Haykin Chapter 4
Baseband Pulse Transmission

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Content

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4.1 Introduction

- In this chapter we study the transmission of digital data (of whatever origin) over a ***baseband channel***.
- Baseband transmission of digital data requires the use of a low-pass channel with a bandwidth large enough to accommodate the essential frequency content of the data stream.
- Typically, however, the channel is ***dispersive*** in that its frequency response deviates from that of an ideal low-pass filter.

4.1 Error Sources in Baseband Transmission

Intersymbol Interference **(ISI)**

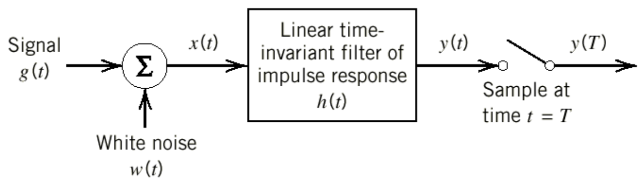
- The result of data transmission over a ***dispersive*** channel is that each received pulse is affected somewhat by adjacent pulses, thereby giving rise to a common form of interference called ***intersymbol interference*** (ISI).
- Intersymbol interference is a major source of bit errors in the reconstructed data stream at the receiver output.
- To correct for it, control has to be exercised over the pulse shape in the overall system.
- Thus much of the material covered in this chapter is devoted to ***pulse shaping*** in one form or another.

4.1 Error Sources in Baseband Transmission

- Another source of bit errors in a baseband data transmission system is the ***channel noise***.
- Naturally, noise and ISI arise in the system simultaneously.

4.2 Matched Filter

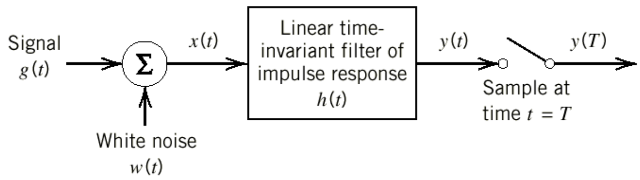
- A fundamental result in communication theory deals with the ***detection*** of a pulse signal of known waveform that is immersed in additive white noise.
- The device for the optimum detection of such a pulse involves the use of a linear-time-invariant filter known as a ***matched filter.***
- It it called a matched filter because its impulse response is matched to the pulse signal.



4.2 Matched Filter

- The filter input *x(t)* consists of a pulse signal g*(t)* corrupted by additive channel noise *w(t),* as shown by

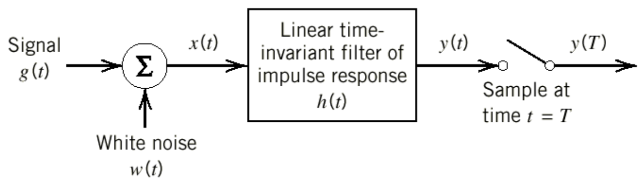
  $x(t) = g(t) + w(t), \quad 0 \le t \le T$, where T is an arbitrary observation interval
- The pulse signal *g(t)* may represent a binary symbol 1 or 0 in a digital communication system.
- The *w(t)* is the sample function of a white noise process of zero mean and power spectral density $N_0/2$.
- It is assumed that the receiver has knowledge of the waveform of the pulse signal *g(t).* The source of uncertainty lies in the noise *w(t).*



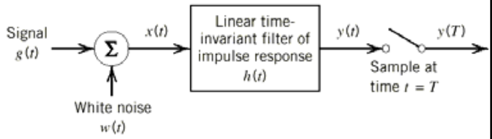
4.2 Matched Filter

- The function of the receiver is to detect the pulse signal *g(t)* in an optimum manner, given the received signal *x(t).*
- To satisfy this requirement, we have to optimize the design of the filter so as to minimize the effects of noise at the filter output in some statistical sense, and thereby enhance the detection of the pulse signal *g(t).*
- Since the filter is linear, the resulting output *y(t)* may be expressed as:

  $$y(t) = g_o(t) + n(t)$$
- where $g_0(t)$ and *n(t)* are produced by the signal and noise components of the input *x(t),* respectively.



