EE 213 ELECTRIC CIRCUITS II

Lecture 6
Frequency Selective Circuits

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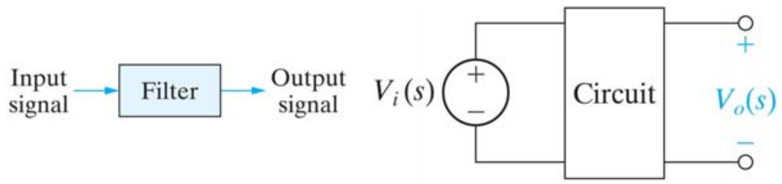
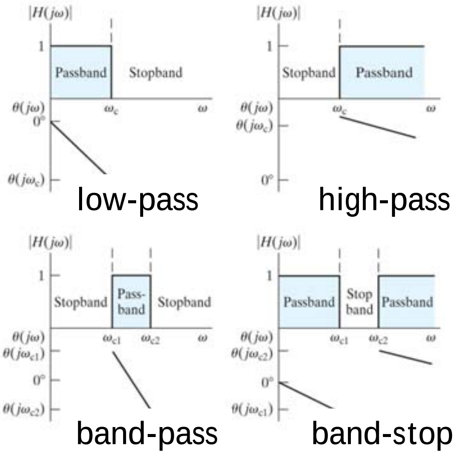
Introduction and Some Terminology



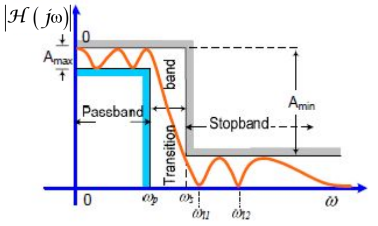
- A **filter** is a frequency selective circuit whose output is formed by the components of the input in a certain frequency band.
- The range of frequencies passed by the filter is called the **passband**; the range of frequencies that are blocked by the filter is called the **stopband**.
- The type of the filter is identified based on its frequency response function $\mathcal{H}(j\omega) = \mathcal{V}_o(j\omega)/\mathcal{V}_i(j\omega)$.

Types of Filters



- ➢ There are basically four types of filters: **low-pass, high-pass, band-pass and band-stop or band-reject**.

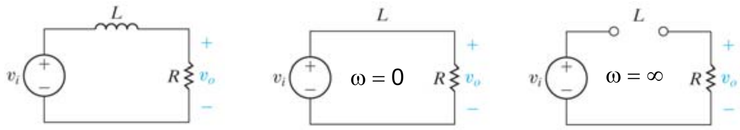
Ideal versus Practial Filters



A Practical Low-Pass Filter

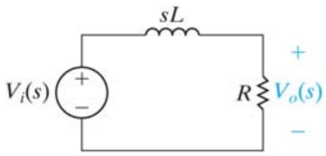
- ➢ Practical filters cannot change their behavior abruptly between the passband and the stopband.
- ➢ Design specifications need to be formulated in terms of the range of allowable gains in the pass/stop-bands and the range of the transition interval.

Low-Pass Filters: Series RL Circuit



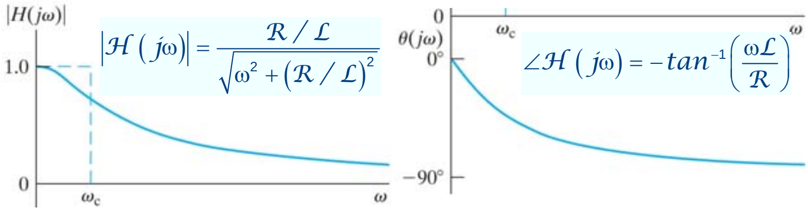
- Consider the series RL circuit with a sinusoidal input of frequency ω.
- For $\omega \ll \mathcal{R}/\mathcal{L}$, the inductor is like short circuit if compared to $\mathcal{R}$ (i.e. $\omega\mathcal{L} \ll \mathcal{R}$) $\Rightarrow v_o \approx v_i$.
- For $\omega \gg \mathcal{R}/\mathcal{L}$, the inductor is like open circuit if compared to $\mathcal{R} \Rightarrow$ (i.e. $\omega\mathcal{L} \gg \mathcal{R}$) $v_o \approx 0$.
- The phase shift in the output decreases from 0 to -90° as ω increases from 0 to ∞.
- This circuit behaves as a low-pass filter.

