MODEL REDUCTION SCHEME OF STATE-AFFINE SYSTEMS

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Abstract

In this paper, a new method for the approximation of discrete time state-affine systems is proposed. The method is based on the diagonalization of proposed generalized controllability and obeservability Gramians. The reduction algorithm employs singular value decomposition to retain states corresponding to dominant singular values of these Gramians. The proposed method can be considered as the generalization of the Moore's balancing reduction approach for linear systems. A numerical example is used to illustrate the effectiveness of the proposed method.

Keywords

Model reduction, balanced realization, state-affine systems.

1. INTRODUCTION

Modeling of physical systems often leads to higher order models. Analysis, simulation and design methods based on this higher order model may eventually lead to complicated structures requiring very complex logic or unreasonable amount of calculation. It is therefore desirable to approximate higher order models by lower ones.

The approximation of higher order complex systems to lower order models attracted the attention of many researchers during the past two decades [1 - 8]. Various model reduction schemes have been proposed in the literature. Early methods were concentrated on the retention of dominant poles in the reduced order model, as in aggregation methods, or the matching of several moments of the original systems, as in Pade approximation methods. However, recently one approach has dramatically changed the status of model reduction. This approach is the balanced realization. Balanced model reduction of linear dynamic systems proved to be a very efficient scheme for the approximation of large scale systems [1]. Meaningful motivations to the state space balanced representation and model reduction via balancing are detailed in [1, 2]. Essentially, strongly controllable and strongly observable states in a balanced representation are retained in the reduced order model as the dominant part of the original high order system. Also, the balanced reduction scheme proved to have several desirable properties [5].

The success of the application of balanced model reduction scheme to several practical systems motivated many researchers to generalize the balancing concept to more general dynamic systems. State space balanced representation and balanced model reduction of bilinear systems have been treated in [5 - 8]. Application to bilinear power systems gave good performance of the approximate system. A balanced reduction algorithm for homogeneous bilinear systems has been developed in [9] and proved to be equivalent to practically balanced linear interconnected sub-systems.

In this work, the balancing concept is defined for an important class of discrete time nonlinear systems, called polynomial affine systems. They are represented in state space form as:

$$x(k+1) = Ax(k) + \sum_{m=1}^{l} u^m(k) N_m x(k) + \sum_{m=1}^{l} u^m(k) B_m \quad (1.a)$$

$$y(k) = C\,x(k) \quad (1.b)$$

where $x(k)$ is an $n \times 1$ state vector, $u(k)$ is a scalar input, $y(k)$ is a scalar output, and $\{A, N_m, B_m, C\}$ are matrices of proper dimensions.

These models represent a major generalization to bilinear systems and represent several practical engineering systems , such as nuclear reactors, heat-transfer processes, and population models [10, 11]. They also occur as a byproduct to the discretization of continuous time bilinear systems [12]. A least square approximation of polynomial affine systems is developed in [12]. In this paper model reduction scheme for such systems is shown to be a simple extension of Moore's balancing for linear systems. The key problem is to develop the controllability and observability maps for such systems. The corresponding "Gramians" then verify a pair of generalized Lyapunov matrix equations.

2. NOTATIONS AND BASIC BACKGROUND MATERIAL

Reachablility and observability concepts of the class of systems in equation (1) have been studied by Tiejun and McCormick [10]. The work of Tiejun and McCormick is considered as a generalization to their counterpart in bilinear systems studied by Isidori [13], D'Alessandro *et al* [14], and Rugh [15]. Results concerning the above mentioned concepts are summarized as follows:

Definition 1. A state x of system (1) is said to be reachable from the origin of the state space if there exists an input signal that maps the origin of the state space into the state x in a finite interval of time.

Definition 2. System (1) is said to reachable if the set of reachable states spans $\Re^n$.

Definition 3. A state x_o of system (1) is said to be unobservable from the origin of the state space if the response $y(t)$ with $x(0) = x_o$ is identical to the response with $x(0) = 0$ for every input signal.

