Solution of Linear State – Space Equations

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t) \qquad \mathrm{x}(\mathrm{t_o}) = x_o$$

The solution of $\dot{x}(t) = A(t)x(t) + B(t)u(t) \qquad \mathrm{x}(\mathrm{t_o}) = x_o$ can be decomposed into two parts:

(a) The zero-input solution, i.e. solution of

$$\dot{x}_{ZI}(t) = A(t)x_{ZI}(t) \qquad \mathrm{x}_{\mathrm{ZI}}(\mathrm{t_o}) = x_o$$

(b) The zero-state solution, i.e.

$$\dot{x}_{ZS}(t) = A(t)x_{ZS}(t) + B(t)u(t) \qquad \mathrm{x}_{\mathrm{ZS}}(\mathrm{t_o}) = 0$$

In other words, *x(t)* which solves

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \qquad \mathrm{x}(\mathrm{t_o}) = x_o$$

is given by $x(t) = x_{ZI}(t) + x_{zs}(t)$

To see this

$$\dot{x}(t) = \dot{x}_{ZI} + \dot{x}_{ZS}$$

$$= A(t)x_{ZI}(t) + A(t)x_{ZS}(t) + B(t)u(t)$$

$$= A(t)(x_{ZI}(t) + x_{ZS}(t)) + B(t)u(t)$$

$$= A(t)x(t) + B(t)u(t)$$

Also $x(t_o) = x_{ZI}(t_o) + x_{ZS}(t_o)$

$$= x_o + 0 = x_o$$

This proves the solution $x(t)$ can be decomposed into two parts.

We will therefore try to find each solution separately.

We begin with the zero-input solution.

We want to solve the differential equation

$$\dot{x}(t) = A(t)x(t)$$

The differential equation has a unique solution for every initial state x_o. Since there are infinitely many possible initial states, then the differential equation has infinitely many solutions.

Theorem :

The set of solutions of $\dot{x}(t) = A(t)x(t) \quad x(t_o) = x_o$ is an n-dimensional vector space, where n is the number of state-variable.

Review of Linear Algebra

Set: It is a collection of elements

EXAMPLES:

R : Set of Real numbers

C : Set of all complex numbers

Z : Set of all integers

Also : Set of {0,1}

: Set of all polynomials of degree <5

: Set of all 2×2 real matrices

Field: A field consists of a set, denoted by $\Im$, of elements called scalars and two operations called addition or + and multiplication or . with the operations defined according to the following axioms:

(a) To every pair of elements α_1 and α_2 the sum $\alpha_1 + \alpha_2$ and the product $\alpha_1 . \alpha_2$ exist in $\Im$.

(b) Addition and multiplication are associative

$$(\alpha_1 + \alpha_2) + \alpha_3 = \alpha_1 + (\alpha_2 + \alpha_3)$$

$$(\alpha_1 \cdot \alpha_2) \cdot \alpha_3 = \alpha_1 \cdot (\alpha_2 \cdot \alpha_3)$$

(c) Addition and multiplication are commutative

$$\alpha_1 + \alpha_2 = \alpha_2 + \alpha_1$$

$$\alpha_1 \cdot \alpha_2 = \alpha_2 \cdot \alpha_1$$

(d) Multiplication is distributive over addition

$$\alpha_1 \cdot (\alpha_2 + \alpha_3) = (\alpha_1 \cdot \alpha_2) + (\alpha_1 \cdot \alpha_3)$$

(e) $\exists$ element in $\Im$, denoted by 0 such that

$$\alpha_1 + 0 = \alpha_1$$

(f) $\exists$ element in $\Im$, denoted by 1 such that

$$1 \cdot \alpha_1 = \alpha_1$$

(g) To each α in $\Im$ there corresponds an element $-\alpha$ in

$\Im$, such that

$$\alpha + (-\alpha) = 0 \qquad \text{(Additive Inverse)}$$

(f) To each α in $\Im$ there corresponds an element α in $\Im$ such that

$$\alpha \cdot \alpha^{-1} = \alpha^{-1} \cdot \alpha = 1 \qquad \text{(Multiplicative Inverse)}$$

Examples

- The set R of all real numbers with the usual addition and multiplication is a field.

- The set C of all complex numbers with the usual addition and multiplication is a field.

Notice that R is a sub field of C

- The set of rational functions with real coefficients with the usual addition and multiplication of polynomials is a field.

The set {0,1} of binary numbers is a field if the rules of addition and multiplication are appropriately defined i.e. if

+	**0**	**1**	.	**0**	**1**
0	0	1	0	0	0
1	1	0	1	0	1

The following sets of familiar numbers do not form fields:

- The set of all positive (negative) real numbers. Since the additive inverse is not in the set.

- The set of all integers. Since the multiplicative inverse is not integer.