Chapter 8. Linear Systems with Random Inputs

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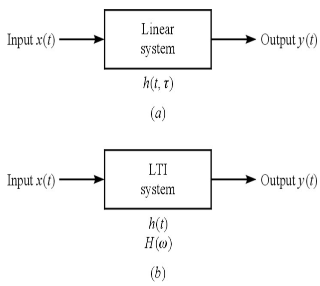
8.2 Random signal response of linear systems

Linear System $\quad y(t) = \int_{-\infty}^{\infty} x(\xi) h(t,\xi) d\xi$

$\delta(t-\xi) \quad \rightarrow \quad h(t,\xi) \qquad$ impulse response

Linear Time-Invariant System (LTI system)



$$y(t) = \int_{-\infty}^{\infty} x(\xi) h(t-\xi) d\xi = \int_{-\infty}^{\infty} h(\xi) x(t-\xi) d\xi$$

$y(t) = x(t) * h(t) = h(t) * x(t) \qquad$ convolution integral

$$Y(\omega) = X(\omega) H(\omega)$$

$$x(t) = e^{j\omega t} \quad \Rightarrow \quad \frac{y(t)}{x(t)} = \frac{\int_{-\infty}^{\infty} h(\xi) e^{j\omega(t-\xi)} d\xi}{e^{j\omega t}} = \int_{-\infty}^{\infty} h(\xi) e^{-j\omega\xi} d\xi = H(\omega)$$

8.2 Random signal response of linear systems

$X(t)$ -- w.s.s. random input $\qquad Y(t)=\int_{-\infty}^{\infty} h(\xi)X(t-\xi)d\xi$

$$E[Y(t)]=E[\int_{-\infty}^{\infty} h(\xi)X(t-\xi)d\xi]=\int_{-\infty}^{\infty} h(\xi)E[X(t-\xi)]d\xi$$

$$=\overline{X}\int_{-\infty}^{\infty} h(\xi)d\xi=\overline{Y}$$

$$R_{YY}(t,t+\tau)=E[Y(t)Y(t+\tau)]$$

$$=E[\int_{-\infty}^{\infty} h(\xi_1)X(t-\xi_1)d\xi_1\int_{-\infty}^{\infty} h(\xi_2)X(t+\tau-\xi_2)d\xi_2]$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t-\xi_1)X(t+\tau-\xi_2)]h(\xi_1)h(\xi_2)d\xi_1 d\xi_2$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\xi_1-\xi_2)h(\xi_1)h(\xi_2)d\xi_1 d\xi_2$$

$X(t)$ w.s.s. $\Rightarrow$ $Y(t)$ w.s.s.

8.2 Random signal response of linear systems

$$R_{YY}(\tau)=\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} R_{XX}(\tau+\xi_1-\xi_2)h(\xi_1)d\xi_1]h(\xi_2)d\xi_2$$

Let $\int_{-\infty}^{\infty} R_{XX}(\tau+\xi_1-\xi_2)h(\xi_2)d\xi_2=g(\tau+\xi_1)=R_{XX}(\tau+\xi_1)*h(\tau+\xi_1)$

$$\Rightarrow R_{YY}(\tau)=\int_{-\infty}^{\infty} g(\tau+\xi_1)h(\xi_1)d\xi_1=g(\tau)*h(-\tau)$$

$$\boxed{\Rightarrow R_{YY}(\tau)=R_{XX}(\tau)*h(\tau)*h(-\tau)}$$

$$\boxed{E[Y(t)^2]=R_{YY}(0)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\xi_1-\xi_2)h(\xi_1)h(\xi_2)d\xi_1 d\xi_2}$$

Example 8.2-1: white noise $X(t)$ $\qquad R_{XX}(\tau)=(N_0/2)\delta(\tau)$

$$E[Y(t)^2]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(N_0/2)\delta(\xi_1-\xi_2)h(\xi_1)h(\xi_2)d\xi_1 d\xi_2$$

$$=(N_0/2)\int_{-\infty}^{\infty} h(\xi_2)^2 d\xi_2$$

8.2 Random signal response of linear systems

$$R_{XY}(t,t+\tau)=E[X(t)Y(t+\tau)]=E[X(t)\int_{-\infty}^{\infty}h(\xi)X(t+\tau-\xi)d\xi]$$
$$=\int_{-\infty}^{\infty}E[X(t)X(t+\tau-\xi)]h(\xi)d\xi$$
$$=\int_{-\infty}^{\infty}R_{XX}(\tau-\xi)h(\xi)d\xi$$
$$=R_{XX}(\tau)*h(\tau)=R_{XY}(\tau)$$

$$R_{YX}(\tau)=R_{XY}(-\tau)=R_{XX}(-\tau)*h(-\tau)=R_{XX}(\tau)*h(-\tau)$$
$$=\int_{-\infty}^{\infty}R_{XX}(\tau-\xi)h(-\xi)d\xi$$

