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A NEW MODEL REDUCTION SCHEME FOR K-POWER BILINEAR SYSTEMS

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ABSTRACT

A model reduction scheme of k-power bilinear systems is proposed in this work. The canonical state space structure of k-power systems is used to simplify a balancing like model reduction scheme for bilinear systems. The derived model reduction algorithm reduces to computational steps similar in complexity to the balanced approximation of linear systems. Controllability and observability gramians turn out to have simple block diagonal structures and their properties are easily derived. The simulation of an 11th order system shows good performances of the reduced order models.

I. INTRODUCTION:

No complete theory of analysis and design of general nonlinear systems is possible due to the lack of definite structural properties. However, a special class of nonlinear systems, bilinear systems, are relatively well studied [1,2].

A subclass of bilinear systems, have the property that the input-output map is homogeneous in the input $u(t)$ of degree k; i.e., the output $y(t)$ has the property

$$y(\alpha u(t)) = \alpha^k y(u(t)) \tag{1}$$

for all scalar α and admissible inputs, $u(t)$. A bilinear system satisfying (1) is called a homogeneous of degree k or a k-power bilinear system and we say that the input-output map is a k-power realizable as the zero state response of an internally bilinear system. Homogeneous systems arise in various areas of engineering. Systems consisting of multiplicative connections of linear subsystems are naturally represented by homogeneous bilinear models [1,3]. Also, successful modeling of hydraulic drives for machine tools and robots by homogeneous bilinear systems is reported in [4]. Homogeneous systems also arise naturally in representing polynomial systems. A polynomial system of degree N is described by a finite sum of homogeneous terms [1]. Homogeneous systems also arise naturally in nonlinear system identification where each homogeneous term is identified separately thus leading to a separate response for each term [5].

Homogeneous bilinear systems often lead to simple design structures extending results and concepts of linear systems. For example, in [6], nonlinear IIR Adaptive Filter using a homogeneous bilinear structure resulted in a cascade connection of linear filters and multipliers. Also, in [7], necessary and sufficient conditions for bounded input-bounded output stability are obtained generalizing linear systems stability criterion.

Motivated by the rich structural properties of homogeneous systems as compared to non-homogeneous systems and the closeness in structure of homogeneous bilinear systems to linear systems, the authors investigate in this paper the reduced order approximation of homogeneous bilinear systems.

Model Order reduction of bilinear systems via the balancing approach has been successfully applied, by the authors, to multi-areas electric power plants [8]. However, the balanced reduction algorithm required the computation of solutions of generalized Lyapunov equations which may be heavy and costly for large dynamic systems. Also, existence and uniqueness of solutions are not transparent.

The purpose of the present paper is two fold. First, it is shown that the balanced model reduction algorithm of homogeneous bilinear systems reduces, unlike the case of general bilinear systems [8], to solving standard linear Lyapunov matrix equations of lower dimensions. Efficient computational algorithms are available [9] to solve these equations. Second, as a consequence of the above it is also shown that, unlike the general bilinear case, existence and uniqueness conditions of solutions of the derived Lyapunov equations for homogeneous bilinear systems, follow trivially from the existing linear theory. These results show again the relative closeness of the structure of homogeneous bilinear systems to that of linear systems. Note that the reduced order model obtained will also be a k-power.

It is to be noted that, to the authors knowledge, the present work represents the first direct development of a model reduction scheme of homogeneous bilinear systems.

The paper is organized as follows: Section II includes mathematical preliminaries. In section III, we review k-power systems. In section IV, the reduced order algorithm of k-power systems is presented. A simulation example is given in section V to illustrate the results.

II. MATHEMATICAL PRELIMINARIES

Bilinear systems are those systems which are linear separately with respect to the state and the control, but not jointly. They can be characterized by the following state variable equations:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} N_i x(t) u_i(t) + Bu(t) \tag{2.a}$$

$$y(t) = Cx(t), \tag{2.b}$$

where $x(t)$ is an n × 1 state vector, $u(t)$ is an m × 1 input vector, u_i is the ith component of $u(t)$, $y(t)$ is an p × 1 output vector and $A, N_1, N_2, \ldots, N_m$, B, and C are real matrices of appropriate size.