Series RL Circuit: Quantitative Analysis



$$\mathcal{H}(s) = \frac{\mathcal{R}}{s\mathcal{L}+\mathcal{R}}; \quad \mathcal{H}(j\omega) = \frac{\mathcal{R}/\mathcal{L}}{j\omega + \mathcal{R}/\mathcal{L}}$$

- The magnitude and phase of the frequency response can be obtained and sketched as follows:



$$|\mathcal{H}(j\omega)| = \frac{\mathcal{R}/\mathcal{L}}{\sqrt{\omega^2 + (\mathcal{R}/\mathcal{L})^2}}$$

$$\angle\mathcal{H}(j\omega) = -tan^{-1}\left(\frac{\omega\mathcal{L}}{\mathcal{R}}\right)$$

Series RL Circuit: Cut-Off Frequency

- The frequency at which the magnitude of the transfer function is decreased from its maximum value by a factor of 1/√2 is called the **cut-off frequency**.

$$\left|\mathcal{H}\left(j\omega_c\right)\right| = \frac{1}{\sqrt{2}}\mathcal{H}_{max} \approx\sim 0.707\mathcal{H}_{max}$$

- ω_c is also called the **half-power frequency**.

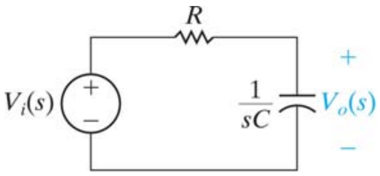
$$\left|\mathcal{V}_o\left(j\omega_c\right)\right| = \left|\mathcal{H}\left(j\omega_c\right)\right|\left|\mathcal{V}_i\right| = \frac{1}{\sqrt{2}}\mathcal{H}_{max}\left|\mathcal{V}_i\right|$$

$$\Rightarrow \mathcal{P}_o\left(\omega_c\right) = \frac{1}{2}\left(\frac{1}{\sqrt{2}}\mathcal{H}_{max}\left|\mathcal{V}_i\right|\right)^2 / \mathcal{R} = \frac{1}{2}\underbrace{\left(\frac{1}{2}\mathcal{H}_{max}^2\left|\mathcal{V}_i\right|^2 / \mathcal{R}\right)}_{\mathcal{P}_o^{max}}$$

- The cut-off frequency for the series RL circuit is:

$$\omega_c = \mathcal{R} / \mathcal{L}$$

Low-Pass Filters: Series RC Circuit



$$\mathcal{H}\left(s\right) = \frac{1}{s\mathcal{R}C + 1}$$

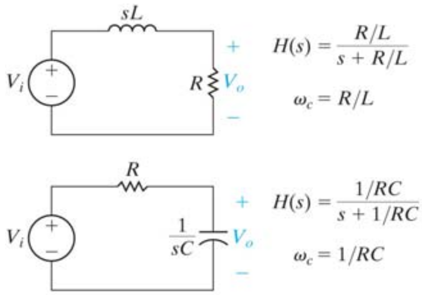
- The magnitude and phase of the frequency response are obtained as follows:

$$\left|\mathcal{H}\left(j\omega\right)\right| = \frac{1}{\sqrt{\left(\omega\mathcal{R}C\right)^2 + 1}}; \quad \angle\mathcal{H}\left(j\omega\right) = -tan^{-1}\left(\omega\mathcal{R}C\right)$$

- The cut-off frequency is given by:

$$\omega_c = \frac{1}{\mathcal{R}C}$$

Low-Pass Filters: General Form



- The transfer function of a **first-order** low-pass filter is:

$$\mathcal{H}(s) = \frac{\omega_c}{s + \omega_c}$$

- This transfer function can also be scaled as $k\mathcal{H}(s)$.

