Mathematical Representation of Linear Systems

- **State Space Model (**Internal Description)

Continuous Time

$$\dot{x}(t) = Ax(t) + Bu(t)$$

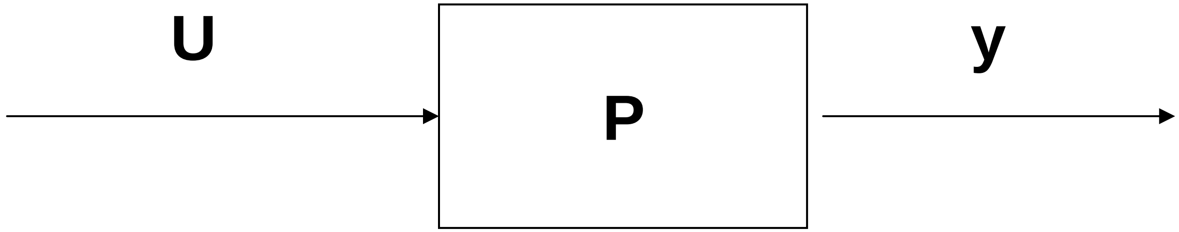
$$y(t) = Cx(t) + Du(t)$$

Discrete Time

$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k) + Du(k)$$

- **Transfer Function (**External Description)



$$y(s)=H(s)u(s)$$

$$y(z)=H(z)u(z)$$

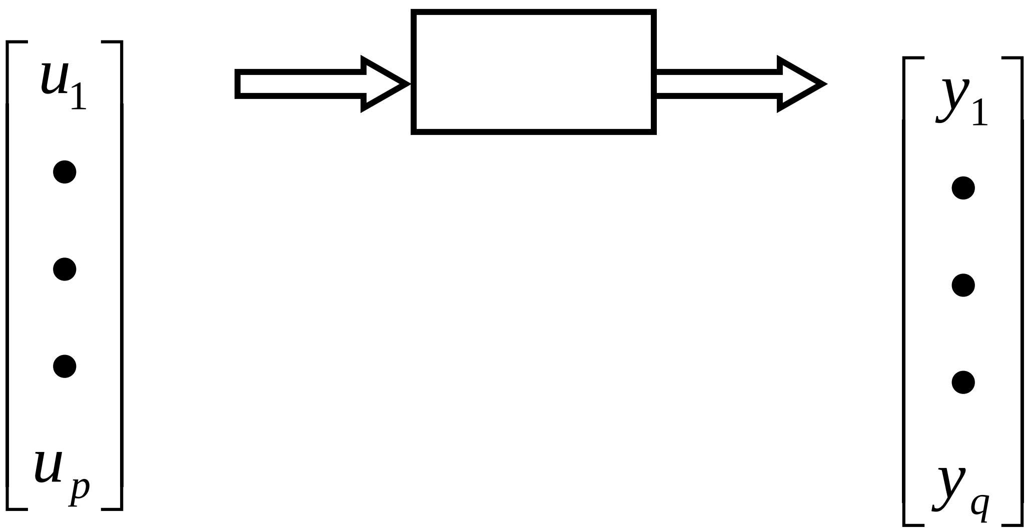
Example: In a network if we are interested in terminal properties we may use the Impedance or Transfer Function. However, if we want the currents and voltages of each branch of the network, then loop or nodal analysis has to be used to find a set of differential equations that describes the network.

- A system is said to be a single-variable if and only if it has only one input and one output (**SISO**)

- A system is said to be a multivariable if and only if it has more than one input or more than one output (**MIMO**)

The Input Output Description

The input-output description of a system gives a mathematical relation between the input and output of the system. In such a case a system may be considered as a black box and we try to get system properties through the input-output pairs.



The time interval in which the inputs and outputs will be defined is from $-\infty$ to ∞. We use u to denote a vector function defined over $(-\infty, \infty)$; u(t) is used to denote the value of u at time t.

If the function u is defined only over $[t_0, t_1)$ we write $u[t_0, t_1)$.

