmatrix} \tag{13}$$

and the finite Hankel matrix $S_{M',\mathrm{M}}$ as the upper left-hand part of the matrix S. The dimension of the finite Hankel matrix is $(2^{\mathrm{M}'}-1) \times (2^{\mathrm{M}}-1)$.

+ [NT: 280×215mm] Ver: 7.51.g/W [16.7.2004–7:21pm] FIRST PROOFS {TandF}Tsys/TSYS-101547.3d (TSYS) Paper: TSYS-101547 Keyword

Two Hankel matrices $S^1_{M',M}$ and $S^2_{M',M}$ are needed in the subsequent development. Those matrices are formed from the matrix $S_{M',M+1}$ as follows:

$$S^1_{M',M} = \text{set of columns } \{2,\ 4-5,\ 8-11,\dots,\ 2^M-(3*2^{\mathrm{M}-1}-1)\} \text{ of the matrix } S_{M',M+1}.$$

Similarly

$$S^2_{M',M} = \text{set of columns } \{3, 6-7, 12-15,\dots, 3*2^{\mathrm{M}-1}-(2^{\mathrm{M}+1}-1)\} \text{ of the matrix } S_{M',M+1}.$$

A bilinear state-space model is obtained from full rank factorization of $S_{M',M}$ (Isidori 1973):

$$S_{M',M} = O_{M'}R_M, \tag{14}$$

where R_M and O_M are the reachability and the observability matrices defined in (3) and (5), respectively.

The state-space matrices A and N are then obtained as least square solutions to

$$S^1_{M',M} = O_{M'}AR_M \tag{15}$$

$$S^2_{M',M} = O_{M'}NR_M. \tag{16}$$

While the matrix b is the first column of R_M and c is the first row of O_M.

4. Calculation of Volterra kernels from the difference equation

The Hankel matrix for linear systems proves to be very useful in many areas of research such as realization theory (Ho and Kalman 1965), model reduction and identification (Kung 1978). The generalized Hankel matrix plays the same role for a bilinear system as does the Hankel matrix for a linear system. Isidori (1973) used the generalized Hankel matrix extensively in the realization problem of bilinear systems. Hsu *et al.* (1983) proposed an algorithm to find a reduced-order model based on the Hankel approach.

The estimation of the kernels of the bilinear systems needed to form the generalized Hankel matrix is a difficult task, as pointed out by many researchers (Diaz and Desrochers 1988). In this section, the kernels, hence the generalized Hankel matrix, will be obtained for bilinear systems characterized by difference equation (1).

Expressions for w_i in terms of the coefficients of the difference equation (1) are obtained by equating the output $y(k)$ of equation (7), for every k, with the recursive expression of $y(k)$ in equation (1). Expressions for the first three terms, $y(k)$, $k=1,2,3$, with zero initial conditions, are:

$$y(1) = b_1u(0) \tag{17}$$

$$y(2) = a_1y(1)+b_1u(1)+b_2u(0)+\beta_{11}y(1)u(1) = \begin{bmatrix} a_1b_1+b_2 & \beta_{11}b_1 \end{bmatrix}\begin{bmatrix} u(0) \\ u(0)u(1)\end{bmatrix} + b_1u(1) \tag{18}$$

$$\begin{aligned} y(3) &= \sum_1^2 a_jy(3-j) + \sum_1^3 b_ju(3-j) + \beta_{11}y(2)u(2) + \sum_1^2 \beta_{2j}y(1)u(3-j) \\ & \big[a_1(a_1b_1+b_2)+a_2b_1+b_3 \quad a_1\beta_{11}b_1+\beta_{22}b_1 \\ & \quad \beta_{11}(a_1b_1+b_2)+\beta_{21}b_1 \quad \beta_{11}\beta_{11}b_1\big] \\ & \begin{bmatrix} u(0) \\ u(0)u(1) \\ u(0)u(2) \\ u(0)u(1)u(2)\end{bmatrix} + \begin{bmatrix} a_1b_1+b_2 & \beta_{11}b_1\end{bmatrix} \\ & \times \begin{bmatrix} u(1) \\ u(1)u(2)\end{bmatrix} + b_1u(2). \end{aligned} \tag{19}$$

