Chapter 3. Operations on One Random Variable - Expectation

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1. Expectation
2. Moments
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5. Computer Generation of One Random Variable

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3.1 Expectation

expectation of X = mean of X = average of X

$$E[X] = \overline{X} = \int_{-\infty}^{\infty} x f_X(x)dx \qquad \text{continuous r.v.}$$

$$E[X] = \overline{X} = \sum_{i=1}^{N} x_i P(x_i) \qquad \text{discrete r.v.}$$

$$f_X(x+a) = f_X(-x+a), \forall x \quad \Rightarrow \quad E[X] = a$$

X r.v. $\Rightarrow$ $Y = g(X)$ r.v. Ex: $Y = g(X) = X^2$

$$P(X=0) = P(X=-1) = P(X=1) = \frac{1}{3} \qquad P(Y=0) = \frac{1}{3} \qquad P(Y=1) = \frac{2}{3}$$

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3.1 Expectation

expectation of a function of a r.v. X

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx \qquad \text{continuous r.v.}$$

$$E[g(X)] = \sum_{i=1}^{N} g(x_i) P(x_i) \qquad \text{discrete r.v.}$$

Ex 3.1-3:

$$f_V(v) = \begin{cases} \frac{2}{5} v e^{-\frac{v^2}{5}}, & v \ge 0 \\ 0, & v < 0 \end{cases} \qquad \text{Rayleigh r.v.}$$

$$\frac{v^2}{5} = \xi \qquad\qquad u' = e^{-\xi}, \quad v = \xi$$

$$E[V^2] = \int_0^{\infty} \frac{2}{5} v^3 e^{-\frac{v^2}{5}} dv = 5\int_0^{\infty} \xi e^{-\xi} d\xi = -5\xi e^{-\xi}\Big|_{\xi=0}^{\infty} + 5\int_0^{\infty} e^{-\xi} d\xi = 5$$

3.1 Expectation

conditional expectation of a r.v. X

$$E[X|B] = \int_{-\infty}^{\infty} x f_X(x|B) dx \qquad \text{continuous r.v.}$$

$$E[X|B] = \sum_{i=1}^{N} x_i P(x_i|B) \qquad \text{discrete r.v.}$$

Ex: $B = \{X \le b\}$

$$f_X(x|X \le b) = \begin{cases} \frac{f_X(x)}{\int_{-\infty}^{b} f_X(x) dx}, & x < b \\ 0, & x \ge b \end{cases} \qquad E[X|X \le b] = \frac{\int_{-\infty}^{b} x f_X(x) dx}{\int_{-\infty}^{b} f_X(x) dx}$$

3.2 Moments

n-th moment of a r.v. X

$$m_n = E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)dx$$ continuous r.v.

$$m_n = E[X^n] = \sum_{i=1}^{N} x_i^n P(x_i)$$ discrete r.v.

$m_0 = 1$ $\quad$ $m_1 = \overline{X}$

n-th central moment of a r.v. X

$$\mu_n = E[(X - \overline{X})^n] = \int_{-\infty}^{\infty} (x - \overline{X})^n f_X(x)dx$$

$\mu_0 = 1$ $\quad$ $\mu_1 = E[X - \overline{X}] = E[X] - E[\overline{X}] = \overline{X} - \overline{X} = 0$

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3.2 Moments

properties of expectation:

(1) $E[c] = c$ $\quad$ c -- constant

(2) $E[ag(X) + bh(X)] = aE[g(X)] + bE[h(X)]$ $\quad$ a, b -- constants

PF:

$$E[c] = \int_{-\infty}^{\infty} cf_X(x)dx = c\int_{-\infty}^{\infty} f_X(x)dx = c$$

$$E[ag(X) + bh(X)] = \int_{-\infty}^{\infty} \{ag(x) + bh(x)\} f_X(x)dx$$

$$= a\int_{-\infty}^{\infty} g(x) f_X(x)dx + b\int_{-\infty}^{\infty} h(x) f_X(x)dx = aE[g(X)] + bE[h(X)]$$

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3.2 Moments

variance of a r.v. X

$$\sigma_X^2 = \mu_2 = E[(X-\bar{X})^2] = E[X^2 - 2\bar{X}X + \bar{X}^2]$$

$$= E[X^2] - 2\bar{X}E[X] + \bar{X}^2 = m_2 - m_1^2$$

standard deviation of a r.v. $X = \sigma_X \; (\geq 0)$

skewness of a r.v. $X = \dfrac{\mu_3}{\sigma_X^3}$

$f_X(x)$ symmetric about $x = \bar{X} \;\Rightarrow\; \mu_3 = 0$

Ex 3.2-1 & Ex3.2-2:

exponential r.v.

