State-Variable Description

Motivation

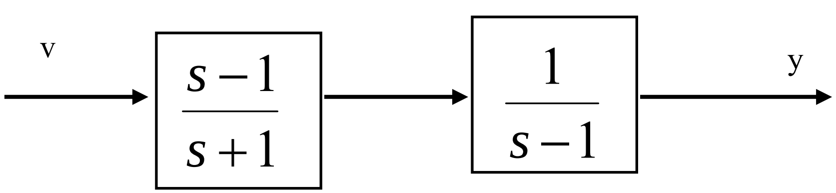
Consider a system with the transfer function

$$H_F(s) = \frac{1}{s-1}$$

Clearly the system is unstable

To stabilize it, we can precede $H_F(s)$ with a compensator

$$H_c(s) = \frac{s-1}{s+1}$$



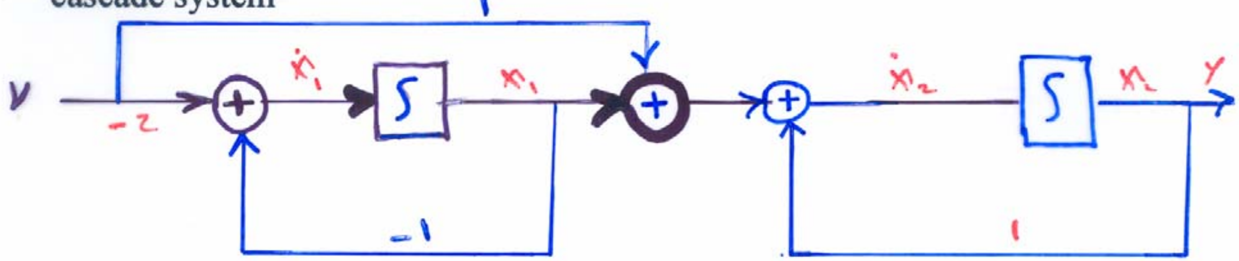
The overall transfer function:

$$H_f(s)\,H_c = \frac{1}{s-1}\quad\frac{s-1}{s+1} = \frac{1}{s+1}$$

This is nice outcome, but unfortunately this technique will not work: After a while the system will burn or saturate.

To see why, let us first set up an analog computer simulation of the cascade system

cascade system



We can write the equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} v \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

There are general methods of solving such so-called state-space equation but it will suffice to proceed as follows:

The first equation is

$$\dot{x}_1 = -x_1 - 2v \quad , \quad x_1(0) = x_{10}$$

Which yields

$$x_1(\mathrm{t}) = \mathrm{e}^{-\mathrm{t}}\, x_{10} - 2\mathrm{e}^{-\mathrm{t}} * v$$

*denotes convolution

The second equation

$$\dot{x}_2 = x_1 + x_2 + v$$

has a solution

$$y(\mathrm{t}) = x_2(\mathrm{t}) = \mathrm{e}^{\mathrm{t}}\, x_{20} + \quad (\mathrm{e}^{\mathrm{t}}\text{-}\mathrm{e}^{-\mathrm{t}})\, x_{10} + \mathrm{e}^{-\mathrm{t}} * v$$

$$\frac{1}{2}$$

$$\Rightarrow y(s) = x_2(s) = \frac{x_{20}}{s-1} + \frac{x_{10}}{(s-1)(s+1)} + \frac{v(s)}{s+1}$$

Therefore the overall transfer function, which has to be calculated with zero initial condition is 1/(s+1) as expected.

Note: However, that unless the initial conditions can always be kept zero, y(.) will grow without bond.

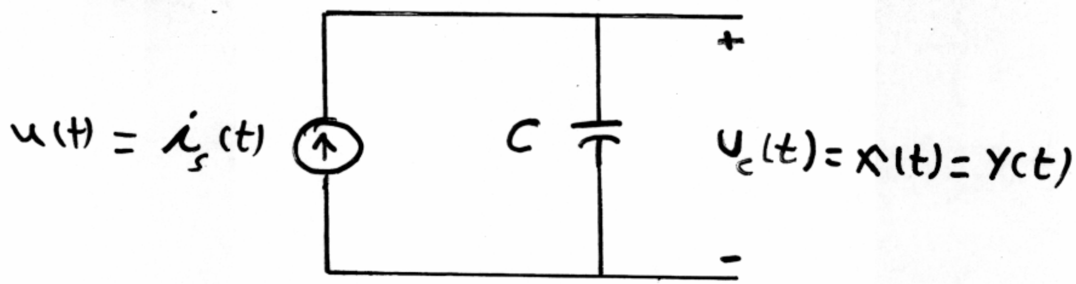
So the input output description of a system is applicable only when the system is initially relaxed

State-Variable

Definition: The state of a system at time t_o is the amount of information at t_o that, together with $u_{[to,\infty\ )}$, determine uniquely the behavior of the system for all $t \geq t_o$

Usually x denotes state, u input, y output

Example



$$y(t) = \frac{1}{C}\int_{-\infty}^{t} u(\tau)d\tau = \frac{1}{C}\int_{-\infty}^{to} u(\tau)d\tau + \frac{1}{C}\int_{to}^{t} u(\tau)d\tau$$

$$= y(t_o) + \frac{1}{C}\int_{to}^{t} u(\tau)d\tau$$

where

$$y(t_o) = \frac{1}{C}\int_{-\infty}^{to} u(\tau)d\tau$$

So if $y(t_o)$ is known, the output after $t \geq to$ can be uniquely determined.