By comparing relations (17–19) with $y(k)$, $k=1,2,3$, in equation (7), the first three kernels are identified as:

$$w_1 = b_1 \tag{20}$$

$$w_2 = \begin{bmatrix} a_1b_1+b_2 & \beta_{11}b_1\end{bmatrix} \tag{21}$$

$$\begin{aligned} w_3 = \big[&a_1(a_1b_1+b_2)+a_2b_1+b_3 \quad a_1\beta_{11}b_1+\beta_{22}b_1 \\ &\beta_{11}(a_1b_1+b_2)+\beta_{21}b_1 \quad \beta_{11}\beta_{11}b_1\big]. \end{aligned} \tag{22}$$

Similarly, the general ith-order kernel is obtained from the following expressions:

$$w_1 = b_1$$

$$w_i = \left[\sum_{j=1}^{i-1} a_j\bar{w}_{i(i-j)} + \bar{b}_i + \sum_{j=2}^{i-1}\sum_{k=j}^{i-1}\beta_{kj}\hat{w}^{kj}_{i(i-k)} \quad \sum_{j=1}^{i-1}\beta_{j1}\bar{w}_{i(i-j)}\right] \quad i\ge 2, \tag{23}$$

where the matrices $\bar{w}_{i(i-j)}$, $\bar{b}_i$ and $\hat{w}^{kj}_{i(i-k)}$ are of dimension $1\times 2^{i-2}$ and are defined as:

$$\bar{w}_{i(i-j)}=\begin{bmatrix} w_{i-j} & 0 & 0 & \dots & 0\end{bmatrix},\quad \text{number of zeros}=2^{i-2}-2^{i-j-1} \tag{24}$$

$$\bar{b}_i=\begin{bmatrix} b_i & 0 & 0 & \dots & 0\end{bmatrix} \tag{25}$$

$$\hat{w}^{kj}_{i(i-k)}=\begin{bmatrix} 0 & 0 & \dots & 0 & w_{i-k} & 0 & 0 & \dots & 0\end{bmatrix}, \tag{26}$$

where w_{i-k} is in the $(1+2^{i-j-1})$ position.

These kernels are used to form the Hankel matrices in (13). Note, for a second-order system ($s=2$), the first two kernels, (20) and (21), are given by Bartee and Georgakis (1992).

5. Low-order Hankel approximation

The Hankel matrix $S_{M',M}$ is generally full rank as it is formed from either measured Volterra kernels or Volterra kernels obtained from unnecessarily high order difference equations. A lower-order approximation scheme that carries the dominant dynamics of the original data and excludes noise modelling is desirable. The robust singular value decomposition is proven to be suitable for such situations (Kung 1978).

Following Isidori (1973), a lower-order bilinear state-space model based on singular value factorization of the Hankel matrix $S_{M',M}$ is obtained as follows:

(1) Construct the generalized finite Hankel matrix $S_{M',M}$.
(2) Form the submatrices $S^1_{M',M}$ and $S^2_{M',M}$ defined in Section 2.
(3) Perform singular value decomposition of the $(2^{M'}-1)\times(2^{M}-1)$ Hankel matrix $S_{M',M}$ and order the singular values of $S_{M',M}$ in a descending manner:

$$S_{M',M}=U\Sigma V^{\mathrm{T}} = \begin{bmatrix} U_1 & U_2\end{bmatrix}\begin{bmatrix}\Sigma_1 & 0\\ 0 & \Sigma_2\end{bmatrix}\begin{bmatrix} V_1^{\mathrm{T}}\\ V_2^{\mathrm{T}}\end{bmatrix}, \tag{27}$$

where, Σ_1 is an $r\times r$ matrix containing all the dominant singular values. While Σ_2 contains the remaining smaller singular values. As a result of the above, $S_{M',M}$ can be approximated as:

$$S_r=U_1\Sigma_1V_1^{\mathrm{T}}=O_rR_r, \tag{28}$$

where

$$O_r=U_1\Sigma_1^{1/2},\qquad R_r=\Sigma_1^{1/2}V_1^{\mathrm{T}}. \tag{29}$$

(4) Form the quadruple state space (A_r,N_r,b_r,c_r) of order r as follows:

$$\begin{aligned} A_r&=\Sigma_1^{-1/2}U_1^{\mathrm{T}}S^1_{M',M}V_1\Sigma_1^{-1/2}\\ N_r&=\Sigma_1^{-1/2}U_1^{\mathrm{T}}S^2_{M',M}V_1\Sigma_1^{-1/2}\\ b_r&=\text{first column of }\Sigma_1^{-1/2}V_1^{\mathrm{T}}\\ c_r&=\text{first row of }U_1\Sigma_1^{-1/2}.\end{aligned} \tag{30}$$

