# An Algorithm to Solve the Inverse Kinematics Problem of a Robotic Manipulator Based on Rotation Vectors

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*ABSTRACT* — **This paper presents an algorithm to solve the inverse kinematics problem for a complex wrist structure six degree of freedom (6DOF) robotic manipulator. The last three rotating axes do not intersect at one point and there are off axes in its coordinate frames. The proposed algorithm based on the rotation vector concept, which is also used to describe the orientation of manipulator end-effector. All the possible solutions of the inverse kinematics problem can be obtained by using the proposed algorithm which is tested practically on the MA2000 robotic manipulator.**

*INDEX TERMS* — ***Kinematics, Manipulator, Vector Analysis.***

## I. Rotation Vectors

The relative orientation representation by the rotation vector are based on the Euler theorem which states that (a displacement of a rigid body with one fixed point can be described as a rotation about some axis). The rotation vector is a vector pointing along this axis, and its magnitude contains information about the rotation angle. Consequently, the rotation vector is only common vector of any two coordinate systems (frames) having the same origin and differing in their orientation. This feature is applied to perform vector transformation between two frames. The direction of the rotation vector defines the axis around which one coordinate system has to be rotated to achieve the same orientation as the other system, while its magnitude defines the amount of the rotation [1]. The rotation vector concepts supposes the existence of a common axis around which one coordinate system has to be rotated to achieve the same orientation as another coordinate system. A vector along this common axis, whose magnitude is defined by the amount of the rotation around this axis, is called the rotation vector R which is given by [1]-[3]:

$$R = u * \tan(\frac{\theta}{2}) = u * f = [R_x, R_y, R_z]^T \quad (1)$$

where u: The unit vector along axis of rotation.
θ: Angle of rotation about the u vector.
f : Magnitude of the rotation vector, f=tan(θ/2).

A vector v undergoing a rotation R to perform a vector $v_b$ can be described as [1],[ 3]:

$$v_b = v + \frac{2R \times (v + R \times v)}{1 + F^2} \quad (2)$$

where R: The rotation vector.
v: The original vector.
$v_b$: The rotated vector (the old vector in the new coordinate frame).
×: Cross product operator.

$$F^2 = \sqrt{R_x^2 + R_y^2 + R_z^2} \quad (3)$$

*Case study 1:* Consider a robotic manipulator arm that is shown in Fig. 1. The gripper orientation depends on the robot joints variables $\theta_1$ and $\theta_2$. The orientation of the gripper vector changes due to the motion of the robotic manipulator. Suppose the robot arm initially at X axis and the gripper vector coordinate are initially $v=[0,2,0]^T$ corresponding to $\theta_1=\theta_2=0$, while the final position of $\theta_1$ and $\theta_2$ are $90^o$ and $60^o$ respectively. Find the final rotated gripper vector?

The system has two rotation vectors $R_1$ and $R_2$ corresponding to the angles of rotation $\theta_1$ and $\theta_2$.

The first rotation $R_1$ is performed around the Z axis and its direction is clockwise. The rotated gripper vector and the rotated rotating unit vector of the second rotating axis due to the first rotation $R_1$ must be computed as follows:

$$u_1 = [0,0,1]^T \quad (4)$$

where $u_1$ is the rotating unit vector of the first rotation operator.

$$R_1 = [0,0,\tan(\frac{\theta_1}{2})]^T = [0,0,1]^T \quad (5)$$

$$v_b = v + \frac{2 * R_1 \times (v + R_1 \times v)}{1 + F_b^2} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} \quad (6)$$

$$F_b^2 = R_{1x}^2 + R_{1y}^2 + R_{1z}^2 = 0 + 0 + 1 = 1 \quad (7)$$

The rotated rotating axis of the second joint due to the rotation about the first joint is computed by:

$$u_{2b} = u_2 + \frac{2 * R_1 \times (u_2 + R_1 \times u_2)}{1 + F_b^2} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad (8)$$