Hence, $y(t_o)$ regarded on the state at time t_o

Linearity

Definition: A system is said to be linear if for every t_o and any two state-input-output pairs

$$\left.\begin{array}{l} x_i(t_o) \\ u_i(t), \quad t \geq t_o \end{array}\right\} \rightarrow y_i(t), \quad t \geq t_o$$

for $i = 1, 2$, we have

$$\left.\begin{array}{l} \alpha_1 x_1(t_o) + \alpha_2 x_2(t_o) \\ \alpha_1 u_1(t) + \alpha_2 u_2(t) \quad t \geq t_o \end{array}\right\} \rightarrow \alpha_1 y_1(t) + \alpha_2 y_2(t), \quad t \geq t_o$$

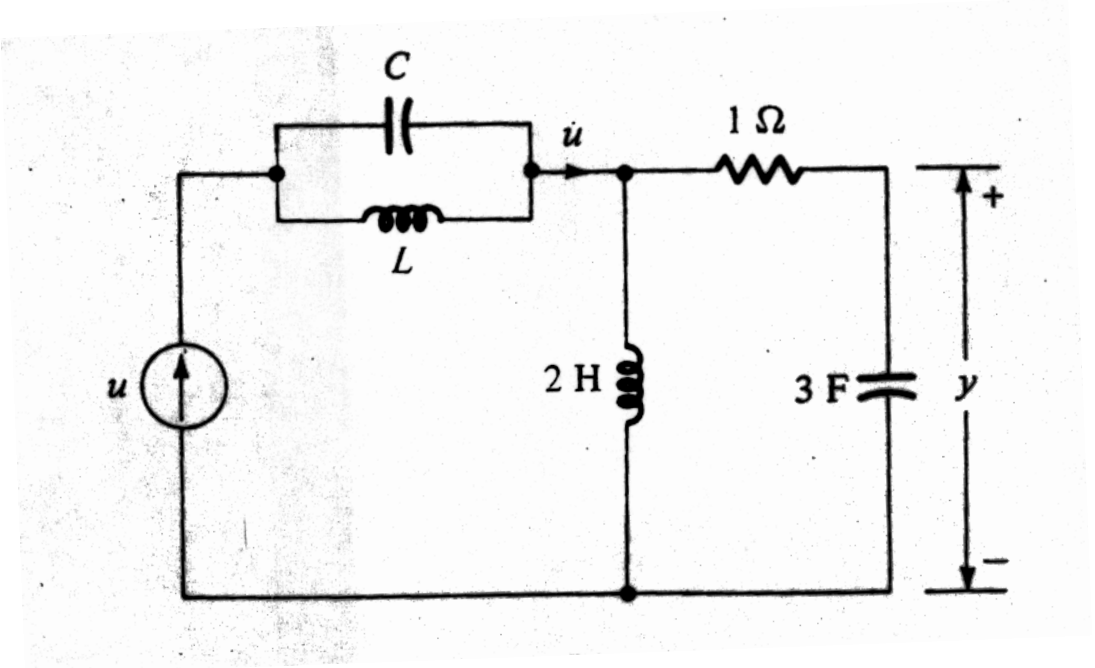
for any real constants α_1 and α_2. Otherwise the system is said to be nonlinear.

- Linearity must hold not only at the output but also at all state variables and must hold for zero initial state and nonzero initial state.

- This definition is different from
  $H(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 H(u_1) + \alpha_2 H(u_2)$

Example

C and L are nonlinear



Because L – C loop is in series connection with the current source, its behavior will not transmit to the output. Hence the above circuit is linear according the input-output definition while it is nonlinear according to the above definition of linearity.

A very important property of any linear system is that the responses of the system can be decomposed into two parts

Output due to $\begin{cases} x(t_o) \\ u(t), \qquad t \geq t_o \end{cases}$

$= \text{output due to} \begin{cases} x(t_o) \\ u(t) \equiv 0, \quad t \geq t_o \end{cases}$

$+ \text{ output due to} \begin{cases} x(t_o) = 0 \\ u(t), \qquad t \geq t_o \end{cases}$

Or

Response = zero-input response + zero-state response

A very broad class of systems can be modeled by

$$\left.\begin{array}{l} \dot{x}_1 = f_1(x_1,\ldots,x_n\, u_1,\ldots,u_p,t) \\ \quad\cdot \\ \quad\cdot \\ \quad\cdot \\ \dot{x}_n = f_n(x_1,\ldots,x_n\, u_1,\ldots,u_p,t) \end{array}\right] \Rightarrow \quad \dot{\underline{x}} = \underline{f}(\underline{x},\underline{u},t)$$

together with

$$\left.\begin{array}{l} y_1 = g_1(x_1,\ldots,x_n\, u_1,\ldots,u_p,t) \\ \quad . \\ \quad . \\ \quad . \\ y_q = g_q(x_1,\ldots,x_n\, u_1,\ldots,u_p,t) \end{array}\right] \Rightarrow \quad \underline{y} = \underline{g}(\underline{x},\underline{u},t)$$

where

$$\underline{x} : \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{bmatrix}, \underline{u} = \begin{bmatrix} u_1 \\ . \\ . \\ . \\ u_p \end{bmatrix}, \textit{and}\ \underline{y} = \begin{bmatrix} y_1 \\ . \\ . \\ . \\ y_q \end{bmatrix}$$

For the special cases:

$$\dot{x} = A(t)x + B(t)u$$

$$y = C(t)x + D(t)u$$