$X(t)$ w.s.s. $\Rightarrow$ $X(t)$ & $Y(t)$ jointly w.s.s.

$$R_{YY}(\tau)=R_{XY}(\tau)*h(-\tau)=R_{YX}(\tau)*h(\tau)$$

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8.2 Random signal response of linear systems

Example 8.2-2: white noise $X(t)$ $\quad R_{XX}(\tau)=(N_0/2)\delta(\tau)$

$$R_{XY}(\tau)=R_{XX}(\tau)*h(\tau)=\int_{-\infty}^{\infty}R_{XX}(\tau-\xi)h(\xi)d\xi$$
$$=\int_{-\infty}^{\infty}(N_0/2)\delta(\tau-\xi)h(\xi)d\xi=(N_0/2)h(\tau)$$

$$R_{YX}(\tau)=R_{XY}(-\tau)=(N_0/2)h(-\tau)$$

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8.4 Spectral characteristics of system response

$$R_{XY}(\tau) = R_{XX}(\tau) * h(\tau) \qquad S_{XY}(\omega) = S_{XX}(\omega)H(\omega)$$

$$R_{YX}(\tau) = R_{XX}(\tau) * h(-\tau) \quad S_{YX}(\omega) = S_{XX}(\omega)H(-\omega) = S_{XX}(\omega)H(\omega)^*$$

$$R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau) = R_{XX}(\tau) * h(\tau) * h(-\tau)$$

$$\boxed{S_{YY}(\omega) = S_{XY}(\omega)H(\omega)^* = S_{XX}(\omega)H(\omega)H(\omega)^* = S_{XX}(\omega)\left|H(\omega)\right|^2}$$

$$h(\tau) \quad \xleftrightarrow{FT} \quad H(\omega)$$

$$h(\tau) \text{ real} \;\Rightarrow\; h(-\tau) \quad \xleftrightarrow{FT} \quad H(-\omega) = H(\omega)^*$$

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8.4 Spectral characteristics of system response

average power $$P_{YY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\left|H(\omega)\right|^2 d\omega$$

Example 8.4-1: Find the average Power at the output, P_{YY}

$$S_{XX}(\omega) = \frac{N_0}{2} \qquad H(\omega) = \frac{1}{1+(j\omega L/R)}$$

$$S_{YY}(\omega) = S_{XX}(\omega)\left|H(\omega)\right|^2 = \frac{N_0/2}{1+(\omega L/R)^2}$$

$$P_{YY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)d\omega = \frac{N_0}{4\pi}\int_{-\infty}^{\infty}\frac{1}{1+(\omega L/R)^2}d\omega$$

$$= \frac{N_0}{4\pi}\int_{-\pi/2}^{\pi/2}\frac{1}{1+\tan^2\theta}\frac{R}{L}\sec^2\theta d\theta = \frac{N_0 R}{4\pi L}\int_{-\pi/2}^{\pi/2} d\theta = \frac{N_0 R}{4L}$$

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8.4 Spectral characteristics of system response

Another method to find the average power:

$$h(t) = (R/L)u(t)e^{-Rt/L} \quad \xleftrightarrow{FT} \quad H(\omega) = \frac{1}{1+(j\omega L/R)}$$

For white noise , $\mathrm{R}_{XX}(\tau) = \frac{N_0}{2}\delta(\tau)$

$$\boxed{P_{YY} = E[Y(t)^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\xi_1 - \xi_2)h(\xi_1)h(\xi_2)d\xi_1 d\xi_2}$$

$$P_{YY} = \frac{N_0}{2}\int_{-\infty}^{\infty} h(t)^2\,dt = \frac{N_0}{2}\int_0^{\infty}(R/L)^2 e^{-2Rt/L}dt = \frac{N_0 R}{4L}e^{-2Rt/L}\Big|_0^{\infty} = \frac{N_0 R}{4L}$$

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