$$f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{x-a}{b}}, & x > a \\ 0, & x < a \end{cases}$$

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3.2 Moments

$$m_1 = E[X] = \int_a^\infty x \frac{1}{b} e^{-\frac{x-a}{b}} dx = a + b$$

$$m_2 = E[X^2] = \int_a^\infty x^2 \frac{1}{b} e^{-\frac{x-a}{b}} dx = (a+b)^2 + b^2$$

$$\sigma_X^2 = \mu_2 = m_2 - m_1^2 = b^2$$

$$m_3 = E[X^3] = \int_a^\infty x^3 \frac{1}{b} e^{-\frac{x-a}{b}} dx = a^3 + 3a^2b + 6ab^2 + 6b^3$$

$$\mu_3 = E[(X-\bar{X})^3] = E[X^3 - 3X^2\bar{X} + 3X\bar{X}^2 - \bar{X}^3] = m_3 - 3m_1m_2 + 3m_1^2m_1 - m_1^3$$

$$= a^3 + 3a^2b + 6ab^2 + 6b^3 - 3(a+b)\{(a+b)^2 + b^2\} + 2(a+b)^3 = 2b^3$$

skewness of a r.v. $X = \dfrac{\mu_3}{\sigma_X^3} = \dfrac{2b^3}{b^3} = 2$

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3.2 Moments

Chebychev's inequality $$P[|X-\bar{X}|\geq\varepsilon]\leq\frac{\sigma_X^2}{\varepsilon^2}$$

$$\sigma_X^2=\int_{-\infty}^{\infty}(x-\bar{X})^2 f_X(x)dx\geq\int_{|x-\bar{X}|\geq\varepsilon}(x-\bar{X})^2 f_X(x)dx$$

$$\geq\varepsilon^2\int_{|x-\bar{X}|\geq\varepsilon} f_X(x)dx=\varepsilon^2 P[|X-\bar{X}|\geq\varepsilon]$$

Markov's inequality

$$P[X<0]=0 \quad\Rightarrow\quad P[X\geq a]\leq\frac{E[X]}{a} \qquad (a>0)$$

Ex 3.2-3:

$$P[|X-\bar{X}|\geq 3\sigma_X]\leq\frac{\sigma_X^2}{9\sigma_X^2}=\frac{1}{9}$$

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3.3 Functions That Give Moments

Characteristic function of r.v. *X*

$$\Phi_X(\omega)=E[e^{j\omega X}]=\int_{-\infty}^{\infty} f_X(x)e^{j\omega x}dx$$

$$f_X(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_X(\omega)e^{-j\omega x}d\omega$$ Fourier transform

$$|\Phi_X(\omega)|\leq\int_{-\infty}^{\infty}|f_X(x)||e^{j\omega x}|dx\leq\int_{-\infty}^{\infty} f_X(x)dx=1=\Phi_X(0)$$

$$\left.\frac{d^n\Phi_X(\omega)}{d\omega^n}\right|_{\omega=0}=\int_{-\infty}^{\infty} f_X(x)j^n x^n e^{j\omega x}dx\Big|_{\omega=0}=j^n\int_{-\infty}^{\infty} f_X(x)x^n dx=j^n E[X^n]$$

$$m_n=(-j)^n\left.\frac{d^n\Phi_X(\omega)}{d\omega^n}\right|_{\omega=0}$$

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3.3 Functions That Give Moments

Moment generating function of r.v. X

$$M_X(v) = E[e^{vX}] = \int_{-\infty}^{\infty} f_X(x) e^{vx} dx$$

$$\left.\frac{d^n M_X(v)}{dv^n}\right|_{v=0} = \int_{-\infty}^{\infty} f_X(x) x^n e^{vx} dx \Big|_{v=0} = \int_{-\infty}^{\infty} f_X(x) x^n dx = m_n$$

Ex 3.3-1 & Ex 3.3-2:

$$f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{x-a}{b}}, & x > a \\ 0, & \text{x} < a \end{cases}$$