- Def: If the output at time t_1 of a system depends only on the input applied at time t_1, the system is called an instantaneous or zero-memory or memoryless system.
- An example for such a system is the resistor.

Def: A system said to have memory if the output at time t_1 depends not only the input applied at t_1, but also on the input applied before and /or after t_1.

Hence, if an input $u[t_1,\infty)$ is applied to a system, unless we know the input applied before t_1, we will obtain different output $y[t_1,\infty)$, although the same input $u[t_1,\infty)$ is applied.

So it is clear that such an input-output pair lacks a unique relation.

Hence in developing the input-output description before an input is applied, the system must be assumed to be relaxed or at rest, and the output is excited solely and uniquely by the input applied thereafter.

A system is said to be relaxed at time t_1 if no energy is stored in the system at that instant. We assume that every system is relaxed at time $-\infty$

Under relaxedness assumption, we can write

$y = Hu$

Where H is some operator or function that specifies uniquely the ouput *y* in terms of the input *u* of the system.

Definition: A system described by the mapping H is said to be linear if

$$H(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 H(u_1) + \alpha_2 H(u_2)$$

For all u_1, u_2, α_1 and α_2 .

The Superposition Principle applies

The linearity condition:

$$H(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 H(u_1) + \alpha_2 H(u_2)$$

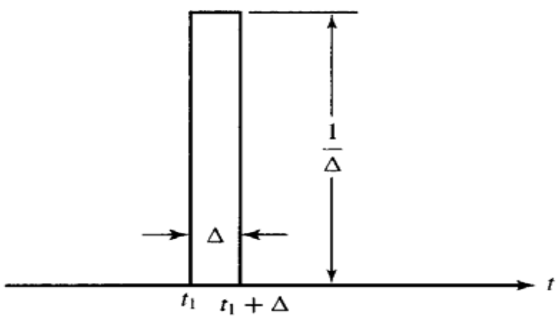
is equivalent to:

1. $H(\alpha u) = \alpha H(u)$ (Homogeneity)

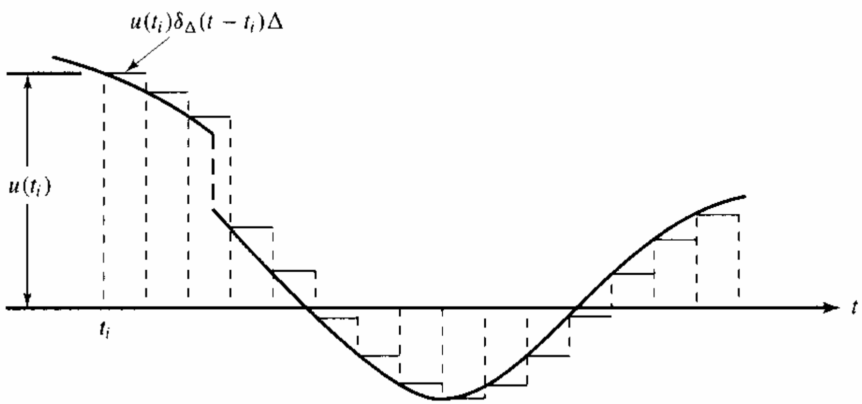
2. $H(u_1 + u_2) = H(u_1) + H(u_2)$ (Additivity)

We can take advantage of the linearity property to calculate the output of a given linear system for any input.

- First, define to be the pulse:



- A given input can be approximated by a sequence of pulses as follows:



$$u(t) \cong \sum_i u(t_i)\delta_\Delta(t-t_i)\Delta$$

$$y = Hu \cong H\left(\sum_i u(t_i)\delta_\Delta(t-t_i)\Delta\right)$$

By linearity

$$y = \sum_i u(t_i)\underbrace{H\left(\delta_\Delta(t-t_i)\right)}_{g_\Delta(t,t_i)}\Delta$$