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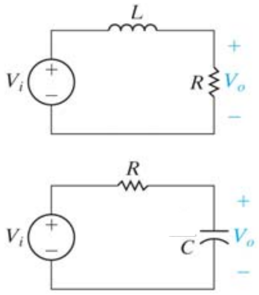
Frequency and Time Domain Relation



$$\tau = \mathcal{L} / \mathcal{R}$$

$$\mathcal{V}_o = \mathcal{V}_o(\infty) + (\mathcal{V}_o(0) - \mathcal{V}_o(\infty))\, exp(-t/\tau)$$

$$\tau = \mathcal{RC}$$

- The time constant is related to the cut-off frequency as

$$\tau = 1/\omega_c$$

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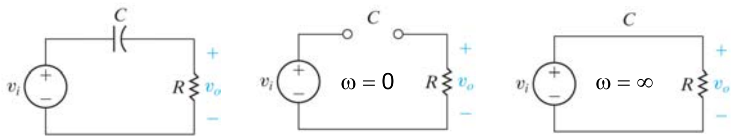
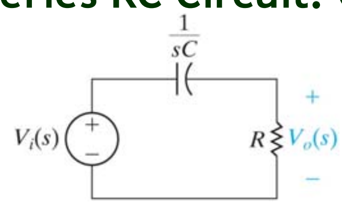
High-Pass Filters: Series RC Circuit



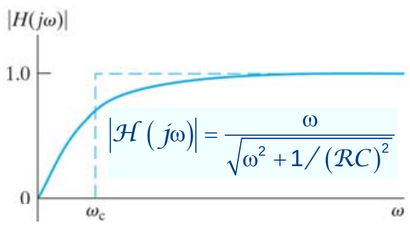
- Consider the series RC circuit with a sinusoidal input of frequency ω.
- For $\omega << 1/(RC)$, the capacitor is like open circuit if compared to R (i.e. $1/(\omega C) >> R$) $\Rightarrow v_o \approx 0$.
- For $\omega >> 1/(RC)$, the capacitor is like short circuit if compared to $R \Rightarrow$ (i.e. $1/(\omega C) << R$) $v_o \approx v_i$.
- The phase shift in the output decreases from 90° to 0 as ω increases from 0 to ∞.
- This circuit behaves as a high-pass filter.

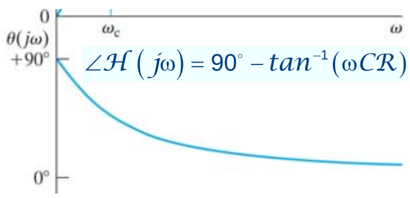
Series RC Circuit: Quantitative Analysis



$$\mathcal{H}(s) = \frac{sCR}{sCR+1}; \quad \mathcal{H}(j\omega) = \frac{j\omega CR}{j\omega CR+1}$$

- The magnitude and phase of the frequency response can be obtained and sketched as follows:





Series RC Circuit: Cut-Off Frequency

- Recall that the cut-off frequency is defined by the following constraint:

$$\left|\mathcal{H}(j\omega_c)\right| = \frac{1}{\sqrt{2}}\mathcal{H}_{max}$$

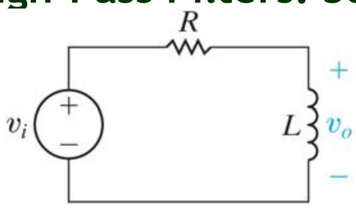
- Since the maximum magnitude is still one, we have:

$$\left|\mathcal{H}(j\omega_c)\right| = \frac{\omega_c}{\sqrt{\omega_c^2 + 1/(\mathcal{R}\mathcal{C})^2}} = \frac{1}{\sqrt{2}} \Rightarrow \frac{\omega_c^2}{\omega_c^2 + 1/(\mathcal{R}\mathcal{C})^2} = \frac{1}{2}$$