4.2 Matched Filter



- A simple way of describing the requirement that the output signal component $g_0(t)$ be considerably greater than the output noise component *n(t)* is to have the filter make the instantaneous power in the output signal $g_0(t)$, measured at time t = T, as large as possible compared with the average power of the output noise *n(t).*
- This is equivalent to maximizing the *peak pulse signal-to-noise ratio,* defined as $$\eta = \frac{|g_o(T)|^2}{E[n^2(t)]}$$
- where $|g_o(T)|^2$ is the instantaneous power in the output signal, *E* is the statistical expectation operator, and $E[n^2(t)]$ is a measure of the average output noise power.
- **The requirement is to specify the impulse response *h(t)* of the filter such that the output signal-to-noise ratio is maximized**.

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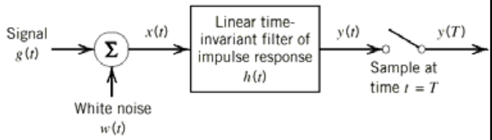
4.2 Matched Filter



Let *G(f)* denote the Fourier transform of the known signal *g(t),* and *H(f)* denote the frequency response of the filter. Then the Fourier transform of the output signal $g_0(t)$ is equal to *H(f)G(f),* and $g_o(t)$ is itself given by the inverse Fourier transform:

$$g_o(t) = \int_{-\infty}^{\infty} H(f)G(f)\exp(j2\pi ft)\,df$$

Hence, when the filter output is sampled at time $t = T$, the signal power will be:

$$|g_o(T)|^2 = \left|\int_{-\infty}^{\infty} H(f)G(f)\exp(j2\pi fT)\,df\right|^2$$

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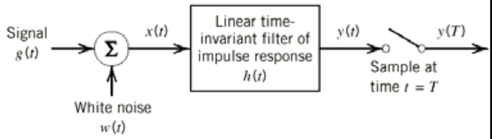
4.2 Matched Filter



- Consider next the effect on the filter output due to the noise *w(t)* acting alone.
- The power spectral density $S_N(f)$ of the output noise *n(t)* is equal to the power spectral density of the input noise *w(t)* times the squared magnitude response $|H(f)|^2$
- Since *w(t)* is white with constant power spectral density $N_0/2$, it follows that:

$$S_N(f) = \frac{N_0}{2}|H(f)|^2$$

- The average power of the output noise *n(t)* is therefore

$$\begin{aligned} E[n^2(t)] &= \int_{-\infty}^{\infty} S_N(f)\, df \\ &= \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\, df \end{aligned}$$

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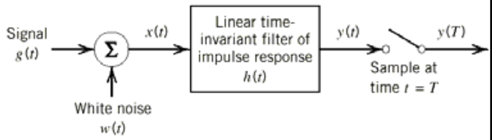
4.2 Matched Filter



- Thus, the peak pulse signal-to-noise ratio is:

$$\eta = \frac{\left|\int_{-\infty}^{\infty} H(f)G(f)\exp(j2\pi fT)\, df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2\, df}$$

- **For a given G(f), what is the frequency response H(f) of the filter that maximizes η ?**
- To find the solution to this optimization problem, we apply a mathematical result known as Schwarz's inequality to the numerator of the above Equation.