It is well known that the input/output representation of system (2) is given by:

$$y(t) = \int_0^t Ce^{A(t-\tau_1)}Bu(\tau_1)d\tau_1 + \sum_{k=2}^{\infty} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} \sum_{j_1, j_2, \ldots, j_k = 1}^{m}$$

$$Ce^{A(t-\tau_1)}N_{j_1}e^{A(\tau_1-\tau_2)}N_{j_2} \ldots N_{j_{k-1}}e^{A(\tau_{k-1}-\tau_k)}b_{j_k}u_{j_1}(\tau_1)\ldots u_{j_k}(\tau_k)d\tau_1, \ldots d\tau_k \tag{3}$$

where b_i is the ith column of B .

Controllability and observability concepts of bilinear systems have been studied by several researchers [10,11]. The controllability and observability mappings $\overline{P}$ and $\overline{Q}$ are defined as follows:

Let

$$P_1(t) = e^{At}B,$$

$$P_i(t_1, \ldots, t_i) = \left[e^{At_i}N_1P_{i-1} \quad e^{At_i}N_2P_{i-1} \quad \ldots \quad e^{At_i}N_mP_{i-1}\right] \qquad i = 2, 3, \ldots \tag{4}$$

then

$$\overline{P} = [P_1 \quad P_2 \quad P_3 \quad \ldots] \tag{5}$$

Similarly, let

$$Q_1(t) = Ce^{At},$$

$$Q_i(t_1, \ldots, t_i) = \left[\left[Q_{i-1}N_1e^{At_i}\right]^T \quad \left[Q_{i-1}N_2e^{At_i}\right]^T \quad \ldots \quad \left[Q_{i-1}N_me^{At_i}\right]^T\right]^T \qquad i = 2, 3, \ldots \tag{6}$$

then

$$\overline{Q} = [Q_1^T \quad Q_2^T \quad Q_3^T \quad \ldots]^T \tag{7}$$

Using (4) and (5) the controllability gramian is defined as

$$P = \sum_{i=1}^{\infty} \int \ldots \int P_iP_i^* dt_1 \ldots dt_i, \tag{8}$$

Similarly the observability gramian is defined as

$$Q = \sum_{i=1}^{\infty} \int \ldots \int Q_i^*Q_i dt_1 \ldots dt_i \tag{9}$$

Theorem 1. For a bilinear system of the form (2), the gramians P and Q satisfy the following generalized algebraic Lyapunov equations:

$$AP + PA^T + \sum_{i=1}^{m} N_iPN_i^T + BB^T = 0, \tag{10}$$

and

$$A^TQ + QA + \sum_{i=1}^{m} N_i^TQN_i + C^TC = 0. \tag{11}$$

Proof: See the Appendix.

The single input single output version of equations (10) and (11) were first reported in [12] without proof. Clearly if $N_i = 0$, for i = 1,2, ...,m then equations (10) and (11) reduce to the normal Lyapunov equations of linear systems, as expected.

Solution for equation (10) can be obtained by rewriting it in a Kronecker product linear matrix equation form as:

$$Gp = c \tag{12}$$

where

$$G = (A \otimes I + I \otimes A + N_1 \otimes N_1 + \ldots + N_m \otimes N_m) \tag{13}$$

and

$$p = vec(P) = (p_{11}, p_{21}, \ldots, p_{n1}, p_{12}, p_{22}, \ldots, p_{n2}, \ldots, p_{1n}, \ldots, p_{nn})^T,$$

$$c = vec(-BB^T).$$

Similar treatment is done for equation (11).

III. *k*-POWER SYSTEMS

Let the bilinear system (2) be a *k*-power and minimal in the sense of D'Alessandro *et al* [10], then it can be transformed into a system with a special structure as in the following Theorem [13].