Theorem 1. The n-dimensional state-affine system (1) is reachable if and only if

$$\text{rank } P = n \qquad (2)$$

where, the $n \times [(l + 1)^n - 1]$ matrix P is defined recursively as follows:

$$P = [P_1, P_2, ..., P_n] \qquad (3.a)$$

$$P_1 = [B_1, B_2, ..., B_l] \qquad (3.b)$$

$$P_i = [AP_{i-1}, N_1 P_{i-1}, ..., N_l P_{i-1}], \qquad i \geq 2 \qquad (3.c)$$

Theorem 2. The n-dimensional state-affine system (1) is observable if and only if

$$\text{rank } Q = n \qquad (4)$$

where the matrix Q is defined as follows:

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{bmatrix}, \qquad Q_1 = C,$$

$$Q_i = \begin{bmatrix} Q_{i-1}A \\ Q_{i-1}N_1 \\ \vdots \\ Q_{i-1}N_l \end{bmatrix} \qquad i \geq 2 \qquad (5)$$

Theorem 3. The n-dimensional state-affine system (1) is minimal if and only if it is reachable and observable.

3. REACHABLITY AND OBSERVABLITY GRAMIANS

Similar to the linear and bilinear cases, the reachability Gramian W_r and the observability Gramian W_o are, respectively, defined as

$$W_r = P_\infty P_\infty^* \qquad (6.a)$$

$$W_o = Q_\infty^* Q_\infty \qquad (6.b)$$

Obviously, system (1) is minimal if and only if W_r and W_o are positive definite.

The Gramians W_r and W_o for system (1) can be computed from the following generalized Lyapunov equations.

Theorem 4. For a discrete state-affine r system of the form (1), W_r and W_o satisfy

$$AW_rA^T + \sum_{i=1}^{l} N_i W_r N_i^T - W_r = -\sum_{i=1}^{l} B_i B_i^T \qquad (7)$$

$$A^T W_o A + \sum_{i=1}^{l} N_i^T W_o N_i - W_o = -C^T C \qquad (8)$$

The proof of the theorem is given in the Appendix.

Clearly if $N_i = 0$, for $i = 1,2, \ldots, l$ then it can be easily observed that W_r and W_o are the reachability and observability Gramians of linear systems and equations (7) and (8) will reduced to the usual Lyapunov equations.

Now we will consider methods to solve the generalized Lyapunov equations (7) and (8). Solutions for equation (7) can be obtained by rewriting it in a Kronecker product linear matrix equation form [16]:

$$Gp = c \qquad (9)$$

Where

$$G = A \otimes I + N_1 \otimes N_1 + \ldots + N_l \otimes N_l \qquad (10)$$

and

$$p = vec\,(P) = [p_{11}, p_{21}, \ldots, p_{n1}, p_{12}, p_{22}, \ldots, p_{n2}, \ldots, p_{1n}, p_{2n}, \ldots, p_{nn}]^T$$

$$c = vec\,(-BB^T).$$

Equation (8) can be similarly solved.

4. Balancing and Model Reduction

In this section, an algorithm for reducing state-affine systems of the form (1) is developed. The algorithm is based on the concept of a balanced realization. In a balanced representation the controllability and observability Gramians, which represent the input-state and state-output maps of the system, respectively, are equal and diagonal. The diagonal entries of these Gramians, called the singular values, measure the degree of controllability and observability of the states. The most controllable and most observable states, corresponding to the largest ordered singular values, are retained in the reduced model. The order is suggested by the magnitudes of the singular values.

Once the controllability Gramian W_r and the observability Gramian W_o have been determined, the balanced realization of system (1) can be obtained by applying the state-space balancing transformation,

$$x_b(k) = T^{-1}x(k), \tag{11}$$

to equation (1). The state-space representation of the new system is,

$$x_b(k+1) = A_b x_b(k) + \sum_{m=1}^{l} u^m(k) N_{bm} x_b(k) + \sum_{m=1}^{l} u^m(k) B_{bm} \tag{12.a}$$

$$y(k) = C_b\, x_b(k) \tag{12.b}$$

where
$A_b = T^{-1}AT$, $N_{bm} = T^{-1}N_mT$, $B_{bm} = T^{-1}B_m$, and $C_b = CT$.