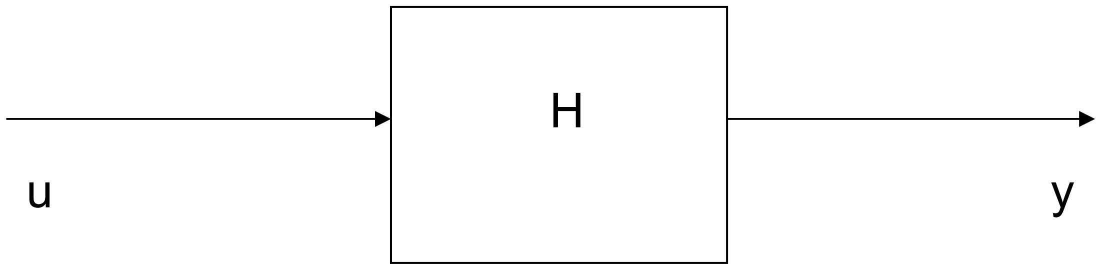
$$y(t) \cong \sum_i u(t_i)g_\Delta(t,t_i)\Delta$$

Taking the limit as $\Delta \to 0$

$$y(t) = \int_{-\infty}^{+\infty} u(\tau)\, g(t,\tau)\, d\tau$$

where $g(t,\tau) := \big(H\delta(t-\tau)\big)(t)$

i.e. $g(t,\tau) =$ the output at time *t* when the input is δ function applied at time τ.

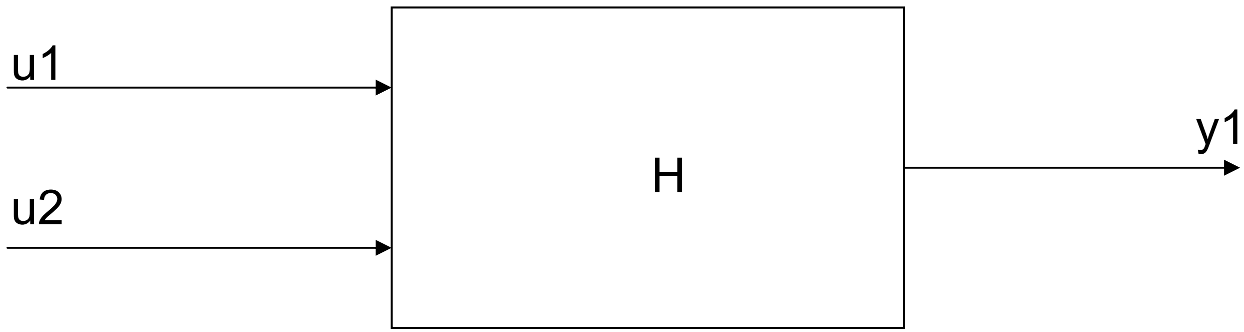


For a linear system H.

$$y(t) = \int_{-\infty}^{\infty} g(t,\tau)u(t)d\tau$$

What about the output for MIMO linear system?

