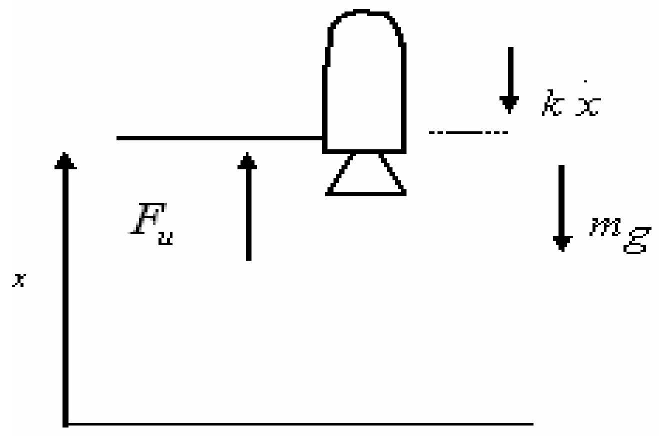
Example 1 (Rocket):



F_u Thrust force
m_g Weight of rocket
$k\dot{x}$ Air friction force

Newton's Law:

$$F_u - mg - k\dot{x} = m\ddot{x}$$

Or

$$m\ddot{x} + k\dot{x} = (F_u - mg)$$

Dividing by *m* $$\ddot{x}+\frac{k}{m}\dot{x}=\frac{F_u}{m}-g$$

Let $x_1 := x$ (Rocket Position)

$x_2 := \dot{x}$ (Rocket Velocity)

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = \ddot{x}(t)$$

$$= -\frac{\mathrm{k}}{\mathrm{m}}\dot{x}(t) + \frac{F_u}{m} - g$$

$$= -\frac{\mathrm{k}}{\mathrm{m}}x_2(t) + \frac{F_u}{m} - g$$

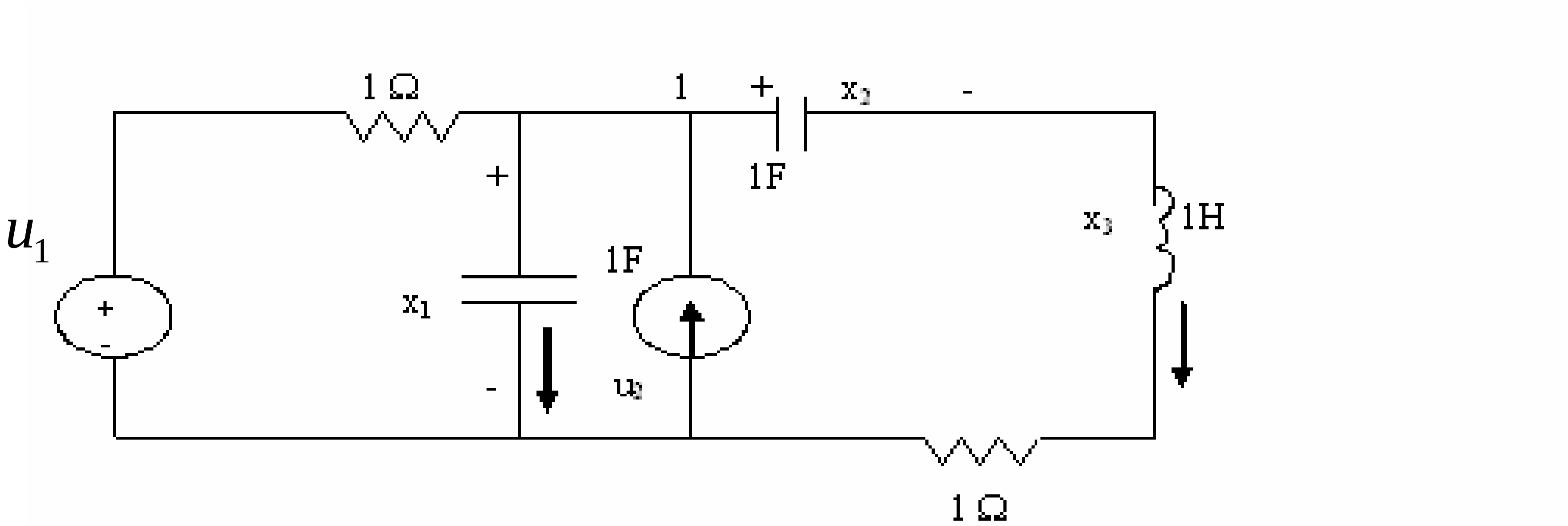
In matrix form:

$$\underbrace{\begin{bmatrix} \dot{x}1(t) \\ \dot{x}2(t) \end{bmatrix}}_{\underline{\dot{x}}(t)} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{k}{m} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\underline{x}} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} \underbrace{\left(\frac{F_u}{m} - g \right)}_{\underline{u}(t)}$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \text{if velocity is output of interest}$$

$$\text{or} \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \text{if position is output of interest}$$

Example 2 (Electric Circuit)



Use: Inductor currents capacitor voltages as state variables.