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Schwarz's inequality

If we have two complex functions Φ_1(x) and Φ_2(x) in the real variable *x,* satisfying the conditions:

$$\int_{-\infty}^{\infty} |\phi_1(x)|^2\, dx < \infty \quad \text{AND} \quad \int_{-\infty}^{\infty} |\phi_2(x)|^2\, dx < \infty$$

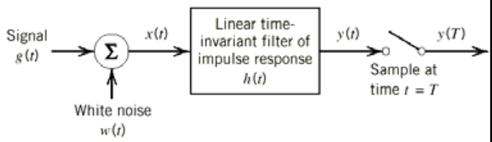
Then

$$\left|\int_{-\infty}^{\infty} \phi_1(x)\phi_2(x)\, dx\right|^2 \le \int_{-\infty}^{\infty} |\phi_1(x)|^2\, dx \int_{-\infty}^{\infty} |\phi_2(x)|^2\, dx$$

The equality in (4.9) holds if, and only if, we have

$$\phi_1(x) = k\phi_2^*(x)$$

4.2 Matched Filter



- Therefore, applying Schwarz's inequality for $\phi_1(x) = H(f)$ and $\phi_2(x) = G(f)\exp(j\pi fT)$,

$$\left|\int_{-\infty}^{\infty} H(f)G(f)\exp(j2\pi fT)\, df\right|^2 \le \int_{-\infty}^{\infty} |H(f)|^2\, df \int_{-\infty}^{\infty} |G(f)|^2\, df$$

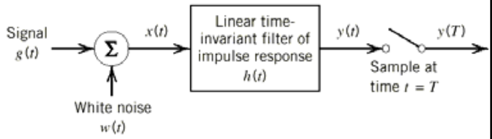
- Thus, the peak pulse signal-to-noise ratio is:

$$\eta \le \frac{2}{N_0}\int_{-\infty}^{\infty} |G(f)|^2\, df$$

- The right-hand side of this relation does not depend on the frequency response H(f) of the filter but only on the signal energy and the noise power spectral density.
- Consequently, the peak pulse signal-to-noise ratio η will be a maximum when H(f) is chosen so that the equality holds; that is,

$$H_{\text{opt}}(f) = kG^*(f)\exp(-j2\pi fT)$$

4.2 Matched Filter



In the time domain, the impulse response of the optimum filter is:

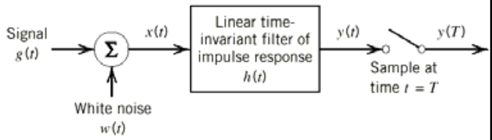
$$h_{opt}(t) = k \int_{-\infty}^{\infty} G^*(f) \exp[-j2\pi f(T - t)]\, df$$

Recall that for real signals g(t), the real part of the spectrum is even and the imaginary part is odd. Thus $G^*(f)=G(-f)$.

$$\begin{aligned} h_{opt}(t) &= k \int_{-\infty}^{\infty} G(-f) \exp[-j2\pi f(T - t)]\, df \\ &= k \int_{-\infty}^{\infty} G(f) \exp[j2\pi f(T - t)]\, df \\ &= kg(T - t) \end{aligned}$$

The impulse response of the optimum filter, except for the scaling factor *k,* is a time-reversed and delayed version of the input signal *g(t).* So, it is ***matched*** to the input signal.

4.2 Matched Filter



- Thus, the peak pulse signal-to-noise ratio will be:

$$\eta_{max} = \frac{2}{N_0} \int_{-\infty}^{\infty} |G(f)|^2\, df$$

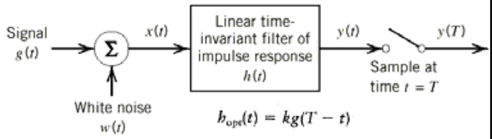
- According to *Rayleigh's energy theorem, the integral of the squared magnitude spectrum of a pulse signal with respect to frequency is equal to the signal energy E*

$$E = \int_{-\infty}^{\infty} g^2(t)dt = \int_{-\infty}^{\infty} |G(f)|^2\, df$$

- Therefore $\eta_{max} = \frac{2E}{N_o}$

- *Thus, the peak pulse signal-to-noise ratio of a matched filter depends only on the ratio of the signal energy to the power spectral density of the white noise at the filter input*