3.3 Functions That Give Moments

$$\Phi_X(\omega) = E[e^{j\omega X}] = \frac{1}{b} e^{\frac{a}{b}} \int_a^{\infty} e^{-(\frac{1}{b} - j\omega)x} dx = \frac{1}{b} e^{\frac{a}{b}} \left. \frac{e^{-(\frac{1}{b} - j\omega)x}}{-(\frac{1}{b} - j\omega)} \right|_{x=a}^{\infty}$$

$$= \frac{1}{b} e^{\frac{a}{b}} \frac{e^{-(\frac{1}{b} - j\omega)a}}{(\frac{1}{b} - j\omega)} = \frac{e^{j\omega a}}{1 - j\omega b}$$

$$\frac{d\Phi_X(\omega)}{d\omega} = \frac{jae^{j\omega a}(1 - j\omega b) + e^{j\omega a} jb}{(1 - j\omega b)^2}$$

$$M_X(v) = E[e^{vX}] = \frac{e^{va}}{1 - vb}$$

$$\frac{dM_X(v)}{dv} = \frac{ae^{va}(1 - vb) + e^{va} b}{(1 - vb)^2}$$

$$m_1 = (-j) \left. \frac{d\Phi_X(\omega)}{d\omega} \right|_{\omega=0} = a + b \qquad m_1 = \left. \frac{dM_X(v)}{dv} \right|_{v=0} = a + b$$

3.3 Functions That Give Moments

Chernoff's inequality

Ex 3.3-3:

$v > 0$

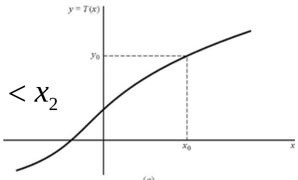
$$P[X \geq a] = \int_a^\infty f_X(x)dx = \int_{-\infty}^\infty f_X(x)u(x-a)dx$$
$$\leq \int_{-\infty}^\infty f_X(x)e^{v(x-a)}dx = e^{-va}M_X(v)$$

3.4 Transformations of a Random Variable

$Y = T(X)$ $\quad\quad$ $f_X(x)$ given $\Rightarrow$ $f_Y(y) = ?$

monotone increasing $\Leftrightarrow$

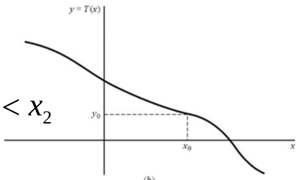
$T(x_1) < T(x_2)$ for any $x_1 < x_2$



(a)

monotone decreasing $\Leftrightarrow$

$T(x_1) > T(x_2)$ for any $x_1 < x_2$



(b)

3.4 Transformations of a Random Variable

Assume monotone increasing $T(\bullet)$ $\quad Y = T(X)$

$$F_Y(y_0) = P[Y \le y_0] = P[X \le x_0] = F_X(x_0)$$

$$\int_{-\infty}^{y_0} f_Y(y)dy = \int_{-\infty}^{T^{-1}(y_0)} f_X(x)dx$$

$$f_Y(y_0) = f_X[T^{-1}(y_0)]\frac{dT^{-1}(y_0)}{dy_0}$$

$$f_Y(y) = f_X[T^{-1}(y)]\frac{dT^{-1}(y)}{dy} = f_X(x)\frac{dx}{dy}$$

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3.4 Transformations of a Random Variable

Assume monotone decreasing $T(\bullet)$ $\quad Y = T(X)$

$$F_Y(y_0) = P[Y \le y_0] = P[X \ge x_0] = 1 - F_X(x_0)$$

$$f_Y(y) = -f_X(x)\frac{dx}{dy}$$

monotone $T(\bullet)$ $\Rightarrow$ $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = f_X(x)\frac{1}{\left|\frac{dy}{dx}\right|}$

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3.4 Transformations of a Random Variable

Ex 3.4-1: $f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x-a_X)^2}{2\sigma_X^2}}$

$Y = T(X) = aX + b$ $\quad f_Y(y) = ?$

monotone $T(\bullet) \quad \Rightarrow$

$$f_Y(y) = f_X(\frac{y-b}{a})\left|\frac{1}{a}\right| = \frac{1}{\sqrt{2\pi a^2\sigma_X^2}} e^{-\frac{(\frac{y-b}{a}-a_X)^2}{2\sigma_X^2}} = \frac{1}{\sqrt{2\pi a^2\sigma_X^2}} e^{-\frac{[y-(aa_X+b)]^2}{2a^2\sigma_X^2}}$$

Y -- gaussian r.v.