The second rotation is done around the shifted Y axis and its direction is anticlockwise so that a negative sign appears on the $R_2$.

$$u_2 = [0,1,0]^T \quad (9)$$

where $u_2$ is the rotating unit vector of the second rotation operator.

$$R_2 = [0, -\tan(\frac{\theta_2}{2}), 0]^T = [0, \frac{-1}{\sqrt{3}}, 0]^T \qquad (10)$$

$$R_{2b} = u_{2b} * (-f_2) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} * (-\tan\frac{60}{2}) = [\frac{1}{\sqrt{3}}, 0, 0]^T \qquad (11)$$

$$v_{bb} = vb + \frac{2 * R_{2b} \times (vb + R_{2b} \times vb)}{1 + F_{bb}^2} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} \qquad (12)$$

The results shows that the gripper vector which coincides originally with the Y axis, point now to the negative X direction.

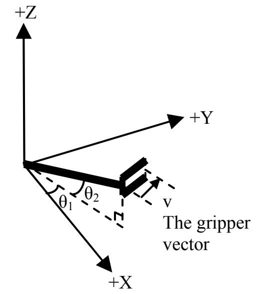


**Fig**. 1. Coordinate of revolute robot with the gripper vector.

## II. The Proposed Algorithm to Solve The Inverse Kinematics

The proposed algorithm divides the main task (solution the inverse kinematics problem) to subtasks to reduce the complexity of the solution of the inverse kinematics of the robotic manipulator.

The first step in the proposed algorithm is to define a set of a Cartesian coordinate frames at the robotic manipulator. Among these coordinate frames there is a one frame is called the global coordinate frame of the system, this frame is fixed at a stationary point to be as a reference coordinate frame for the other frames which are called the local frames. At each joint in the robotic manipulator there is a local coordinate frame. The origin of this frame is fixed at the central point of that joints and the orientation of this frame is the same as the orientation of the global frame (coincidence).

The rotating axis of each joint in the manipulator driven by an actuator is called the rotating axis of that joint of the manipulator. For each rotating axis there is a unit vector u along the axis of rotation of that joint which can be defined as $u = [u_x, u_y, u_z]^T$, where $u_x$, $u_y$ and $u_z$ are the unit vectors of the rotation in the local coordinate frame of that joint which has the same orientation of the global coordinate frame.

The position of the end-effector (gripper) of the robotic manipulator can be defined as: $p = [p_x, p_y, p_z]^T$, where the $p_x$, $p_y$ and $p_z$ are the coordinates of the point p in the global XYZ coordinate frame.

The orientation of the end-effector of the robotic manipulator can be described by defining the gripper vectors (components of the gripper) of the manipulator. The number of the gripper vectors is equal to the rotating axes in the gripper structure. Each gripper vector is defined with respect to the local coordinate frame of its joint. There are two methods to describe the orientation of the gripper in this algorithm. In the first method the description of the gripper is achieved by direct description of the desired gripper vectors, so that it is called direct orientation description. The indirect orientation description is the second method of orientation description. In this method the desired orientation of the gripper is described by defining the rotation angles about each rotating axis in the gripper structure starting from the actual (present) orientation of the gripper to the new orientation.

*Case study 2:* Suppose the end-effector of a robotic manipulator with three joints as illustrated in Fig. 2 The first local coordinate frame is fixed at point (A) with the same orientation of the global coordinate system. The first rotation unit vector $u_1$ can be defined as $u_1 = [1,0,0]^T$ because the rotation of this joint at this orientation is done about the $X_1$ axis. The first gripper vector will be $v_1 = [0,0,a]^T$ where (a) is the length of the first link in the gripper structure. The second local coordinate frame is fixed at point (B). The second rotation unit vector $u_2$ is given by $u_2=[1,0,0]^T$; while the second gripper vector is $v_2=[0,0,b]^T$ where (b) is the length of the second link in the gripper structure. The third local coordinate is fixed at point (C) which gives the third rotation unit vector $u_3=[0,0,1]^T$, and the third gripper vector $v_3=[0,d,d]^T$ or $v_3=[0,-d,d]^T$ ($v_3$ is even symmetrical about the third rotating axis $u_3$).