The controllability and observability Gramians of the new system are given by,

$$W_{rb} = T^{-1}W_rT^{-T} \tag{13}$$

$$W_{ob} = T^{T}\, W_o\, T \tag{14}$$

Moreover, these Gramians are equal and diagonal. Normally the Gramians of the balanced system have the following special arrangement:

$$W_{rb} = W_{ob} = \Sigma = \mathrm{diag}\,[\,\sigma_1, \sigma_2, ..., \sigma_n]$$
$$\sigma_1 \geq \sigma_2 \geq ... \sigma_1 > 0 \tag{15}$$

The σ_i called the Hankel singular values of the system, are determined by

$$\sigma_i = [\lambda_i(W_rW_o)]^{1/2} \tag{16}$$

where λ_i denotes the ith eigenvalue of W_rW_o.

An efficient algorithm for computation of a balanced representation for linear systems developed by Laub *et. al.* [17] is modified in this paper to compute a balanced representations of the nonlinear systems. The algorithm is summarized as follows,

i. Use eqs. (7 and 8) to find the controllability and observability Gramians.
ii. Compute Cholesky factors of Gramians: Let L_r and L_o denote the lower triangular Cholesky factors of W_r and W_o , that is ,

$$W_r = L_rL_r^T, \qquad W_o = L_oL_o^T \tag{17}$$

iii. Compute the singular value decomposition of the product of the Cholesky factors:

$$L_o^TL_r = U\Sigma V^T \tag{18}$$

iv. Form the balancing transformation

$$T = L_rV\Sigma^{-1/2} \tag{19}$$

v. Form the balanced state-space matrices

$$A_b = T^{-1}AT \tag{20}$$

$$N_{bm} = T^{-1}N_mT, \quad m = 1, 2, ..., l \tag{21}$$

$$B_{bm} = T^{-1}B_m, \quad m = 1, 2, ..., l \tag{22}$$

$$C_b = CT \tag{23}$$

To obtain a reduced order model, let the matrices given in equation (12) be partitioned as

$$\begin{bmatrix} x_{b1}(k+1) \\ x_{b2}(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_{b1}(k) \\ x_{b2}(k) \end{bmatrix} + \sum_{m=1}^{l} u^m(k)\begin{bmatrix} N_{11m} & N_{12m} \\ N_{21m} & N_{22m} \end{bmatrix}\begin{bmatrix} x_{b1}(k) \\ x_{b2}(k) \end{bmatrix} + \sum_{m=1}^{l} u^m(k)\begin{bmatrix} B_{1m} \\ B_{2m} \end{bmatrix} \tag{24.a}$$

$$y(k) = \begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} x_{b1}(k) \\ x_{b2}(k) \end{bmatrix} \tag{24.b}$$

where the vector $x_{b1} \in \Re^{r}$ contains the most controllable and observable states, and the vector $x_{b2} \in \Re^{n-r}$ contains the least controllable and observable states. Also, let $\boldsymbol{\Sigma}$ be partitioned similarly:

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \qquad (25)$$

where Σ_1 = diag [σ_1, ..., σ_r], and Σ_2 = diag [σ_{r+1}, ..., σ_n]. If σ_r / σ_{r+1} »1, then the subsystem given by

$$x_r(k+1) = A_{11}x_r(k) + \sum_{m=1}^{l} u^m(k)N_{11m}x_r(k) + \sum_{m=1}^{l} u^m(k)B_{1m} \qquad (26.a)$$

$$y_r(k) = C_1x_r(k) \qquad (26.b)$$

is the reduced order model of the full order balanced system which will contain only the most controllable and observable parts of the system.