- Hence the cut-off frequency for the series RC circuit is:

$$\omega_c = 1/(\mathcal{R}\mathcal{C})$$

High-Pass Filters: Series RL Circuit



$$\mathcal{H}(s) = \frac{s}{s + \mathcal{R}/\mathcal{L}}$$

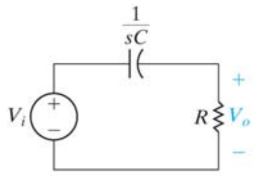
- The magnitude and phase of the frequency response are obtained as follows:

$$\left|\mathcal{H}(j\omega)\right| = \frac{\omega}{\sqrt{\omega^2 + (\mathcal{R}/\mathcal{L})^2}}; \quad \angle\mathcal{H}(j\omega) = 90^\circ - tan^{-1}(\omega\mathcal{L}/\mathcal{R})$$

- The cut-off frequency is given by:

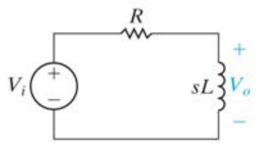
$$\omega_c = \frac{\mathcal{R}}{\mathcal{L}}$$

High-Pass Filters: General Form



$$H(s) = \frac{s}{s + 1/RC}$$

$$\omega_c = 1/RC$$



$$H(s) = \frac{s}{s + R/L}$$

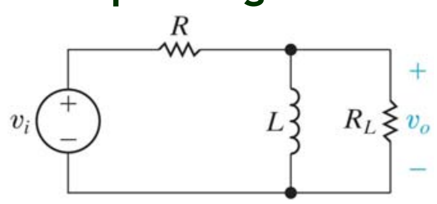
$$\omega_c = R/L$$

- The transfer function of a **first-order** high-pass filter is:

$$\mathcal{H}(s) = \frac{s}{s + \omega_c}$$

- This transfer function can also be scaled as $k\mathcal{H}(s)$.

Example: High-Pass Filter with a Load



$$\mathcal{H}(s) = \frac{sLR_L}{sL(R + R_L) + RR_L}$$

- Determine the type of the filter: high-pass
- Find the cut-off frequency:

$$\mathcal{H}(s) = \underbrace{\frac{R_L}{R + R_L}}_{k} \frac{s}{s + \frac{RR_L}{L(R + R_L)}} \Rightarrow \omega_c = \frac{RR_L}{L(R + R_L)}$$

- The gain of the filter decreases with decreasing load.

Band-Pass and Band-Stop Filters

- These are filters that pass inputs within a certain frequency interval, while filtering out signals whose frequencies lie outside the pass-band.
- Band-stop filters perform a complementary function, i.e. they filter out (pass) signals whose frequencies lie within (outside) a certain frequency interval.
- Both filters are characterized by five parameters. Only two of these can be specified independently.

Cut-off Frequencies	:	ω_{c1}, ω_{c2}
Center (Resonant) Frequency	:	$\omega_0 = \sqrt{\omega_{c1}\omega_{c2}}$
Bandwidth	:	$\beta = \omega_{c2} - \omega_{c1}$
Quality Factor	:	$Q = \dfrac{\omega_0}{\beta}$

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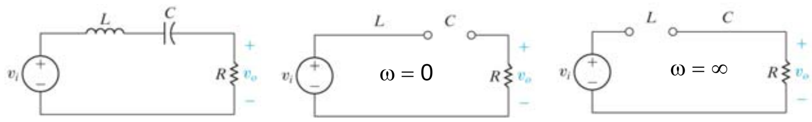
Band-Pass Filters: Series RLC Circuit



- Consider the series RLC circuit with a sinusoidal input of frequency ω.
- For $\omega = 0$, the inductor/capacitor behave as short/open circuit respectively $\Rightarrow v_o = 0$.
- For $\omega = \infty$, the inductor/capacitor behave as open/short circuit respectively $\Rightarrow v_o = 0$.
- At resonance inductor + capacitor behaves as short circuit $\Rightarrow v_o = v_i$
- This circuit behaves as a band-pass filter.