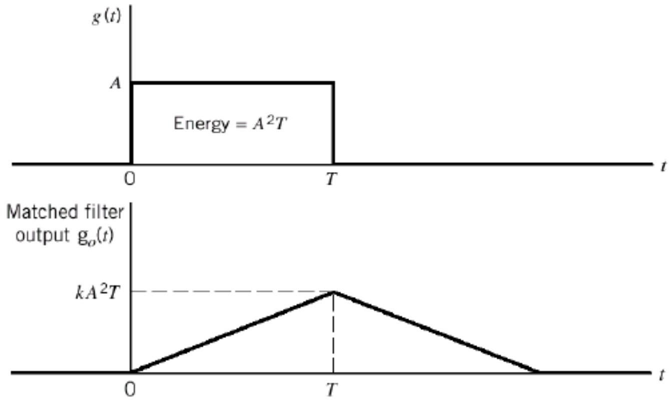
4.2 Matched Filter



$h_{opt}(t) = kg(T - t)$

$H_{opt}(f) = kG^*(f) \exp(-j2\pi fT)$

Example: Find the matched filter output $g_o(t)$ for the following signal:



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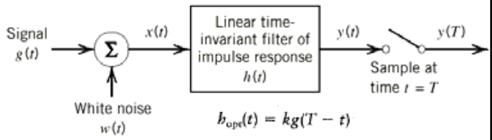
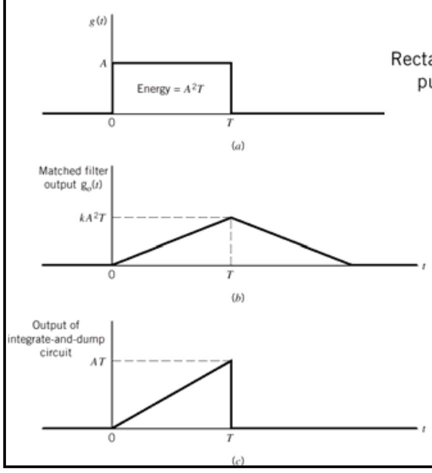
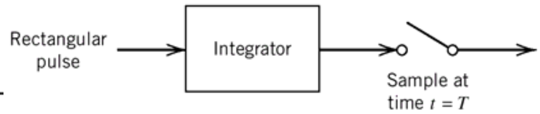
4.2 Matched Filter



$h_{opt}(t) = kg(T - t)$

For the special case of a rectangular pulse, the matched filter may be implemented using a circuit known as the *integrate-and-dump circuit*





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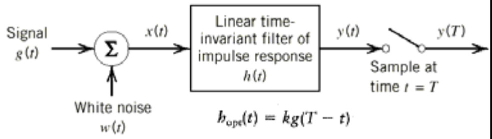
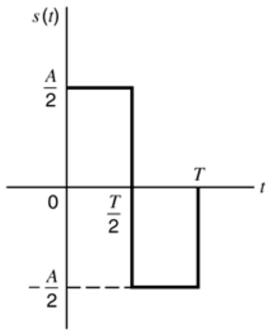
4.2 Matched Filter



$h_{opt}(t) = kg(T-t)$

$H_{opt}(f) = kG^*(f)\exp(-j2\pi fT)$

Example2: Consider the signal s(t)

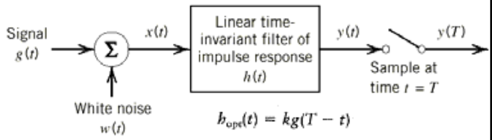


a) Determine the impulse response of a filter matched to this signal and sketch it as a function of time.

b) Plot the matched filter output as a function of time.

c) What is the peak value of the output?