2 Input, 1 Ouput

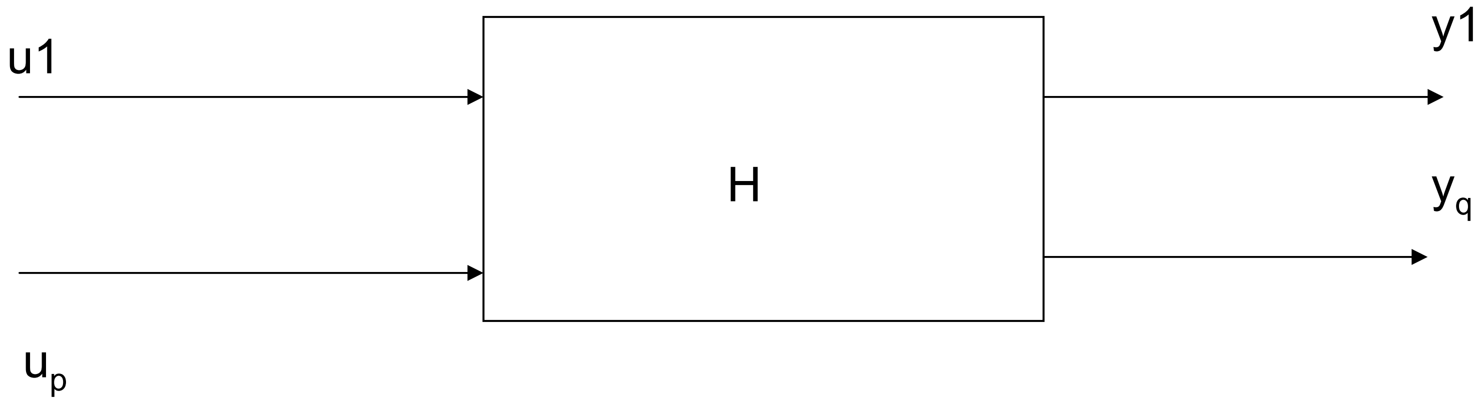


$$y_1 = H\left(\begin{bmatrix} u_1 \\ 0 \end{bmatrix}\right) + H\left(\begin{bmatrix} 0 \\ u_2 \end{bmatrix}\right)$$

$$= \mathrm{H}_{11}u_1 + H_{12}u_2$$

$$= \int_{-\infty}^{\infty} g_{11}(t,\tau)u_1(\tau)dt + \int_{-\infty}^{\infty} g_{12}(t,\tau)u_2(\tau)dt$$

For a general MIMO system



$$y_1(t) = \int_{-\infty}^{\infty} g_{11}(t,\tau)u_1(\tau)d\tau + \ldots + \int_{-\infty}^{\infty} g_{1p}(t,\tau)u_p(\tau)d\tau$$

$$\vdots$$

$$y_q(t) = \int_{-\infty}^{\infty} g_{q1}(t,\tau)u_1(\tau)d\tau + \ldots. + \int_{-\infty}^{\infty} g_{qp}(t,\tau)u_p(\tau)d\tau$$

In a matrix form

$$\underline{y}(t) = \int_{-\infty}^{\infty} G(t,\tau)\underline{u}(\tau)dt$$

"Input-Output Description for the linear system"
Where

$$G(t,\tau) = \begin{bmatrix} g_{11}(t,\tau) & \cdots & g_{1p}(t,\tau) \\ \vdots & & \\ g_{q1}(t,\tau) & \cdots & g_{qp}(t,\tau) \end{bmatrix}$$

- **Causality**

- **Definition**: A system is said to be causal or nonanticipatory if the output of the system at time t does not depend on the input applied after time t; it depends only on the input applied before and at time t.

For a linear system,

$$y(t) = \int_{-\infty}^{+\infty} g(t,\tau)u(\tau)d\tau$$

Where $g(t,\tau)$ is the response to an impulse applied at time τ .

If the linear system is causal, $g(t,\tau)=0$ for $t < \tau$.

Thus, for a (1) causal (2) linear system, the output is related to the input by:

$$y(t) = \int_{-\infty}^{t} g(t,\tau)u(\tau)d\tau$$

For a linear system

$$\underline{y}(t) = \int_{-\infty}^{\infty} G(t,\tau)u(\tau)d\tau$$

$$= \underbrace{\int_{-\infty}^{t_o} G(t,\tau)\underline{u}(\tau)d\tau}_{\text{If the system is releaxed}} + \int_{t_o}^{\infty} G(t,\tau)\underline{u}(\tau)d\tau$$

at $t = t_0$, this term is zero

for $t \geq t_0$

Therefore, for a linear system which is <u>relaxed at</u> t_0

$$\underline{y}(t) = \int_{t_o}^{\infty} G(t,\tau)\underline{u}(\tau)d\tau \qquad t \geq t_o$$

For a linear system which is <u>causal</u> and <u>relaxed at</u> t_0

$$\underline{y}(t) = \int_{t_o}^{t} G(t,\tau)\underline{u}(\tau)d\tau \qquad t \geq t_o$$