Theorem 2 . Let (2) be a minimal bilinear realization of a *k*-power. Then there exists a minimal bilinear realization of the form

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \dot{z}_k(t) \end{bmatrix} = \begin{bmatrix} A_1 & & & & \\ & A_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & A_k \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ \cdot \\ \cdot \\ \cdot \\ z_k(t) \end{bmatrix}$$

$$+ \left(\sum_{i=1}^{m} u_i(t) \begin{bmatrix} 0 & & & & \\ N_{1i} & 0 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & N_{k-1i} & 0 \end{bmatrix} \right) \begin{bmatrix} z_1(t) \\ z_2(t) \\ \cdot \\ \cdot \\ \cdot \\ z_k(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} u(t) \tag{14.a}$$

$$y(t) = [0 \quad 0 \quad . \quad . \quad . \quad 0 \quad C_k] \begin{bmatrix} z_1(t) \\ z_2(t) \\ \cdot \\ \cdot \\ \cdot \\ z_k(t) \end{bmatrix} \tag{14.b}$$

For the *k*-power system representation given by equation (14), the generalized Lyapunov equations (10) and (11) simplify to the following:

Theorem 3. Let the matrices $A, N_1, N_2, \ldots, N_m$, B, and C of *k*-power system be in the form (14.a) and (14.b) then the controllability and observability gramians of the *k*-power system are given by

$$P = diag[P_{11}, P_{22}, \ldots, P_{kk}] \tag{15}$$

$$Q = diag[Q_{11}, Q_{22}, \ldots, Q_{kk}] \tag{16}$$

where P_{ii} and Q_{ii} are solutions of the following Lyapunov equations:

$$A_1P_{11} + P_{11}A_1^T + B_1B_1^T = 0, \tag{17}$$

$$A_jP_{jj} + P_{jj}A_j^T + \sum_{i=1}^{m} N_{(j-1)i}P_{(j-1)(j-1)}N_{(j-1)i}^T = 0, \qquad j = 2, 3, \ldots, k \tag{18}$$

$$A_k^TQ_{kk} + Q_{kk}A_k + C_k^TC_k = 0, \tag{19}$$

$$A_j^TQ_{jj} + Q_{jj}A_j + \sum_{i=1}^{m} N_{ji}^TQ_{(j+1)(j+1)}N_{ji} = 0, \qquad j = k-1, k-2, \ldots, 2, 1 \tag{20}$$

Proof. In what follows the proof of equation (15) will be given. The proof of (16) is analogous to (15).
Using (14.a) substitute for $A, N_1, N_2, \ldots, N_m$, and B in (10). That will give the diagonal elements of P as in equations (17) and (18). The off diagonal elements of P are given by the following two sets:

$$A_1 P_{1j} + P_{1j} A_j^T = 0 \qquad j = 2, 3, \ldots, k \tag{21}$$

$$A_j P_{jl} + P_{jl} A_l^T + \sum_{i=1}^{m} N_{(j-1)i} P_{(j-1)(l-1)} N_{(l-1)i}^T = 0,$$
$$j = 2, 3, \ldots, k-1 \qquad l = j+1, j+2, \ldots, k \tag{22}$$

Moreover, $P_{1j} = 0$, for $j = 2, 3, \ldots, k$, is the unique solution of (21) if and only if the matrices A_1 and $-A_j$ have no eigenvalues in common [14]. The same argument is repeatedly used for equation (22) leading to all off diagonal elements equal to zero.

Remark 1: From the above theorem, the computation of the generalized Lyapunov equations for k-power systems reduces to solving the standard Lyapunov equations (17) and (19), then replacing these solutions into equations (18) and (20) respectively. These equations are again standard Lyapunov equations arising in linear systems.

Remark 2: It is shown in [7] that for a k-power BIBO system, every A_j $j = 1, 2, \ldots, k$, is asymptotically stable. Therefore [14] every $P_{jj} \geq 0$, and $Q_{jj} \geq 0$ for $j = 1, 2, \ldots, k$ and in turn $P \geq 0$, and $Q \geq 0$. Furthermore it can be shown from (17-20) that the obtained reduced model is asymptotically stable under simple linear controllability and observability conditions.

IV. *k*-POWER MODEL REDUCTION ALGORITHM

In this section, an algorithm for reducing k-power systems is developed. The algorithm is based on the concept of a balanced realization. Model reduction based on balanced realization has been extensively studied for linear systems [15-18]. In a balanced representation the controllability and observability gramians, which represent the input-state and state-output maps, respectively, of the system, are equal and diagonal. The diagonal entries of these gramians, called the singular values, measure the degree of controllability and observability of the states in this representation.