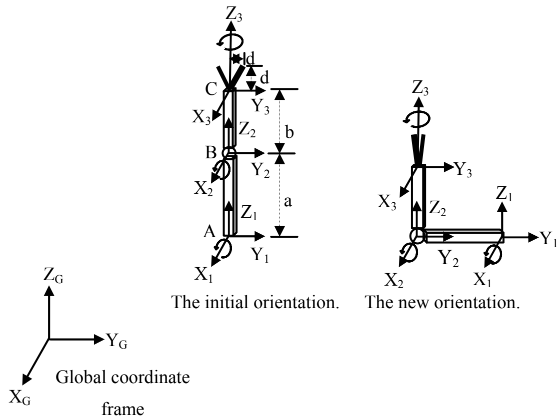


**Fig. 2.** The gripper with old and new orientations.

By using the first method of the orientation description the new gripper orientation can be described by defining the new (desired) gripper vectors $v_1$, $v_2$, and $v_3$. If it is assumed the new gripper vectors are given by $v_1=[0,-$

$a,0]^T$, $v_2=[0,0,b]^T$ ,and $v_3=[d,0,d]^T$, then the new position of the first link in the gripper structure lies at the negative part of the $Y_1$ axis, while the second link lies at the positive part of the $Z_2$ axis and the third link lies in the $X_3Z_3$ plane at point x=±d and z=d.

The description of the new gripper orientation using the second method of the orientation description can be done by defining the rotation angles about each one of the three rotating axes in the gripper structure. Thus the new orientation can be obtained from the first orientation by rotating by an angle $+\pi/2$ around the first rotating axis which coincident to $X_1$ axis, then rotating by an angle $-\pi/2$ about the second rotating axis which coincident to $X_2$ axis and finally rotating by an angle $\pm\pi/2$ about the third rotating axis which coincident to $Z_3$ axis. So that the rotation angles are $\varphi_1=+\pi/2$, $\varphi_2=-\pi/2$, and $\varphi_3=\pm\pi/2$ where $\varphi_1$, $\varphi_2$ and $\varphi_3$ are the rotation angles of the first, second and third joints in the gripper.

## III. Computation Of The Desired Griper Vector

When the indirect orientation description of the gripper vector is used to define the orientation of the desired gripper vector, it is important to describe the gripper orientation in the direct orientation description. There are two main steps to compute the desired gripper vectors $v_d$ in the direct form from the indirect form. The first step computes the actual (present) gripper orientation and actual (present) gripper rotating unit vectors. The second step represents the computation of the desired gripper vector by applying the rotation operator about each rotating axis in the actual (present) gripper orientation.

As starting point to compute the actual orientation of the gripper vector, the reset position and orientation of the manipulator must be defined. It is recommended to make the reset position and orientation of the manipulator at the manipulator structure with the all joints variables are set to zero. According to the manipulator reset position the reset values of the components of the gripper vector and the rotating unit vectors of the gripper rotating axes must be defined.

Now, by using the present values of the manipulator joints variables, the $i^{th}$ actual (present) gripper rotating unit vector $u_{ai}$ can be computed by rotating its value at the reset position $u_{0i}$ about all the previous rotating axes in the manipulator by using the rotation vectors concept. Symbolically it can be written as:

$$u_{ia} = (....(((u_{i0})_{R_{k+i-1}})_{R_{k+i-2}})_{R_{k+i-3}}....)_{R_1} \quad (13)$$

where i=1,2,.....I(I: total number of gripper rotating axes).

k: Number of rotating axes in the arm structure.

$u_{ia}$: The $i^{th}$ actual (present) gripper rotating unit vector.

$u_{i0}$: The $i^{th}$ gripper rotating unit vector at the manipulator reset position.