The reduced-order model given in (30) exhibits a balanced behaviour. A system is said to be balanced if the reachability and observability gramians are equal and diagonal. A balanced realization procedure for linear systems has been developed by Moor (1981) and Pernebo and Silverman (1982). The balanced realization procedures were extended to bilinear systems by Hsu *et al.* (1983) and Al-Baiyat *et al.* (1994). The reduced-order system in (30) can be shown to be balanced by using the quadruple (A_r,N_r,b_r,c_r) given in (30) and the definitions of the reachability and the observability gramians as follows:

$$\begin{aligned}\Xi_R&=R_rR_r^{\mathrm{T}}\\ &=\Sigma_1^{1/2}V_1^{\mathrm{T}}V_1(\Sigma_1^{1/2})^{\mathrm{T}}\\ &=\Sigma_1\end{aligned} \tag{31}$$

$$\begin{aligned}\Xi_O&=O_r^{\mathrm{T}}O_r\\ &=\Sigma_1^{1/2}U_1^{\mathrm{T}}U_1(\Sigma_1^{1/2})^{\mathrm{T}}\\ &=\Sigma_1.\end{aligned} \tag{32}$$

Hence

$$\Xi_R=\Xi_O=\Sigma_1. \tag{33}$$

6. Example

The reduced-order scheme given above will be thoroughly evaluated on a continuous steered tank reactor (CSTR) system given by Bartee and Georgakis (1992). Experimental data from the CSTR system were fitted to a bilinear difference equation through nonlinear regression to give the following model (Bartee and Georgakis 1992):

$$\begin{aligned} y(k)=\;&1.3187y(k-1)-0.2214y(k-2)-0.1474y(k-3)\\ &-8.6337u(k-1)+2.9234u(k-2)+1.2493u(k-3)\\ &-0.0858y(k-1)u(k-1)+0.0050y(k-2)u(k-1)\\ &+0.0602y(k-2)u(k-2)+0.0035y(k-3)u(k-1)\\ &-0.0281y(k-3)u(k-2)+0.0107y(k-3)u(k-3).\end{aligned} \tag{34}$$

The Volterra kernels are computed using the algorithm developed in Section 4. Three Hankel matrices of 3×4, 5×6 and 7×8 blocks are formed as in equation (13) from the obtained kernels. The rank of the Hankel matrices of the different sizes is five, which leads to bilinear realizations of order five (Isidori 1973). However, a lower-order model can be obtained by using the method given above.

The approximate reduced-order scheme given above will be evaluated for different data length, i.e. the number of kernels used to form the Hankel matrix.

From the Hankel matrices of 3×4 blocks, a second- and third-order reduced model were computed. The state-space matrices of a second-order model are as follows:

$$A_{r2}=\begin{bmatrix}0.9325 & 0.1358\\ -0.1141 & 0.5248\end{bmatrix}\quad N_{r2}=\begin{bmatrix}-0.0726 & 0.0442\\ 0.0455 & 0.2804\end{bmatrix}$$
$$b_{r2}=\begin{bmatrix}2.9740\\ 0.4475\end{bmatrix}\quad c_{r2}=\begin{bmatrix}-2.9739 & 0.4968\end{bmatrix}. \tag{35}$$

Similarly, the state-space matrices of a third-order model are:

$$A_{r3}=\begin{bmatrix}0.9325 & 0.1358 & 0.0112\\ -0.1141 & 0.5248 & -0.0099\\ 0.0105 & 0.3005 & -0.1758\end{bmatrix}$$
$$N_{r3}=\begin{bmatrix}-0.0726 & 0.0442 & -0.0717\\ 0.0455 & 0.2804 & -0.5200\\ 0.0348 & 0.1474 & -0.2729\end{bmatrix}$$
$$b_{r3}=\begin{bmatrix}2.9740\\ 0.4475\\ -0.1469\end{bmatrix}\quad c_{r3}=[-2.9739\quad 0.4968\quad 0.0819]. \tag{36}$$