The most controllable and most observable states, corresponding to the largest ordered singular values, are retained in the reduced model. The order is suggested by the magnitudes of the singular values.

Once the controllability gramian P and the observability gramian Q have been determined, the balanced realization of system (14) can be obtained by applying the state-space balancing transformation

$$x_b(t) = T^{-1} x(t), \tag{23}$$

to equation (14). The state-space representation of the new system is:

$$\dot{x}_b(t) = A_b x_b(t) + \sum_{i=1}^{m} N_{bi} x_b(t) u_i(t) + B_b u(t),$$
$$y(t) = C_b x_b(t), \tag{24}$$

Where

$A_b = T^{-1}AT$, $N_{bi} = T^{-1}N_i T$, $B_b = T^{-1}B$, and $C_b = CT$.
The controllability and observability gramians of the new system are given by:

$$P_b = T^{-1} P T^{-T} \tag{25}$$

$$Q_b = T^T Q T \tag{26}$$

Moreover, these gramians are equal and diagonal. Normally the gramians of the balanced system has additionally the following special arrangement:

$$P_b = Q_b = \Sigma = diag[\sigma_1, \sigma_2, \ldots, \sigma_n]$$
$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0 \tag{27}$$

The σ_i, called the Hankel singular values of the system, are determined by

$$\sigma_i = (\lambda_i(PQ))^{\frac{1}{2}} \tag{28}$$

where $\lambda_i(PQ)$ denotes the ith eigenvalue of PQ.

However, in order for the balanced system to have the same structure as the k-power system (14), the Hankel singular values of the balanced system are arranged in descending order for the subsystems as follows:

$$P_b = Q_b = \Sigma = diag[\Sigma_1, \Sigma_2, \ldots, \Sigma_k] \tag{29}$$

where

$$\Sigma_j = diag[\sigma_{j1}, \sigma_{j2}, \ldots, \sigma_{j\nu}] \qquad j = 1, 2, \ldots, k \tag{30}$$
$$\sigma_{j1} \geq \sigma_{j2} \geq \ldots \geq \sigma_{j\nu} > 0$$

ν is the dimension of jth subsystem.

An efficient algorithm for the computation of a balanced representation for linear systems developed by Laub *et. al.* [9] will be modified in this paper to compute a balanced k-power system. The algorithm is summarized as follows:

i. Use Eqs. (17 -20) to find the controllability and observability gramians of the subsystems.
ii. Compute Cholesky factors of P_{jj} and Q_{jj}:
Let L_{rj} and L_{oj} denote the lower triangular Cholesky factors of P_{jj} and Q_{jj}, i.e.,
$$P_{jj} = L_{rj} L_{rj}^T, \quad Q_{jj} = L_{oj} L_{oj}^T. \tag{31}$$
iii. Compute the singular value decomposition of the product of the Cholesky factors:
$$L_{oj}^T L_{rj} = U_j \Sigma_j V_j^T. \tag{32}$$
iv. Form the balancing transformation for the subsystems
$$T_j = L_{rj} V_j \Sigma_j^{-1/2}. \tag{33}$$
v. Form the balancing transformation
$$T = diag[T_1, T_2, \ldots, T_k] \tag{34}$$
vi. Form the balanced state-space matrices
$$A_b = T^{-1}AT, \qquad N_{bi} = T^{-1}N_i T, \qquad i=1, 2, \ldots, m \tag{35}$$
$$B_b = T^{-1}B \qquad C_b = CT. \tag{36}$$

It is a simple matter to show that the balanced system (24) has the same structure as the system given in (14) with

$$A_{bj} = T_j^{-1} A_j T_j, \qquad B_{b1} = T_1^{-1} B_1, \qquad C_{bk} = C_k T_k, \qquad j = 1, 2, \ldots, k$$
$$N_{bji} = T_{j+1}^{-1} N_{ji} T_j \qquad j = 1, 2, \ldots, k-1 \qquad i = 1, 2, \ldots, m \tag{37}$$