$(u)_{R_j}$ : Rotate the vector u about the $j^{th}$ rotating axis of the manipulator.

$$(u)_{R_j} = u + \frac{2R_j \times (u + R_j \times u)}{1 + F_j^2} \quad (14)$$

$$R_j = [R_{jx}, R_{jy}, R_{jz}] = u_{0j} \tan(\frac{\theta_j}{2}) = u_{0j} f_j \quad (15)$$

$$F_j^2 = R_{jx}^2 + R_{jy}^2 + R_{jz}^2 \quad (16)$$

The $i^{th}$ actual gripper vector $v_{ia}$ can be computed in the same manner but the rotation of the $i^{th}$ gripper vector at the reset position $v_{i0}$ is perform starting from the $i^{th}$ rotating axis in the gripper ending at the first rotating axis in the arm structure. Symbolically it can be written as:

$$v_{ia} = (....(((v_{i0})_{R_{k+i}})_{R_{k+i-1}})_{R_{k+i-2}}....)_{R_1} \quad (17)$$

where $v_{ia}$: The $i^{th}$ actual (present) gripper vector.

$v_{i0}$: The $i^{th}$ gripper vector at the manipulator reset position.

$(v)_{R_j}$ : Rotate the gripper vector v about the $j^{th}$ rotating axis of the manipulator.

$$(v)_{R_j} = v + \frac{2R_j \times (v + R_j \times v)}{1 + F_j^2} \quad (18)$$

$$R_j = [R_{jx}, R_{jy}, R_{jz}] = u_{0j} \tan(\frac{\theta_j}{2}) = u_{0j} f_j \quad (19)$$

$$F_j^2 = R_{jx}^2 + R_{jy}^2 + R_{jz}^2 \quad (20)$$

The first desired gripper vector can be computed by rotating the first actual gripper vector about the first rotating unit vector by the value of $\varphi_1$.

$$v_{1d} = v_{1a} + \frac{2R_1 \times (v_{1a} + R_1 \times v_{1a})}{1 + F_1^2} \quad (21)$$

$$R_1 = [R_{1x}, R_{1y}, R_{1z}] = u_{1a} \tan(\frac{\phi_1}{2}) \quad (22)$$

$$F_1^2 = R_{1x}^2 + R_{1y}^2 + R_{1z}^2 \quad (23)$$

The second desired gripper vector is computed by rotating the rotated second actual gripper vector $v_{2r}$ about the rotated second rotating unit vector $u_{2r}$ by the value of $\varphi_2$. The rotated second actual gripper vector $v_{2r}$ and the rotated second rotating gripper unit vector $u_{2r}$ are the results of applying the rotation operator on the second actual gripper vector $v_{2a}$ and the second actual rotating unit vector $u_{2a}$ about the first actual rotating unit vector $u_{a1}$ by the value of $\varphi_1$ respectively.

$$v_{2r} = v_{2a} + \frac{2R_1 \times (v_{2a} + R_1 \times v_{2a})}{1 + F_1^2} \quad (24)$$

$$u_{2r} = u_{2a} + \frac{2R_1 \times (u_{2a} + R_1 \times u_{2a})}{1+F_1^2} \quad (25)$$

$$v_{2d} = v_{2r} + \frac{2R_2 \times (v_{2r} + R_2 \times v_{2r})}{1+F_2^2} \quad (26)$$

$$R_2 = [R_{2x}, R_{2y}, R_{2z}] = u_{2r} \tan(\frac{\phi_2}{2}) \quad (27)$$

$$F_2^2 = R_{2x}^2 + R_{2y}^2 + R_{2z}^2 \quad (28)$$

The same procedure is applied to compute the other components of the gripper vector.