5. EXAMPLE

The algorithm developed in the previous sections is applied to a seventh order system. This example is solely to illustrate the results and to evaluate the derived model reduction algorithm. The matrices *A, N_1, N_2, B_1, B_2,* and *C* of the model are as follow:

$$A = \begin{bmatrix} 0.077 & 0 & 0 & 0 & 0 & 0 & -0.044 \\ 0 & 0.149 & -0.043 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.467 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.005 & 0.152 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.391 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.360 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.413 \end{bmatrix}$$

$$N_1 = 10^{-1} \times \begin{bmatrix} 0.632 & 0.093 & 0.644 & 0.417 & 0.825 & 0.194 & 0.726 \\ 0.888 & 0.373 & 0.721 & 0.059 & 0.779 & 0.401 & 0.329 \\ 0.859 & 0.519 & 0.636 & 0.642 & 0.801 & 0.284 & 0.190 \\ 0.766 & 0.789 & 0.668 & 0.440 & 0.490 & 0.463 & 0.899 \\ 0.260 & 0.396 & 0.017 & 0.601 & 0.125 & 0.793 & 0.138 \\ 0.484 & 0.657 & 0.797 & 0.614 & 0.405 & 0.396 & 0.567 \\ 0.463 & 0.782 & 0.472 & 0.180 & 0.890 & 0.421 & 0.555 \end{bmatrix}$$

$$N_2 = 10^{-1} \times \begin{bmatrix} 0.001 & 0.599 & 0.672 & 0.834 & 0.372 & 0.721 & 0.869 \\ 0.001 & 0.181 & 0.554 & 0.343 & 0.248 & 0.120 & 0.226 \\ 0.681 & 0.736 & 0.188 & 0.237 & -0.881 & 0.350 & 0.206 \\ 0.640 & 0.624 & 0.188 & 0.609 & 0.010 & 0.529 & 0.089 \\ 0.281 & 0.729 & 0.071 & 0.250 & 0.169 & 0.156 & 0.193 \\ 0.368 & 0.083 & 0.342 & 0.684 & 0.865 & 0.729 & 0.559 \\ 0.601 & 0.072 & 0.838 & 0.690 & 0.215 & 0.139 & 0.612 \end{bmatrix}$$

$$B_1 = [0.795 \quad 0.696 \quad 0.753 \quad 0.670 \quad 0.633 \quad 0.0564 \quad 0.598]^T$$

$$B_2 = [0.223 \quad 0.319 \quad 0.700 \quad 0.117 \quad 0.763 \quad 0.526 \quad 0.554]^T$$

$$C = [0.588 \quad 0.330 \quad 0.703 \quad 0.143 \quad 0.162 \quad 0.485 \quad 0.08602]$$

Using the balancing algorithm we compute the following Hankel singular values of the system:

Σ = diag[3.6662, 0.1782, 0.0675, 0.0282, 0.0075, 0.0016, 0.0011].

From the above Hankel singular values, a third order as well as a second order reduced models were computed. The state space matrices of the third order model are,

$$A_{br} = \begin{bmatrix} 0.3208 & 0.1753 & 0.0153 \\ 0.1299 & 0.2071 & 0.0173 \\ -0.0264 & 0.0077 & 0.3732 \end{bmatrix}$$

$$N_{br1} = \begin{bmatrix} 0.3658 & -0.0648 & 0.1102 \\ -0.0409 & -0.0061 & 0.0027 \\ -0.0770 & -0.0198 & -0.1199 \end{bmatrix}$$

$$N_{br2} = \begin{bmatrix} 0.2593 & -0.1066 & -0.0357 \\ -0.0392 & -0.0284 & -0.0096 \\ -0.0674 & -0.0194 & 0.0176 \end{bmatrix}$$

$$B_{br1} = \begin{bmatrix} 1.2359 \\ -0.2273 \\ 0.1046 \end{bmatrix} \qquad B_{br2} = \begin{bmatrix} 1.0075 \\ 0.2115 \\ 0.0684 \end{bmatrix}$$

$$C_{br} = [1.5960 \quad -0.0146 \quad -0.0828].$$

Similarly, the matrices of the second order reduced model are found to be

$$A_{br} = \begin{bmatrix} 0.3208 & 0.1752 \\ 0.1299 & 0.2071 \end{bmatrix} \quad N_{br1} = \begin{bmatrix} 0.3658 & -0.0648 \\ -0.0409 & -0.0061 \end{bmatrix}$$

$$N_{br2} = \begin{bmatrix} 0.2593 & -0.1066 \\ -0.0392 & -0.0284 \end{bmatrix}$$