4.2 Matched Filter



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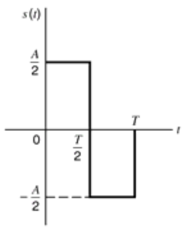
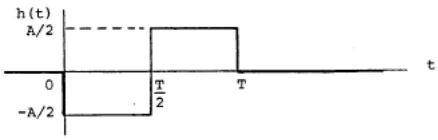
Example2: Consider the signal s(t)



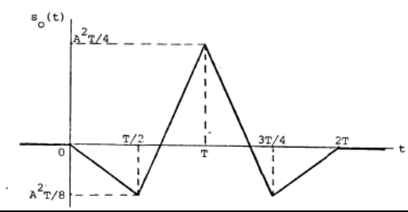
a) Determine the impulse response of a filter matched to this signal and sketch it as a function of time.



b) Plot the matched filter output as a function of time.

c) What is the peak value of the output?

Peak value = $A^2T/4$



4.3 Error Rate Due to Noise

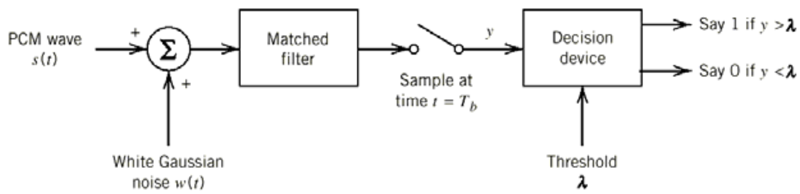
- Consider a binary PCM system based on *polar non-return-to-zero (NRZ) signaling.*
- In this form of signaling, symbols 1 and 0 are represented by positive and negative rectangular pulses of equal amplitude and equal duration.
- The channel noise is modeled as *additive white Gaussian noise w(t)* of zero mean and power spectral density $N_0/2$.
- In the signaling interval $0 \le t \le T_b$, the received signal is thus written as follows:

$$x(t) = \begin{cases} +A + w(t), & \text{symbol 1 was sent} \\ -A + w(t), & \text{symbol 0 was sent} \end{cases}$$

- where T_b is the ***bit duration***, and A is the ***transmitted pulse amplitude***.

4.3 Error Rate Due to Noise

The structure of the receiver used to perform this decision-making process is:



There are two possible kinds of error to be considered:

- Symbol 1 is chosen when a 0 was actually transmitted; we refer to this error as an *error of the first kind.*
- Symbol 0 is chosen when a 1 was actually transmitted; we refer to this error as an *error of the second kind.*

4.3 Error Rate Due to Noise

- To determine the average probability of error, we consider these two situations separately.
- Suppose that symbol 0 was sent. Then

  $x(t) = -A + w(t), \qquad 0 \le t \le T_b$

- The matched filter output, sampled at time $t = T_b$, is:

$$y = \int_0^{T_b} x(t)\, dt$$
$$= -A + \frac{1}{T_b}\int_0^{T_b} w(t)\, dt$$

- which represents the sample value of a random variable Y

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4.3 Error Rate Due to Noise

- Since the noise $w(t)$ is white and Gaussian, we may characterize the random variable Y as follows:
  - The random variable Y is Gaussian distributed with a mean of $-A$.
  - The variance of the random variable Y is

$$\sigma_Y^2 = E[(Y + A)^2]$$
$$= \frac{1}{T_b^2} E\left[\int_0^{T_b}\int_0^{T_b} w(t)w(u)\, dt\, du\right]$$
$$= \frac{1}{T_b^2}\int_0^{T_b}\int_0^{T_b} E[w(t)w(u)]\, dt\, du$$
$$= \frac{1}{T_b^2}\int_0^{T_b}\int_0^{T_b} R_W(t, u)\, dt\, du$$

- where $R_w(t, u)$ is the autocorrelation function of the white noise $w(t)$.
- Since $w(t)$ is white with a power spectral density $N_0/2$, we have

$$R_W(t, u) = \frac{N_0}{2}\delta(t-u)$$ Where $\delta(t-\mu)$ is a time-shifted delta function