By using the Hankel matrix of 5×6 blocks, a different second- and as well as a third-order reduced model were computed. The state-space matrices of a second-order model are:

$$A_{r2}=\begin{bmatrix}0.9146 & 0.1059\\ -0.1010 & 0.5696\end{bmatrix}\quad N_{r2}=\begin{bmatrix}-0.0721 & 0.0560\\ 0.0242 & 0.2094\end{bmatrix}$$
$$b_{r2}=\begin{bmatrix}3.0192\\ 0.7277\end{bmatrix}\quad c_{r2}=[-3.0194\quad 0.7047]. \tag{37}$$

Similarly, the state-space matrices of a third-order model are:

$$A_{r3}=\begin{bmatrix}0.9146 & 0.1059 & 0.0132\\ -0.1010 & 0.5696 & 0.0941\\ 0.0153 & 0.3671 & -0.1157\end{bmatrix}$$
$$N_{r3}=\begin{bmatrix}-0.0721 & 0.0560 & -0.0761\\ 0.0242 & 0.2094 & -0.3374\\ 0.0310 & 0.1150 & -0.1883\end{bmatrix}$$
$$b_{r3}=\begin{bmatrix}3.0192\\ 0.7277\\ -0.2354\end{bmatrix}$$
$$c_{r3}=[-3.0194\quad 0.7047\quad 0.1356]. \tag{38}$$

Finally, by taking the Hankel matrix of 7×8 blocks, a different second- and as well as a third-order reduced model were computed. The state-space matrices of a second-order model are:

$$A_{r2}=\begin{bmatrix}0.9060 & -0.0927\\ 0.0921 & 0.5877\end{bmatrix}\quad N_{r2}=\begin{bmatrix}-0.0728 & -0.0604\\ -0.0150 & 0.1779\end{bmatrix}$$
$$b_{r2}=\begin{bmatrix}3.0528\\ -0.8872\end{bmatrix}\quad c_{r2}=[-3.0536\quad -0.8266]. \tag{39}$$

Similarly, the state-space matrices of a third-order model are:

$$A_{r3}=\begin{bmatrix}0.9060 & -0.0927 & 0.0106\\ 0.0921 & 0.5877 & -0.1095\\ 0.0209 & -0.3601 & -0.0987\end{bmatrix}$$
$$N_{r3}=\begin{bmatrix}-0.0728 & -0.0604 & -0.0777\\ -0.0150 & 0.1779 & 0.2904\\ 0.0299 & -0.0949 & -0.1609\end{bmatrix}$$
$$b_{r3}=\begin{bmatrix}3.0528\\ -0.8872\\ -0.3024\end{bmatrix}$$
$$c_{r3}=[-3.0536\quad -0.8266\quad 0.1574]. \tag{40}$$

Figure 1 shows the step responses of the original system given in (34), the second and third reduced-order models obtained from the Hankel matrix of 3×4 blocks. It is clear that the reduced-order model gives comparable approximation for the original system. Figure 2 shows the step responses of the original system, the second

2

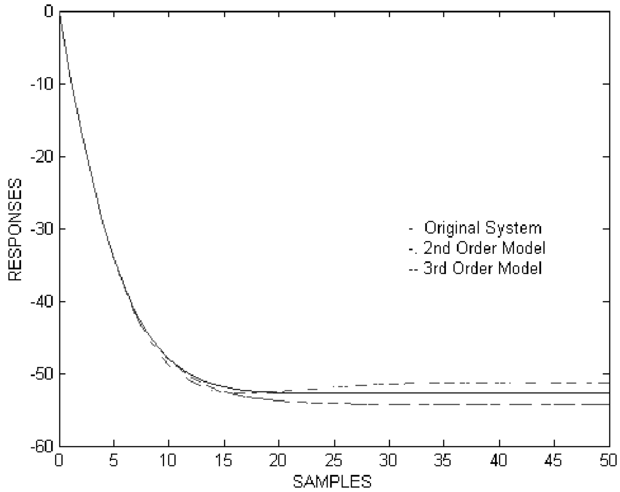

Figure 1. Step response of the original, second and third reduced systems from the Hankel matrix of size 3×4 blocks.