$$B_{br1} = \begin{bmatrix} 1.2359 \\ -0.2273 \end{bmatrix} \qquad B_{br2} = \begin{bmatrix} 1.0075 \\ 0.2115 \end{bmatrix}$$

$$C_{br} = [1.5960 \quad -0.0146].$$

The step response of the original 7^{th} order system, the third order reduced model, and the second order reduced model is shown in Figure 1. The third order model response is essentially superimposed on the original response and, therefore, represents a good approximation to the original system in both transient and steady state behavior. For the 2nd order reduction, the transient response is good approximations of the original response. There is, however, an offset in steady state. This offset is also present in the balanced model reduction of linear systems, as it is well known that the model reduction scheme based on balancing leads generally to good transient performance and may give poor low frequency approximation. One can introduce frequency weighting to improve the low frequency approximation [18].

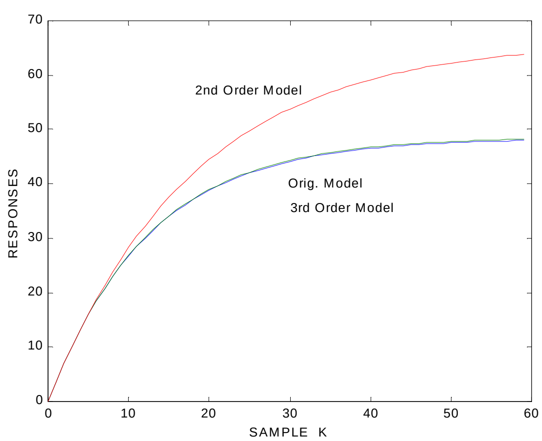

Figure 1. Comparison of the outputs for unit step inputs

6. CONCLUSIONS

A new model reduction method for discrete time state-affine systems has been proposed in this paper. The method is based on the diagonalization of proposed generalized controllability and obeservability Gramians. The generalized controllability and obeservability Gramians can be obtained from solving generalized Lyapunov equations. The reduction algorithm employs singular value decomposition to retain states corresponding to dominant singular values of these Gramians. The algorithm is illustrated through an example.

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APPENDIX

Proof of Theorem 4

From (6.a)

$$W_r = P_\infty P_\infty^* = [P_1 \quad P_2 \quad \cdot \quad \cdot \quad \cdot][P_1 \quad P_2 \quad \cdot \quad \cdot \quad \cdot]^*$$

$$= P_1 P_1^* + P_2 P_2^* + \cdots$$

Using equation 3 gives

$$W_r = B_1 B_1^T + B_2 B_2^T + \cdots + B_l B_l^T$$
$$+ [AP_1 \quad N_1 P_1 \quad \cdot \quad \cdot \quad N_l P_1][AP_1 \quad N_1 P_1 \quad \cdot \quad \cdot \quad N_l P_1]^T$$
$$+ [AP_2 \quad N_1 P_2 \quad \cdot \quad \cdot \quad N_l P_2][AP_2 \quad N_1 P_2 \quad \cdot \quad \cdot \quad N_l P_2]^T + \cdots$$
$$= \sum_{i=1}^{l} B_i B_i^T + AP_1 P_1^T A^T + \sum_{i=1}^{l} N_i P_1 P_1^T N_i^T +$$
$$AP_2 P_2^T A^T + \sum_{i=1}^{l} N_i P_2 P_2^T N_i^T \cdots$$

$$= \sum_{i=1}^{l} B_i B_i^T + A(P_1 P_1^T + P_2 P_2^T + \cdots)A^T +$$
$$N_1(P_1 P_1^T + P_2 P_2^T + \cdots)N_1^T +$$
$$N_2(P_1 P_1^T + P_2 P_2^T + \cdots)N_2^T + \cdots$$

$$= \sum_{i=1}^{l} B_i B_i^T + AW_r A^T + N_1 W_r N_1^T + N_2 W_r N_2^T + \cdots$$
$$= \sum_{i=1}^{l} B_i B_i^T + AW_r A^T + \sum_{i=1}^{l} N_i W_r N_i^T + \cdots$$

which is equation (7).
Proof of (8) is analogous to that of (7).

Q.E.D.