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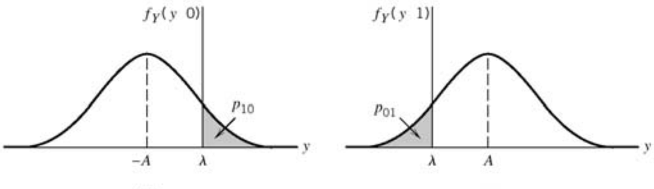
4.3 Error Rate Due to Noise

- Therefore, the variance of Y is:

$$\sigma_Y^2 = \frac{1}{T_b^2}\int_0^{T_b}\int_0^{T_b}\frac{N_0}{2}\delta(t-u)\,dt\,du$$
$$= \frac{N_0}{2T_b}$$

- The conditional probability density function of the random variable Y, given that symbol 0 was sent, is:

$$f_Y(y|0) = \frac{1}{\sqrt{\pi N_0/T_b}}\exp\left(-\frac{(y+A)^2}{N_0/T_b}\right)$$



(a) (b)

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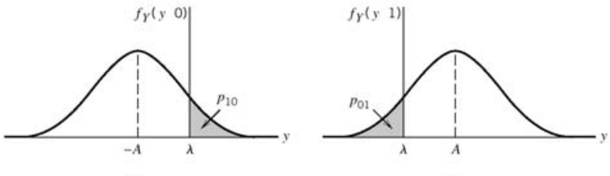
4.3 Error Rate Due to Noise

- Let p_{10} denote the *conditional probability of error, given that symbol 0 was sent.*

$$p_{10} = P(y > \lambda\,|\,\text{symbol 0 was sent})$$
$$= \int_\lambda^\infty f_Y(y|0)\,dy$$
$$= \frac{1}{\sqrt{\pi N_0/T_b}}\int_\lambda^\infty \exp\left(-\frac{(y+A)^2}{N_0/T_b}\right)dy$$



(a) (b)

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Complementary Error Function

- *complementary error function* is defined as:

$$\mathrm{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^{\infty} \exp(-z^2)\, dz$$

- which is closely related to the Gaussian distribution.
- For large positive values of *u,* we have the following *upper bound* on the complementary error function:

$$\mathrm{erfc}(u) < \frac{\exp(-u^2)}{\sqrt{\pi}u}$$

- Relation to Q-Function:

$$Q(v) = \frac{1}{\sqrt{2\pi}} \int_v^{\infty} \exp\left(-\frac{x^2}{2}\right) dx$$

$$Q(v) = \frac{1}{2}\,\mathrm{erfc}\left(\frac{v}{\sqrt{2}}\right)$$

$$\mathrm{erfc}(u) = 2Q(\sqrt{2}u)$$

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Error Function

$$\mathrm{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^{\infty} \exp(-z^2)\, dz$$

TABLE A6.6 *The error function*[a]

u	*erf(u)*	*u*	*erf(u)*
0.00	0.00000	1.10	0.88021
0.05	0.05637	1.15	0.89612
0.10	0.11246	1.20	0.91031
0.15	0.16800	1.25	0.92290
0.20	0.22270	1.30	0.93401
0.25	0.27633	1.35	0.94376
0.30	0.32863	1.40	0.95229
0.35	0.37938	1.45	0.95970
0.40	0.42839	1.50	0.96611
0.45	0.47548	1.55	0.97162
0.50	0.52050	1.60	0.97635
0.55	0.56332	1.65	0.98038
0.60	0.60386	1.70	0.98379
0.65	0.64203	1.75	0.98667
0.70	0.67780	1.80	0.98909
0.75	0.71116	1.85	0.99111
0.80	0.74210	1.90	0.99279
0.85	0.77067	1.95	0.99418
0.90	0.79691	2.00	0.99532
0.95	0.82089	2.50	0.99959
1.00	0.84270	3.00	0.99998
1.05	0.86244	3.30	0.999998

[a]The error function is tabulated extensively in several references; see for example, Abramowitz and Stegun (1965, pp. 297–316).