To obtain a reduced order model, let the matrices given in equation (37) be partitioned as

$$A_{bj} = \begin{bmatrix} A_{bj11} & A_{bj12} \\ A_{bj21} & A_{bj22} \end{bmatrix} \qquad N_{bji} = \begin{bmatrix} N_{bji11} & N_{bji12} \\ N_{bji21} & N_{bji22} \end{bmatrix}$$

$$B_{bl} = [B_{bl1}^T \quad B_{bl2}^T]^T \qquad C_{bk} = [C_{bk11} \quad C_{bk12}]$$

Also, let Σ_j be partitioned in a similar way:

$$\Sigma_j = \begin{bmatrix} \Sigma_{j1} & 0 \\ 0 & \Sigma_{j2} \end{bmatrix},$$

where $\Sigma_{j1} = \mathrm{diag}[\sigma_{j1}, \ldots, \sigma_{jr}]$ and $\Sigma_{j2} = \mathrm{diag}[\sigma_{j(r+1)}, \ldots, \sigma_{jv}]$. If $\sigma_{jr}/\sigma_{j(r+1)} \gg 1$ for $j = 1, 2, \ldots, k$, then the reduced order model of the full order model is given by

$$\dot{x}_{br}(t) = A_{br}x_{br}(t) + \sum_{i=1}^{m} N_{bri}x_{br}u_i(t) + B_{br}u(t)$$

$$\hat{y}(t) = C_{br}x_{br}(t)$$

where

$A_{br} = diag\,[A_{b111}, A_{b211}, \ldots, A_{k11}]$

$$N_{bri} = \begin{bmatrix} 0 & & & & \\ N_{b1i11} & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & N_{b(k-1)i11} & 0 \end{bmatrix} \qquad B_{br} = \begin{bmatrix} B_{b11} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$C_{br} = [0 \quad 0 \quad . \quad . \quad . \quad 0 \quad C_{bk11}]$$

V. EXAMPLE

In this section, the preceding algorithm will be applied to an eleventh order 2-power system with two inputs and two outputs. This example is to solely illustrate the results and interpretion of the derived model reduction algorithm. The matrices A, N_1, N_2, B, and C of the model are as follows:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \qquad N_1 = \begin{bmatrix} 0 & 0 \\ N_{11} & 0 \end{bmatrix}$$

$$N_2 = \begin{bmatrix} 0 & 0 \\ N_{12} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \qquad C = [0 \quad C_2]$$

where

$A_1 = diag\,[-0.462, -0.951, -0.633, -0.439, -0.825, -0.689]$,

$A_2 = diag\,[-0.7022, -0.987, -0.954, -0.851, -0.289]$,

$$N_{11} = \begin{bmatrix} 0.177 & 0.234 & 0.795 & 0.056 & 0.117 & 0.330 \\ 0.828 & 0.102 & 0.696 & 0.598 & 0.763 & 0.703 \\ 0.158 & 0.219 & 0.753 & 0.227 & 0.526 & 0.143 \\ 0.988 & 0.635 & 0.670 & 0.319 & 0.554 & 0.162 \\ 0.257 & 0.696 & 0.633 & 0.700 & 0.588 & 0.485 \end{bmatrix}$$

$$N_{12} = \begin{bmatrix} 0.860 & 0.135 & 0.431 & 0.476 & 0.2539 & 0.721 \\ 0.813 & 0.529 & 0.259 & 0.388 & 0.672 & 0.945 \\ 0.557 & 0.311 & 0.370 & 0.279 & 0.676 & 0.461 \\ 0.739 & 0.588 & 0.393 & 0.0783 & 0.514 & 0.940 \\ 0.316 & 0.518 & 0.447 & 0.370 & 0.729 & 0.322 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.214 & 0.269 \\ 0.081 & 0.692 \\ 0.389 & 0.284 \\ 0.952 & 0.777 \\ 0.948 & 0.784 \\ 0.390 & 0.422 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.282 & 0.011 & 0.983 & 0.820 & 0.398 \\ 0.194 & 0.192 & 0.244 & 0.136 & 0.601 \end{bmatrix}$$

The singular values of the 11th order 2-power system are:

$\Sigma = diag\,[5.4152, 0.1528, 0.0253, 0.0143, 0.0020, 0.0006, 5.8521, 0.6788, 0.0176, 0.0063, 0.0016]$