## IV. General Steps Of The Proposed Algorithm To Solve The Inverse Kinematics Problem

The solution of the inverse kinematics problem can be found by the following steps:

1. The first step, a Cartesian (rectangular right handed) coordinate system $X_0Y_0Z_0$ is assigned to the base of the manipulator. This coordinate represents the global coordinate of the manipulator system.
2. The robotic manipulator structure will be divided into two main parts the first part (arm) includes the major three axes and it extends to the rotating axis of the fourth link at point $o_b$. This point ($o_b$) represents the end point of the major three links with the off axis (if it exists) due to the fourth link structure. While the second part (wrist) includes all the remainder of the robotic manipulator structure (from the virtual point $o_b$ to the end of the end-effector).
3. Derivation (by using any mathematical method) the possible sets of the joints variables (angles) of the arm structure that attain the end of arm (point $o_b$) to a certain position (that will be computed later in step 8). Note that this mathematical calculation is simple because it treats only three links and it represents a translation operation so that the orientation of the end point of arm ($o_b$) is not important when the end point of the arm reaches the desired position.
4. For each of the three major rotating axes in the arm structure, the rotation vector with respect to the base (global) coordinate frame must be defined.
5. For the wrist structure, there are a Cartesian coordinate system at the beginning of each rotating axis in the wrist structure with the same orientation of the global coordinate system. These coordinate frames are used to define the rotating unit vector of each rotating axis in the gripper structure. Also there are another Cartesian coordinate frames with the same orientation of the global coordinate frame at the beginning of each component of the gripper vector. These coordinate frames are used to describe the gripper vectors.
6. Computation the rotation vectors of the gripper structure for each rotating axis. Also the initial gripper vectors and the initial gripper rotating unit vectors at the reset position of the manipulator must be defined.
7. Tf the indirect method is used to describe the gripper orientation, the new (desired) griper vectors $v_{id}$ must computed as described in the previous section. Also the total desired gripper vector $v_d$ must be computed as:

$$v_d = \sum_{i=1}^{i=n} v_{id} \quad (29)$$

where n is the number of gripper vectors.

8. Determination (by using the equations which were derived in step 3) the required position of point $o_b$ from the desired gripper vector $v_d$ (which is computed in step 7) and the desired position of the gripper p as:

$$o_b = p - v_d \quad (30)$$

9. Determination the sets of the possible solutions of the joints variables of arm structure of the robotic manipulator that is required to attain the point $o_b$ to the new position (which is computed in step 8).
10. Repetition of steps 11, 12, 13, and 14 for each set of solutions of the arm (first part) joints variables that are computed in step 9.
11. Repetition steps 12, 13, and 14 for each joint in the wrist structure, starting with i=1.
12. Using the values of the arm joint variables (that are computed in step 9) and the initial position of the $i^{th}$ gripper vector, calculate the intermediate values of the $i^{th}$ rotated gripper vector $v_{im}$ and the $i^{th}$ intermediate rotating unit vector $u_{im}$ if the arm is moved form the reset position to the new position as computed in step 8.
13. From the desired gripper vectors $v_{id}$ and the intermediate gripper vectors $v_{im}$ calculate the required value of $\theta_i$ which makes $v_{im}$ coincident to $v_{id}$ by using the following formulas:

$$v_{id} = v_{im} + \frac{2R_i \times (v_{im} + R_i \times v_{im})}{1+F_i^2} \quad (31)$$

$$F_i^2 = R_{ix}^2 + R_{iy}^2 + R_{iz}^2 \quad (32)$$

$$R_i = [R_{ix}, R_{iy}, R_{iz}] = u_{im} \tan(\frac{\theta_i}{2}) = u_{im} f_i \quad (33)$$

substituted (31) in (33) and solve for $f_i$ , yields:

$$v_{id} - v_{im} = \frac{2u_{im} f_i \times (v_{im} + u_{im} f_i \times v_{im})}{1+F_i^2} \quad (34)$$

$$\text{then } \theta_i = 2\tan^{-1}(f_i) \quad (35)$$

14. If $i \neq I$ ( no. of the links in the wrist structure), then i = i +1, and append the computed value of the angle $\theta_i$ in step 13 to the arm joint variable and apply the steps 11-14 to compute $\theta_{i+1}$, and so on.
15. The total joints variables will be $[\theta_1, \theta_2, \ldots \theta_m, \theta_4, \theta_5, \ldots \theta_n]$, where $\theta_1$, $\theta_2$ …$\theta_m$ are computed in step 9 and

$\theta_4$, $\theta_5$ …$\theta_n$ are computed in steps 11 to 14, and m, n are the number of joints of the arm and wrist respectively.