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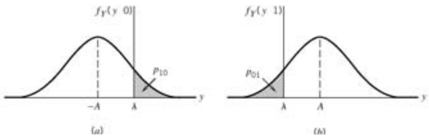
4.3 Error Rate Due to Noise



$$p_{10} = P(y > \lambda \,|\, \text{symbol 0 was sent})$$
$$= \int_{\lambda}^{\infty} f_Y(y\,|\,0)\, dy$$
$$= \frac{1}{\sqrt{\pi N_0/T_b}} \int_{\lambda}^{\infty} \exp\left(-\frac{(y+A)^2}{N_0/T_b}\right) dy$$

- Define a new variable z as:

$$z = \frac{y + A}{\sqrt{N_0/T_b}}$$

- Thus

$$p_{10} = \frac{1}{\sqrt{\pi}} \int_{(A+\lambda)/\sqrt{N_0/T_b}}^{\infty} \exp(-z^2)\, dz$$
$$= \frac{1}{2}\,\mathrm{erfc}\left(\frac{A+\lambda}{\sqrt{N_0/T_b}}\right)$$

- Similarly, *conditional probability of error, given that symbol 1 was sent is*.

$$p_{01} = \frac{1}{\sqrt{\pi}} \int_{(A-\lambda)/\sqrt{N_0/T_b}}^{\infty} \exp(-z^2)\, dz$$
$$= \frac{1}{2}\,\mathrm{erfc}\left(\frac{A-\lambda}{\sqrt{N_0/T_b}}\right)$$

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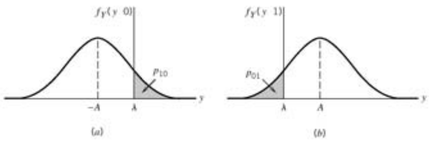
4.3 Error Rate Due to Noise



- Let p_a and p_1 denote the *a priori* probabilities of transmitting symbols 0 and 1, respectively.
- Hence, the *average probability of symbol error* P_e in the receiver is given by:

$$P_e = p_0 p_{10} + p_1 p_{01}$$
$$= \frac{p_0}{2}\,\mathrm{erfc}\left(\frac{A+\lambda}{\sqrt{N_0/T_b}}\right) + \frac{p_1}{2}\,\mathrm{erfc}\left(\frac{A-\lambda}{\sqrt{N_0/T_b}}\right)$$

- What is the optimal value of the threshold λ that minimizes the error probability P_e ?
- We need to derive P_e and equate it to zero.
- For this optimization we use *Leibniz's rule*

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Leibniz's Rule

$$\mathrm{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^{\infty} \exp(-z^2)\, dz$$

- Consider the integral $\int_{a(u)}^{b(u)} f(z, u)\, dz$
- *Leibniz's rule* states that the derivative of this integral with respect to *u* is

$$\frac{d}{du} \int_{a(u)}^{b(u)} f(z, u)\, dz = f(b(u), u)\frac{db(u)}{du} - f(a(u), u)\frac{d(a(u)}{du} + \int_{a(u)}^{b(u)} \frac{\partial f(z, u)}{\partial u}\, dz$$

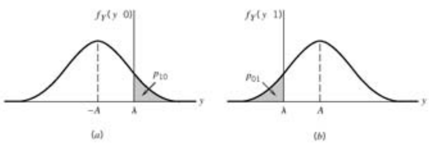
- For the problem at hand, we note from the definition of the complementary error function that: $f(z, u) = \frac{2}{\sqrt{\pi}} \exp(-z^2)$

$$a(u) = u$$
$$b(u) = \infty$$

- The application of Leibniz's rule to the complementary error function thus yields

$$\frac{d}{du}\mathrm{erfc}(u) = -\frac{1}{\sqrt{\pi}} \exp(-u^2)$$

4.3 Error Rate Due to Noise



- Hence, differentiating P_e with respect to λ by making use of the Leibniz's rule, then setting the result equal to zero and simplifying terms, we obtain the optimum threshold as:

$$\lambda_{\mathrm{opt}} = \frac{N_0}{4AT_b} \log\left(\frac{p_0}{p_1}\right)$$

- For the special case when symbols 1 and 0 are equiprobable, we have

$$p_1 = p_0 = \frac{1}{2}$$

- And that leads to $\lambda_{\mathrm{opt}} = 0$

4.3 Error Rate Due to Noise

- This result is intuitively satisfying as it states that, for the transmission of **equiprobable** binary symbols, we should choose the threshold at the midpoint between the pulse heights —*A* and *+A* representing the two symbols 0 and 1.
- Note that for this special case we also have $p_{01} = p_{10}$
- A channel for which the conditional probabilities of error p_{01} and p_{10} are equal is said to be *binary symmetric.*

4.3 Error Rate Due to Noise

- Correspondingly, for **equiprobable binary polar NRZ PCM**, the average probability of symbol error

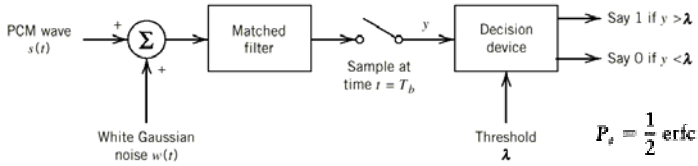
$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\frac{A}{\sqrt{N_0/T_b}}\right)$$

- Since *transmitted signal energy per bit* is defined as $E_b = A^2 T_b$
- Accordingly, we may finally formulate the average probability of symbol error to be:

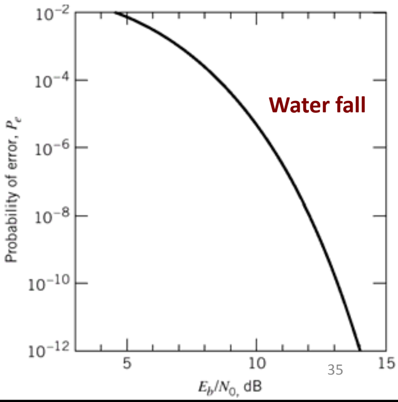
$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

- which shows that *the average probability of symbol error in a binary symmetric channel depends solely on E_b / N_0, the ratio of the transmitted signal energy per bit to the noise spectral density.*

4.3 Error Rate Due to Noise



$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$



- Using the upper bound on the complementary error function, we may correspondingly bound the average probability of symbol error for the PCM receiver

$$P_e < \frac{\exp(-E_b/N_0)}{2\sqrt{\pi E_b/N_0}}$$

- The PCM receiver therefore exhibits an ***exponential*** improvement in the average probability of symbol error with increase in E_b/N_0

4.3 Error Rate: Example

Q) A binary PCM system using polar NRZ signaling operates just above the error threshold with an average probability of error equal to 10^{-5}. Suppose that the signaling rate is doubled. Find the new value of the average probability of error. You may use Table A6.6 to evaluate the complementary error function.

A) For a binary PCM system, with NRZ signaling, the average probability of error is

$$P_e = \frac{1}{2}\,erfc\left(\sqrt{\frac{E_b}{N_o}}\right)$$

The signal energy per bit is $E_b = A^2 T_b$, where A is the pulse amplitude and T_b is the bit duration.

If the signaling rate is doubled, the bit duration T_b is reduced by half.

➔ E_b is reduced by half

4.3 Error Rate: Example (cont)

- Let $u = \sqrt{\frac{E_b}{N_o}}$, then for $P_e = 10^{-6} = \frac{1}{2} erfc(u)$, we get u=3.3
- Now when the signaling rate is doubled, the new value of P_e is:

$$P_e = \frac{1}{2} erfc\left(\frac{u}{\sqrt{2}}\right)$$
$$= \frac{1}{2} erfc(2.33) = 10^{-3}$$

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