The fourth order reduced model corresponding to states x_1, x_2, x_7, x_8 is computed, and its state space matrices are found as:

$$A_{br} = \begin{bmatrix} -0.637 & -0.191 & 0.000 & 0.000 \\ -0.170 & -0.659 & 0.000 & 0.000 \\ 0.000 & 0.000 & -0.538 & 0.243 \\ 0.000 & 0.000 & 0.350 & -0.630 \end{bmatrix}$$

$$N_{br1} = \begin{bmatrix} 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \\ 0.769 & 0.650 & 0.000 & 0.000 \\ -0.189 & -0.086 & 0.000 & 0.000 \end{bmatrix}$$

$$N_{br2} = \begin{bmatrix} 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \\ 0.755 & 0.164 & 0.000 & 0.000 \\ -0.349 & -0.128 & 0.000 & 0.000 \end{bmatrix}$$

$$B_{br} = \begin{bmatrix} 1.819 & 1.895 \\ 0.058 & 0.445 \\ 0.000 & 0.000 \\ 0.000 & 0.000 \end{bmatrix} \qquad C_{br} = \begin{bmatrix} 0.000 & 0.000 & 2.024 & -0.916 \\ 0.000 & 0.000 & 1.481 & 0.130 \end{bmatrix}$$

Similarly a second order reduced model corresponding to states x_1, x_7 is chosen. The matrices of the second reduced order model are found to be

$$A_{br} = \begin{bmatrix} -0.637 & 0.000 \\ 0.000 & -0.538 \end{bmatrix}, \qquad N_{br1} = \begin{bmatrix} 0.000 & 0.000 \\ 0.769 & 0.000 \end{bmatrix},$$

$$N_{br2} = \begin{bmatrix} 0.000 & 0.000 \\ 0.756 & 0.000 \end{bmatrix}, \qquad B_{br} = \begin{bmatrix} 1.819 & 1.895 \\ 0.000 & 0.000 \end{bmatrix},$$

$$C_{br} = \begin{bmatrix} 0.000 & 2.024 \\ 0.000 & 1.481 \end{bmatrix}$$

Responses y_1 and y_2 are shown in figures 1 and 2 for input u_1 and in figures 3 and 4 for input u_2, for the original 11th order 2-power system, the fourth order reduced model and the 2nd order reduced model. The fourth order model responses are essentially superimposed on the original responses and therefore represent a good approximation to the original system in both transient and steady state behavior. For the 2nd order reduction, the transient responses are good approximations of the original responses. There is, however, an offset in steady state particularly for the second output y_2. This offset is also present in the balanced model reduction of linear systems, as it is well known that the model reduction scheme based on balancing leads generally to good transient performance and may give poor low frequency approximation. One can introduce frequency weighting to improve the low frequency approximation [19].

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APPENDIX
PROOF OF THEOREM 1

In what follows the proof of equation (10) will be given. The proof of (11) is analogues to (10).
Let

$$\overline{P}_1(t) = \int e^{At_1} BB^T e^{A^T t_1} dt_1,$$

$$\overline{P}_i = \int \sum_{j=1}^{m} e^{At_i} N_j P_{i-1} N_j^T e^{At_i} dt_i \qquad i = 2, 3, \ldots$$

To show that the P in (8) is a solution of (10). Consider

$$AP + PA^T = A\left[\sum_{i=1}^{\infty} \int \cdots \int P_i P_i^* dt_1 \ldots dt_i\right] + \left[\sum_{i=1}^{\infty} \int \cdots \int P_i P_i^* dt_1 \ldots dt_i\right] A^T$$

$$= A\left[\int e^{At_1} BB^T e^{A^T t_1} dt_1 + \sum_{k=2}^{\infty} \int \sum_{j=1}^{m} e^{At_k} N_j \overline{P}_{k-1} N_j^T e^{A^T t_k} dt_k\right]$$

$$+ \left[\int e^{At_1} BB^T e^{A^T t_1} dt_1 + \sum_{k=2}^{\infty} \int \sum_{j=1}^{m} e^{At_k} N_j \overline{P}_{k-1} N_j^T e^{A^T t_k} dt_k\right] A^T$$