## IIV. SOLUTION OF INVERSE KINEMATICS PROBLEM OF MA2000 MANIPULATOR USING PROPOSED ALGORITHM

The proposed algorithm is tested practically on the MA2000 robotic manipulator, which is a 6DOF with complex wrist structure as shown in Figs. 3, 4.

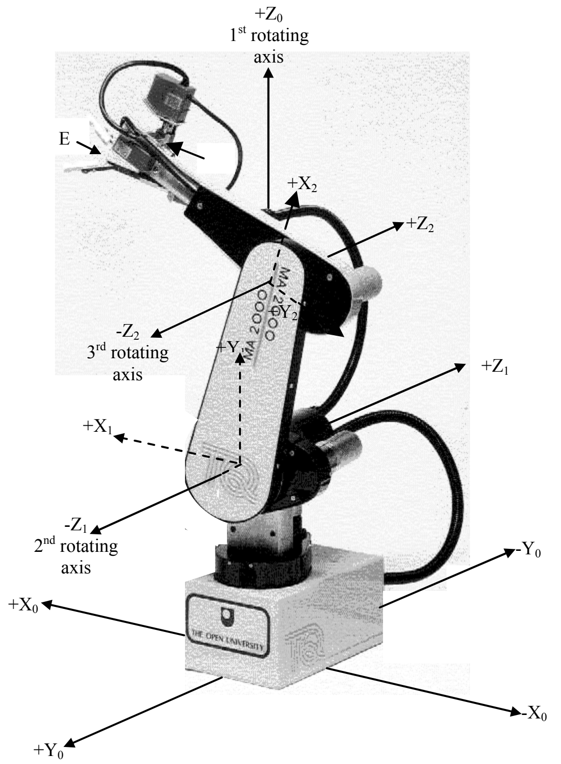


**Fig. 3.** MA2000 robotic manipulator with the major coordinate frames.

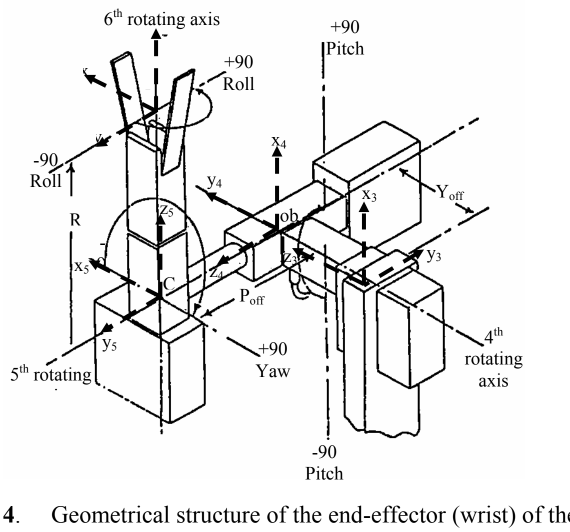


**Fig. 4.** Geometrical structure of the end-effector (wrist) of the MA2000 robotic manipulator.

The schematic diagram of the MA2000 at the reset position is shown in the Fig. 5. If it is required to move the gripper to a new position $[-25,-10,50]^T$cm with respect to the base (global) coordinate frame. The desired orientation of the gripper can be obtained by rotates the gripper $180^o$ about the pitch rotating axis, $225^0$ about the rotated yaw rotating axis and finally $135^0$ about the rotated roll rotating axis. The problem is to determine the values of the six joint variables of the MA2000 that attain the central point of the gripper to the desired position with the desired orientation.