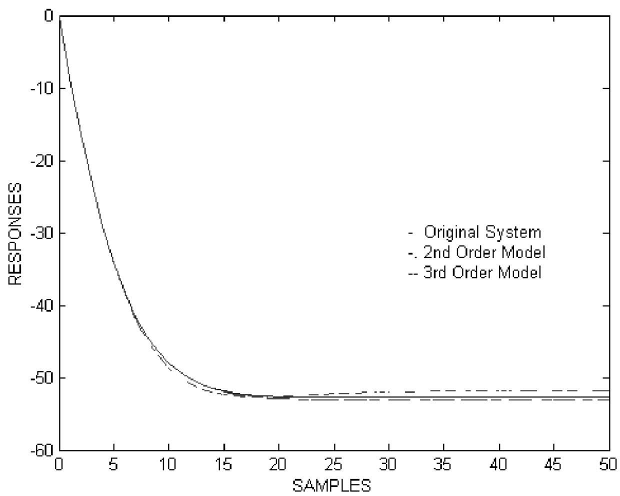

Figure 2. Step response of the original, second and third reduced systems from the Hankel matrix of size 5×6 blocks.

and third reduced-order models obtained from the Hankel matrix of 5×6 blocks. Clearly, the third-order model gives better approximation than the second-order model. Figure 3 shows the step responses of the original system, the second and third reduced-order models obtained from the Hankel matrix of 7×6 blocks. In this case, both the second- and third-order models give an excellent approximation of the original model. Clearly, as the Hankel matrix increases in size, the reduced-order model will give better approximation of the original system (figure 4).

7. Conclusions

In this paper, a simple expression for the Volterra kernels has been derived for the class of discrete time bilinear systems described by a input–output difference equation. This important result permitted the formation of a generalized Hankel matrix that led to an algorithm for generating a reduced-order bilinear state-space model. The algorithm is based on the singular value decomposition of the Hankel matrix. Several Hankel matrices were studied under different data length cases and different orders of the state-space models were found. Finally, a numerical example was employed to illustrate the effectiveness of the proposed algorithm.

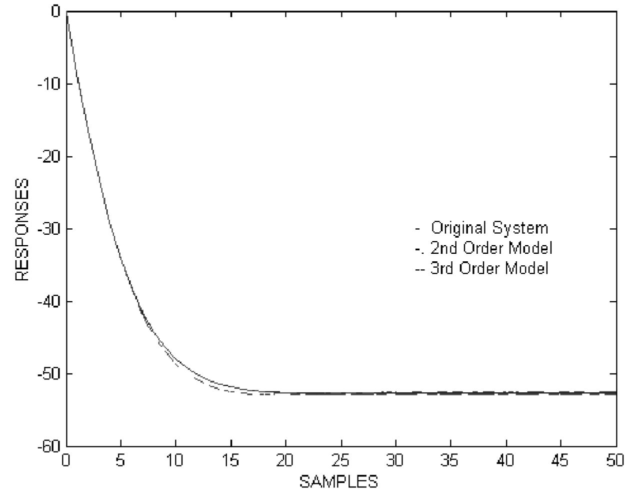

Figure 3. Step response of the original, second and third reduced systems from the Hankel matrix of size 7×8 blocks.

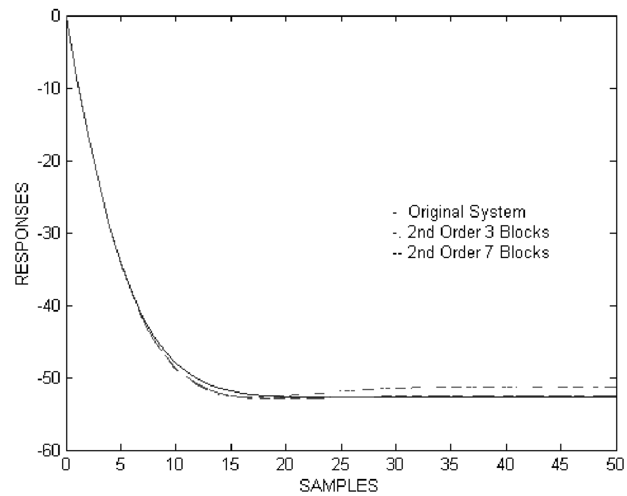

Figure 4. Step response of the original, second reduced systems from the Hankel matrices of size 3×4 and 7×8 blocks.

Acknowledgement

The author acknowledges the support of King Fahd University of Petroleum and Minerals.

References

Al-Baiyat, S. A., Bettayeb, M., and Al-Saggaf, U. M., 1994, *International Journal of Systems Science*, **25**, 1631–1642.

Ang, E. J., A Sreeram, V., and Liu, W. Q., 1995, *IEEE Transactions on Automatic Control*, **40**, 2153–2158.