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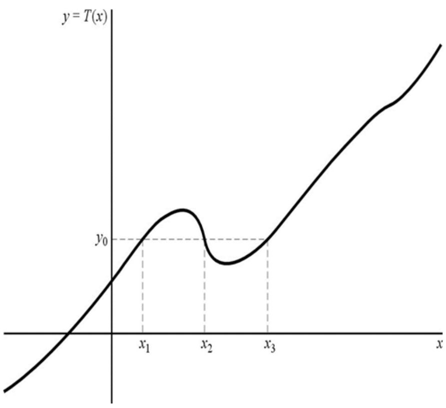
3.4 Transformations of a Random Variable

nonmonotone $T(\bullet)$

$Y = T(X)$

$$f_Y(y) = \sum_n \frac{f_X(x_n)}{\left|\frac{dT(x)}{dx}\right|_{x=x_n}}$$



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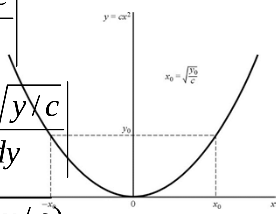
3.4 Transformations of a Random Variable

Ex 3.4-2:

$Y = T(X) = cX^2$ nonmonotone

$$f_Y(y) = f_X(\sqrt{y/c})\left|\frac{d\sqrt{y/c}}{dy}\right| + f_X(-\sqrt{y/c})\left|\frac{-d\sqrt{y/c}}{dy}\right|$$

$$= \frac{f_X(\sqrt{y/c}) + f_X(-\sqrt{y/c})}{2\sqrt{cy}}, \qquad y \geq 0$$



3.4 Transformations of a Random Variable

Transformation of a discrete r.v.

$Y = T(X)$ $\qquad P(Y = y) = P(X \in \{x : T(x) = y\})$

Ex 3.4-3:

$$P(X = -1) = 0.1, \quad P(X = 0) = 0.3, \quad P(X = 1) = 0.4, \quad P(X = 2) = 0.2$$

$$Y = T(X) = 2 - X^2 + \frac{X^3}{3}$$

$$T(-1) = \frac{2}{3}, \quad T(0) = 2, \quad T(1) = \frac{4}{3}, \quad T(2) = \frac{2}{3}$$

$$P(Y = \frac{2}{3}) = 0.3, \quad P(Y = \frac{4}{3}) = 0.4, \quad P(Y = 2) = 0.3$$

3.5 Computer Generation of One Random Variable

computer simulation

X -- uniform on (0,1)

$$F_X(x) = \begin{cases} 0, & x \le 0 \\ x, & 0 < x \le 1 \\ 1, & x \ge 1 \end{cases}$$

$Y = T(X)$ -- monotone increasing

$$F_Y[y = T(x)] = F_X(x) = x, \quad 0 < x < 1$$

$$y = T(x) = F_Y^{-1}(x), \quad 0 < x < 1$$

Ex 3.5-1:

$$F_Y(y) = 1 - e^{-\frac{y^2}{b}} = x, \quad 0 < x < 1$$

$$y = T(x) = F_Y^{-1}(x) = \sqrt{-b\ln(1-x)}, \quad 0 < x < 1$$

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3.5 Computer Generation of One Random Variable

Ex 3.5-2: X -- uniform on (0,1) $Y = T(X)$

$$F_Y(y) = \begin{cases} 0, & y \le -a \\ 0.5 + \frac{1}{\pi}\sin^{-1}(\frac{y}{a}), & -a < y < a \\ 1, & y \ge a \end{cases}$$

$$y = T(x) = F_Y^{-1}(x) = a\sin[\pi(x - 0.5)], \quad 0 < x < 1$$

Ex 3.5-3: MATLAB program

Rayleigh r.v.

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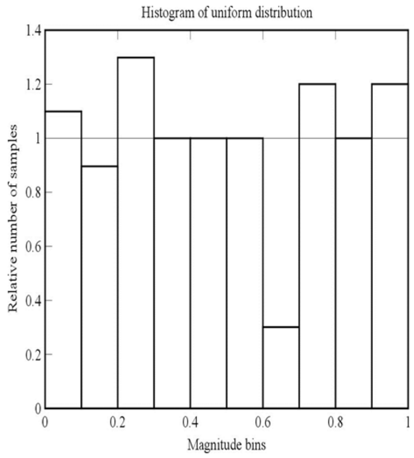
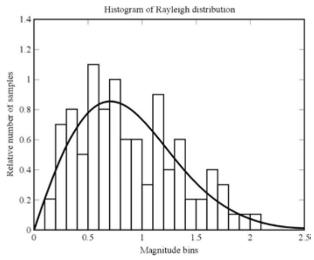
3.5 Computer Generation of One Random Variable





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