By applying the steps in previous section, four possible sets of solutions of the inverse kinematics of the MA2000 manipulator can be found. Thus there are four sets of the joints variables [$\theta_1$ , $\theta_2$ , $\theta_3$ , $\theta_4$ , $\theta_5$ , $\theta_6$ ] that make the MA2000 attains the desired position with the desired orientation which are given in table 1. The sketch of MA2000 with these joints variables (solutions) are shown in Figs. 6, 7, 8, and 9.

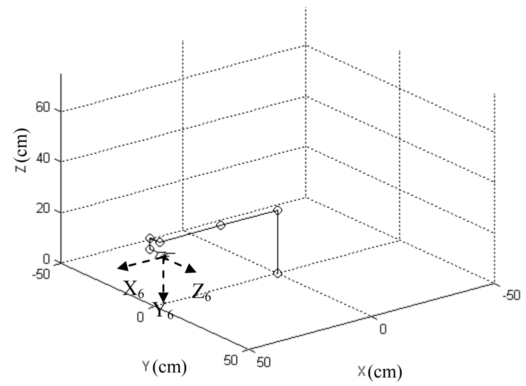


**Fig. 5.** Sketch of the MA2000 at the reset position.

TABLE I
THE FOUR SETS OF SOLUTIONS OF THE INVERSE KINEMATICS PROBLEM FOR MA2000 MANIPULATOR THAT ATTAINED IT TO POINT [-25,-10,50 cm].

| | $\theta_1$(deg.) | $\theta_2$(deg.) | $\theta_3$(deg.) | $\theta_4$(deg.) | $\theta_5$(deg.) | $\theta_6$(deg.) |
|---|---|---|---|---|---|---|
| 1st set | 197.35 | -7.89 | 79.77 | 108.11 | 27.64 | 135 |
| 2nd set | 197.35 | 75.87 | -79.77 | 183.90 | 27.64 | 135 |
| 3rd set | -1.22 | 187.89 | -79.77 | 71.88 | 226.22 | 135 |
| 4th set | -1.22 | 104.12 | 79.77 | 356.09 | 226.22 | 135 |

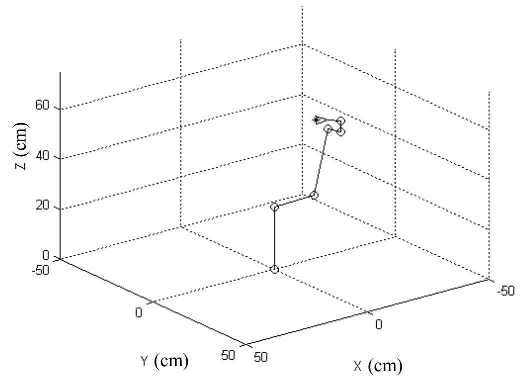


**Fig. 6** Sketch of MA2000 manipulator at joint variables $[197.3564^0, -7.8977^0, 79.7788^0, 08.1189^0, 27.6436^0, 135^0]$.

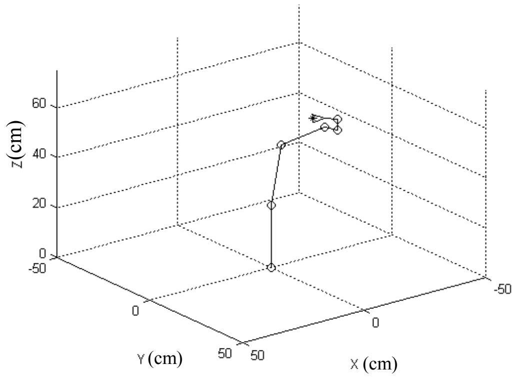


**Fig. 7** Sketch of MA2000 manipulator at joint variables $[197.3564^0, 75.8702^0, -79.7788^0, 183.9086^0, 436^0, 135^0]$.