Atassi, A. N., and Khalil, H. K., 2001, *IEEE Transactions on Automatic Control*, **AC 46**, 742–746.

Bartee, J. F., and Georgakis, C., 1992, *Proceeding of the 1992 American Control Conference*, Chicago, IL, USA, pp. 2576–2580.

3 Bettayeb, M., 1981, PhD dissertation, Department of Electrical Engineering, University of Southern California, Los Angeles.

Brockett, R. W., 1976, *IEEE Proceedings*, **64**, 61–72.

D'Alessandro, P., Isidori, A., and Ruberti, A., 1974, *SIAM Journal*
4 *on Control*, **12**, 517–535.

Diab, M., Liu, W. Q., and Sreeram, V., 1998, *Optimal Control Applications and Methods*, **19**, 147–167.

Diab, M., Sreeram, V., and Liu, W. Q., 1997, *Proceedings of the 36th Conference on Decision and Control*, San Diego, CA, USA, pp. 3292–3297.

Diaz, H., and Desrochers, A. A., 1988, *Automatica*, **24**, 629–641.

Enns, D. F., 1984, *Proceedings of the 23rd IEEE Conference on Decision and Control*, Las Vegas, NV, USA, pp. 127–132.

Haber, R., and Unbehauen, H., 1990, *Automatica*, **26**, 651–677.

Ho, B. L., and Kalman, R. E., 1965, *Proceeding of the 3rd Allerton Conference on Circuit and System Theory*, Monticello, IL, USA, pp. 449–459.

Hsu, C. S., Desai, U. B., and Crawley, 1983, *Proceedings of the 22nd IEEE Conference on Decision and Control*, San Antonio, TX, USA, pp. 783–788.

6.7 Huang, X. X., Yan, W., and Teo, K. L., 2001, *IEEE Transactions on Automatic Control*, **46**, 1279–1284.

Isidori, A., 1973, *IEEE Transactions on Automatic Control*, **18**, 626–631.

Kung, S. Y., 1978, *Proceeding of the 12th Asilomar Conference on Circuits, Systems and Computers*, Pacific Grove, CA, USA, pp. 705–714.

Levin, A., and Narendra, K., 1995, *Neural Computation*, **7**, 349–357.

Mohler, R. R., 1991, *Nonlinear Systems*, Vol. II (New York: Prentice-Hall).

Moor, B. C., 1981, *IEEE Transactions on Automatic Control*, **26**, 17–31.

Muscato, G., 2000, *IEEE Transactions on Automatic Control*, **45**, 5
339–344.

Pernebo, L., and Silverman, L. M., 1982, *IEEE Transactions on Automatic Control*, **27**, 382–387.

Priestley, M. B., 1988, *Non-linear and Non-Stationary Time Series Analysis* (London: Academic Press).

Rugh, W. J., 1981, *Nonlinear System Theory* (New York: Johns 7
Hopkins University Press).

Serrani, A., Isidori, A., and Marconi, L., 2001, *IEEE Transactions on Automatic Control*, **AC 46**, 1178–1194.

Spanos, J. T., Milman, M. H., and Mingori, D. L., 1992, 8
Automatica, 897–909.

Sreeram, V., and Agathoklis, P., 1991, *International Journal of Control*, **53**, 129–144.

Wang, G., Sreeram, V., and Liu, W. Q., 1999, *IEEE Transactions on Automatic Control*, **AC 44**, 1734–1737.

Xiao, C., Sreeram, V., Liu, W. Q., and Venetsanopoulos, A. N., 1997, *Proceedings of the American Control Conference*, Albuquerque, NM, USA, pp. 3567–3571.

Zhang, L., and Lam, J., 2002, *Automatica*, **38**, 205–216.

Zhang, L., Lam, J., Huang, B., and Yang, G. H., 2003, *International Journal of Control*, **76**, 414–427.

AUTHOR QUERIES

JOURNAL ID: TSYS-101547

QUERY NUMBER	QUERY
1	Please check that the author name Unbehaven is listed as Unbehauen
2	Please check that Figures 1–4 are of poor quality
3	Please provide title
4	Please check that there is no part title in Journal references
5	Please check if the author name Muscato, G. is OK?
6	Please check if the initial W. for the author Yan is OK?
7	Please provide citation for the references Huang et al., 2001 and Rugh, 1981
8	Please provide volume number