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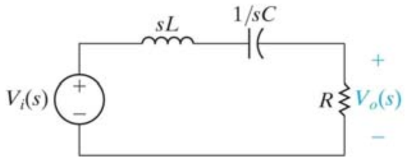
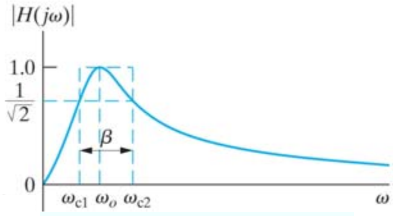
Series RLC Circuit: Quantitative Analysis

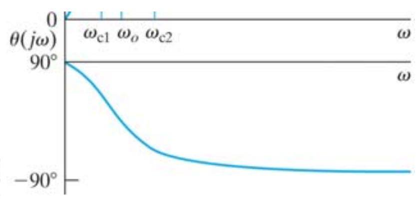


$$\mathcal{H}(s) = \frac{\beta s}{s^2 + \beta s + \omega_0^2}$$

$$\omega_0^2 = 1/(\mathcal{L}\mathcal{C}),\ \ \beta = \mathcal{R}/\mathcal{L}$$

- The magnitude and phase are sketched as follows:





$$|\mathcal{H}(j\omega)| = \frac{\beta\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + \beta^2\omega^2}}$$

$$\angle\mathcal{H}(j\omega) = 90^\circ - tan^{-1}\left(\frac{\beta\omega}{\omega_0^2 - \omega^2}\right)$$

Series RLC Circuit: Cut-off Frequencies

- Recall that at the cut-off frequencies we have:

$$|\mathcal{H}(j\omega_c)| = \frac{1}{\sqrt{\frac{1}{\beta^2}\left(\frac{\omega_0^2}{\omega_c} - \omega_c\right)^2 + 1}} = \frac{1}{\sqrt{2}}\mathcal{H}_{max} = \frac{1}{\sqrt{2}}$$

- This implies that the cut-off frequencies satisfy:

$$\frac{\omega_0^2}{\omega_c} - \omega_c = \pm\beta \Rightarrow \omega_c^2 \pm \beta\omega_c - \omega_0^2 = 0$$

- For + and - sign, we obtain –respectively- the smaller and the larger cut-off frequencies as:

$$\omega_{c1} = -\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_0^2},\quad \omega_{c2} = \frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_0^2}$$

- One can confirm that: $\omega_0 = \sqrt{\omega_{c1} \cdot \omega_{c2}}$

Series RLC Circuit: Bandwidth and QF

- The bandwidth of the series RLC circuit is given by:

$$BW = \omega_{c2} - \omega_{c1} = \beta = \frac{\mathcal{R}}{\mathcal{L}}$$

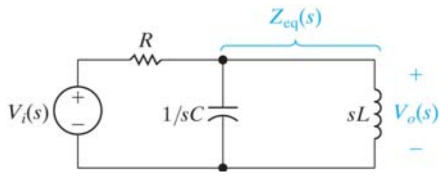
- The **Quality Factor** is the ratio of the center frequency to the bandwidth:

$$\mathcal{Q} = \frac{\omega_0}{\beta} = \frac{1}{\mathcal{R}}\sqrt{\frac{\mathcal{L}}{\mathcal{C}}}$$

- The cut-off frequencies can be expressed in terms of the quality factor and the center frequency as:

$$\omega_{c1} = \omega_0\left(-\frac{1}{2\mathcal{Q}} + \sqrt{\left(\frac{1}{2\mathcal{Q}}\right)^2 + 1}\right), \quad \omega_{c2} = \omega_0\left(\frac{1}{2\mathcal{Q}} + \sqrt{\left(\frac{1}{2\mathcal{Q}}\right)^2 + 1}\right)$$

Band-Pass Filters: Parallel RLC Circuit



$$\mathcal{H}(s) = \frac{\beta s}{s^2 + \beta s + \omega_0^2}$$

$$\omega_0^2 = 1/(\mathcal{LC}), \quad \beta = 1/(\mathcal{RC})$$

- The bandwith is given by:

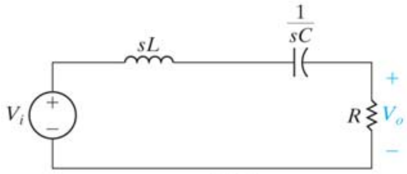
$$BW = \omega_{c2} - \omega_{c1} = \beta = \frac{1}{\mathcal{RC}}$$

- The quality factor is obtained as:

$$\mathcal{Q} = \frac{\omega_0}{\beta} = \mathcal{R}\sqrt{\frac{\mathcal{C}}{\mathcal{L}}}$$

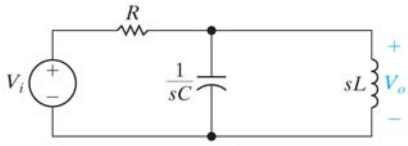
- The cut-off frequencies are expressed in the same way as functions of ω_0 and $\mathcal{Q}$.

Band-Pass Filters: General Form



$$H(s) = \frac{(R/L)s}{s^2 + (R/L)s + 1/LC}$$

$$\omega_o = \sqrt{1/LC} \qquad \beta = R/L$$



$$H(s) = \frac{s/RC}{s^2 + s/RC + 1/LC}$$

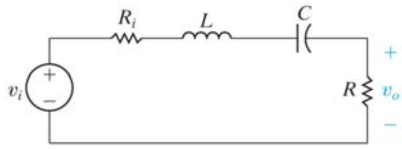
$$\omega_o = \sqrt{1/LC} \qquad \beta = 1/RC$$

2nd-order band-pass filter:

$$\mathcal{H}(s) = \frac{k\beta s}{s^2 + \beta s + \omega_0^2}$$

where k is a scaling gain

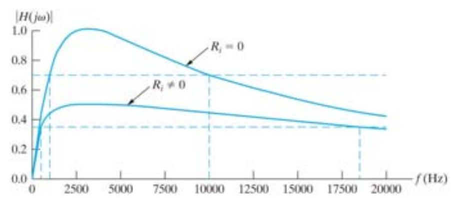
Band-Pass Filter with a Nonideal Source



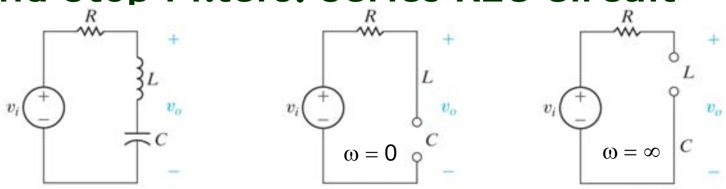
- The transfer function from v_i to v_o is given by:

$$\mathcal{H}(s) = \frac{k\beta s}{s^2 + \beta s + \omega_0^2}; \quad \beta = \frac{\mathcal{R} + \mathcal{R}_i}{\mathcal{L}}, \quad \omega_0 = \frac{1}{\mathcal{LC}}, \quad k = \frac{\mathcal{R}}{\mathcal{R} + \mathcal{R}_i}$$

- $\mathcal{R}_i > 0 \Rightarrow$ increased bandwidth and $k < 1$.

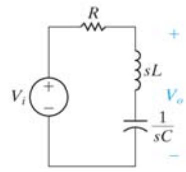


Band-Stop Filters: Series RLC Circuit



- Consider the series RLC circuit with a sinusoidal input of frequency ω.
- For $\omega = 0$ and $\omega = \infty$, inductor + capacitor behaves as open circuit $\Rightarrow v_o = v_i$.
- At resonance, inductor + capacitor behaves as short circuit $\Rightarrow v_o = 0$.
- This circuit behaves as a band-pass filter. Such filters are also referred to as **band-reject or notch** filters.