$$= A \int e_1^{At} BB^T e^{A^T t_1} dt_1 + \int e^{At_1} BB^T e^{A^T t_1} dt_1 A^T + A\left[\sum_{k=2}^{\infty} \int \sum_{j=1}^{m} e^{At_k} N_j \overline{P}_{k-1} N_j^T e^{A^T t_k} dt_k\right]$$

$$+ \left[\sum_{k=2}^{\infty} \int \sum_{j=1}^{m} e^{At_k} N_j \overline{P}_{k-1} N_j^T e^{A^T t_k} dt_k\right] A^T$$

$$= \int \frac{d}{dt_1}\left(e^{At_1} BB^T e^{A^T t_1} dt_1 A\right) + \sum_{k=2}^{\infty} \sum_{j=1}^{m} \int \frac{d}{dt_k}\left(e^{At_k} N_j \overline{P}_{k-1} N_j^T e^{A^T t_k}\right) dt_k$$

Now by integrating between 0 and ∞ and by assuming that A is stable then we have

$$AP + PA^T = -BB^T - \sum_{k=2}^{\infty} \sum_{j=1}^{m} N_j \overline{P}_{k-1} N_j^T$$

$$= -BB^T - \sum_{j=1}^{m} N_j \left(\sum_{k=2}^{\infty} \overline{P}_{k-1}\right) N_j^T$$

$$= -BB^T - \sum_{j=1}^{m} N_j P N_j^T$$

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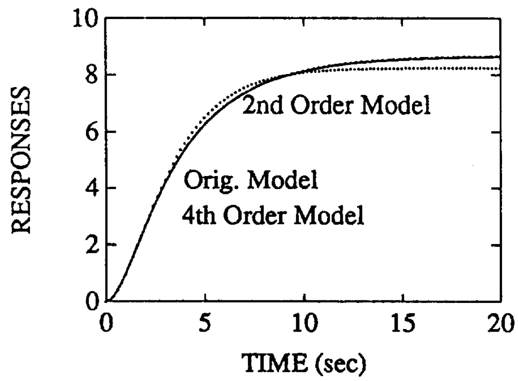

Figure 1. Step response y_1 to input u_1

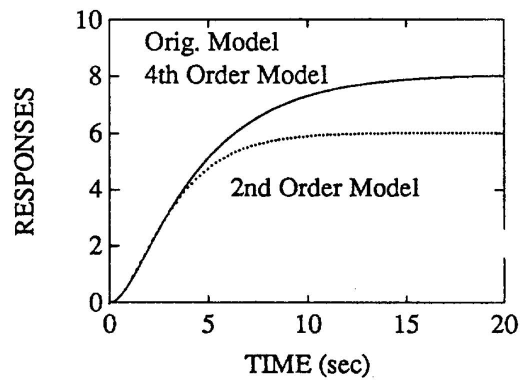

Figure 2. Step response y_2 to input u_1

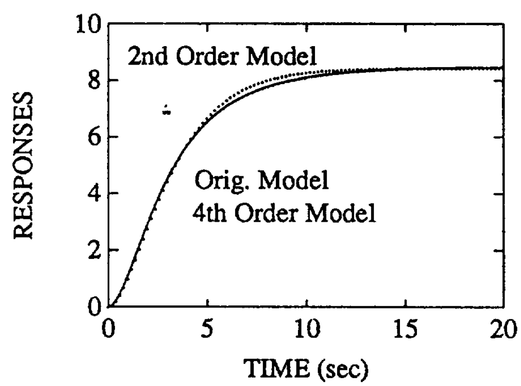

Figure 3. Step response y_1 to input u_2

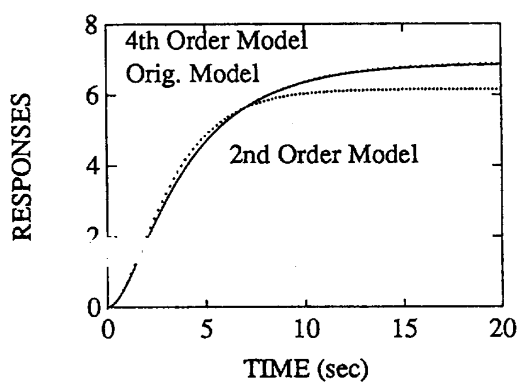

Figure 4. Step response y_2 to input u_2