Summing current at node (1)

$$(u_1 - x_1) - \dot{x}_1 + u_2 - x_3 = 0 \qquad \text{(I)}$$

$$\dot{x}_2 = x_3 \qquad \text{(II)}$$

KVL for the right most loop

$$\dot{x}_3 + x_3 - x_1 + x_2 = 0 \qquad \text{(III)}$$

In matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Example 4

$$\dddot{y}(t) + a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = u(t)$$

Let $x_1 := y$

$$x_2 = \dot{y}$$

$$x_3 = \ddot{y}$$

Then $\dot{x}_1 = x_2$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = \dddot{y}(t) = -a_2 \ddot{y}(t) - a_1 \dot{y}(t) - a_0 y(t) + u(t)$$

$$= -a_2 x_3 - a_1 x_2 - a_0 x_1 + u(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}^{b} u$$

$$y = \overbrace{[1 \quad 0 \quad 0]}^{C} \begin{Bmatrix} x1 \\ x2 \\ x3 \end{Bmatrix} + \overbrace{[0]}^{D} u$$

Example 5:

Find a state space representation for the system

$$y(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} u(s)$$

or equivalently,

$$\dddot{y}(t) + a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = b_2 \ddot{u}(t) + b_1 \dot{u}(t) + b_0$$

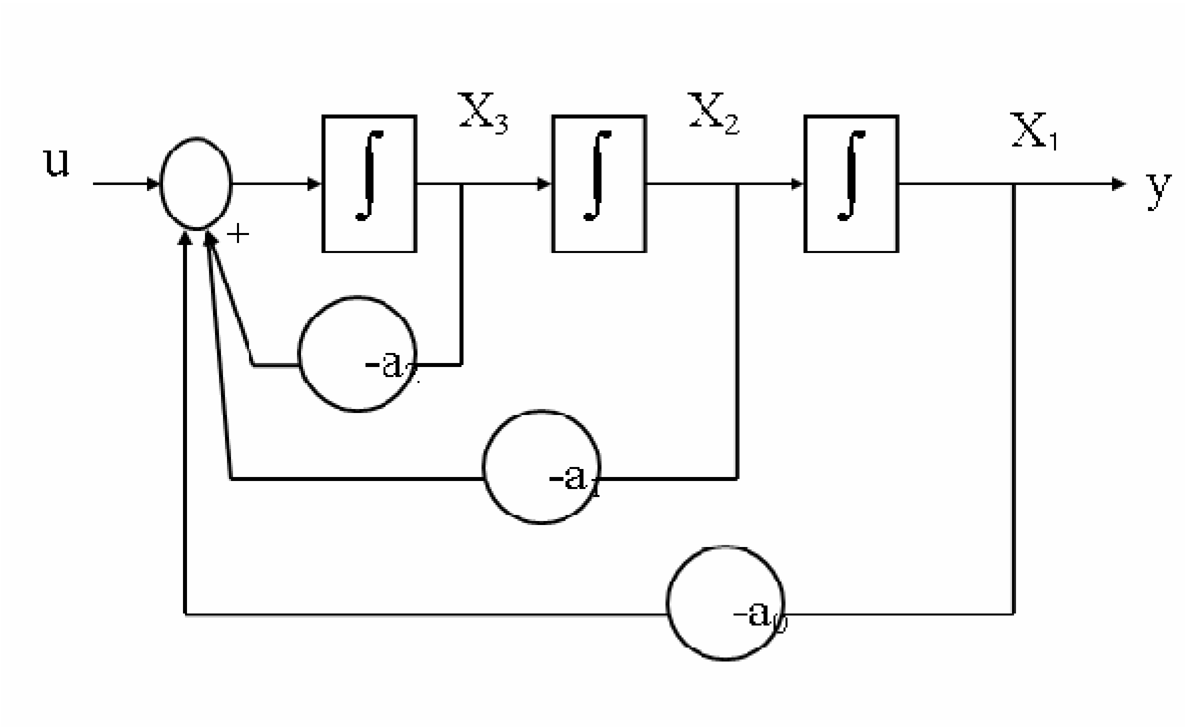
To get a state space representation, we solve the problem in two steps:

1. Letting $$\tilde{y}(s) = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} u(s)$$

We obtain a state space description with $\tilde{y}(s)$ as output:

$$(s^3 + a_2 s^2 + a_1 s + a_0)\tilde{y}(s) = u(s) \qquad \text{or}$$

$$\dddot{\tilde{y}}(t) + a_2 \ddot{\tilde{y}}(t) + a_1 \dot{\tilde{y}}(t) + a_0 \tilde{y}(t) = u(t)$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}^{b} u$$

2. $$y(s) = (b_2 s^2 + b_1 s + b_0)\tilde{y}(s)$$

or

$$y(t) = b_2 \ddot{\tilde{y}}(t) + b_1 \dot{\tilde{y}}(t) + b_0 \tilde{y}(t)$$

$$y(t) = b_2 x_3 + b_1 x_2 + b_0 x_1$$

$$y(t) = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$