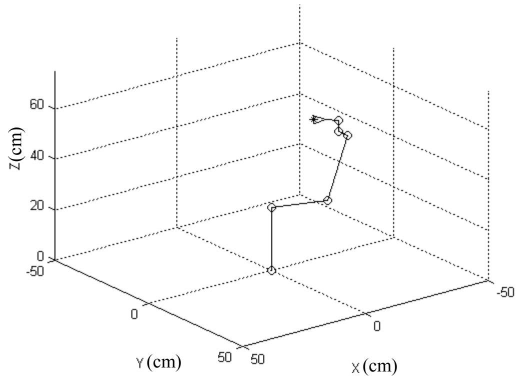


**Fig.8.** Sketch of MA2000 manipulator at joint variables $[-1.2296^0, 187.8977^0, -79.7788^0, 71.8811^0, 226.2296^0, 135^0]$.

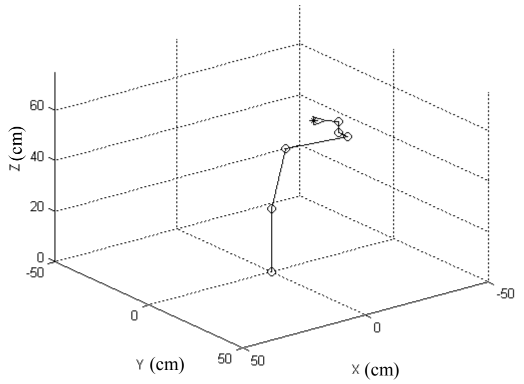


**Fig. 9.** Sketch of MA2000 manipulator at joint variables $[-1.2296^0, 104.1298^0, 79.7788^0, 356.0914^0, 226.2296^0, 135^0]$.

It is important to know that there are not always four sets of solutions. Some times there are three, two, one, or no solutions in solving the inverse kinematics problem of MA2000 manipulator. For example if the desired position of gripper of MA2000 is $[4.4,-5,81]^T$cm with new orientation corresponding to rotation by angles $90^0$, $90^0$ and $0^0$ about pitch, yaw and roll axes with respect to reset position of MA2000. The algorithm gives only a single solution (due to physical structure of the manipulator) for the inverse kinematics problem of the manipulator which equals to $[0^0,90^0,0^0,0^0,90^0,0^0]$ as it is shown in Fig. 10.

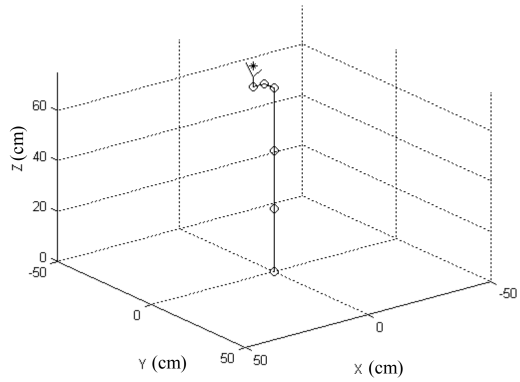


**Fig. 10.** Sketch of the MA2000 robotic manipulator at the joint variables $[0^0, 90^0, 0^0, 0^0, 90^0, 0^0]$

## VIII. CONCLUSIONS

The proposed algorithm to solve the inverse kinematics of the robotic manipulators has the following features:

The algorithm is based on the rotation vectors concept.

1. It can be applied to most types of the robotics manipulators include the complex structure robotic manipulators which have high DOF.
2. It gives all the possible sets of the joints variables of robotic manipulator that attain the end-effector of the manipulator to the desired position and orientation.
3. It has a less computation complexly than other method which uses the algebraic solution by using trigonometric (non linear) equations or the numerical techniques to find the solution of the inverse kinematics problem. Thus the execution time and the computational rate of the proposed algorithm are less than other methods such as the numerical method so that this algorithm is more suitable to use in the real time applications.

The algorithm is used practically to solve the inverse kinematics problem of the MA2000 robotic manipulator. The results give four sets of possible solutions by using the proposed algorithm. Some times there are one, two, three, or no solutions depending on the physical structure of the manipulator and the desired position and orientation of the end-effector of the manipulator.

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