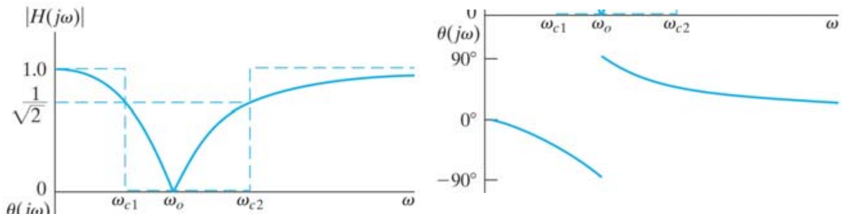
Series RLC Circuit: Quantitative Analysis



$$\mathcal{H}(s) = \frac{s^2 + \omega_0^2}{s^2 + \beta s + \omega_0^2}$$

$$\omega_0^2 = 1/(\mathcal{L}\mathcal{C}), \ \beta = \mathcal{R}/\mathcal{L}$$

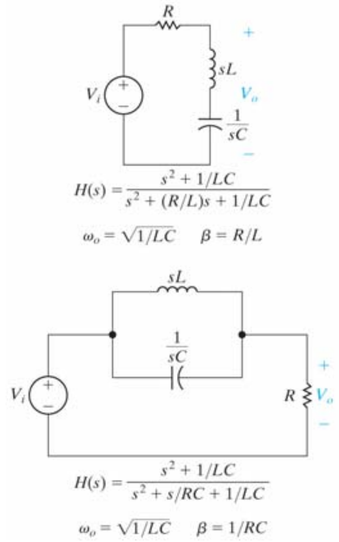
- The magnitude and phase are sketched as follows:



$$|\mathcal{H}(j\omega)| = \frac{|\omega_0^2 - \omega^2|}{\sqrt{(\omega_0^2 - \omega^2)^2 + \beta^2\omega^2}}$$

$$\angle\mathcal{H}(j\omega) = -tan^{-1}\left(\frac{\beta\omega}{\omega_0^2 - \omega^2}\right)$$

Band-Stop Filters: General Form



2nd-order band-stop filter:

$$\mathcal{H}(s) = \frac{k\left(s^2 + \omega_0^2\right)}{s^2 + \beta s + \omega_0^2}$$

where k is a scaling gain.

Filter parameters are same as those of a band-pass filter.

Example: Notch Filter Synthesis

- Design a notch filter to eliminate sinusoidal signals of frequency within 250±150 Hz from an input signal. Use a series RLC circuit with $\mathcal{R} = 100\ \Omega$.
- Choose the center frequency as:

$$\omega_0 = \sqrt{\omega_{c1}\omega_{c2}} = 2\pi\sqrt{f_{c1}f_{c2}} = 2\pi\sqrt{100 \cdot 400} = 400\pi \text{ rad/s}$$

- Set the quality factor to:

$$\mathcal{Q} = \frac{\omega_0}{\beta} = \frac{400\pi}{2\pi(400 - 100)} = \frac{2}{3}$$

- The inductance and capacitance must be chosen as:

$$\mathcal{L} = \frac{\mathcal{R}}{\beta} = \frac{100}{600\pi} \approx 53 \text{ mH}, \quad C = \frac{1}{\omega_0^2 \mathcal{L}} = \frac{\beta}{\omega_0^2 \mathcal{R}} = \frac{600\pi}{(400\pi)^2 \cdot 100} \approx 119 \text{ mF}$$

Bode Plots

- Bode plots or diagrams are graphical representations of the frequency response in logarithmic scale.
- For a given transfer function $\mathcal{H}(s)$, a Bode diagram consists of:
- ❑ Amplitude (magnitude) plot of $\mathcal{H}(j\omega)$; and
- ❑ Phase angle plot of $\mathcal{H}(j\omega)$.

  with respect to the frequency.
- In order to cover a wider frequency range and obtain linear sketches, the frequency axis is represented in logarithmic scale.
- The magnitudes are also displayed in the logarithmic scale, with a particularly defined unit named decibel.

The Decibel Scale

- The amplitude is plotted in units of **decibel (dB)**:

$$\mathcal{A}_{\mathrm{dB}} = 20\,log_{10}\,\mathcal{A}$$

- Below is a table of actual amplitudes and the corresponding values in decibels:

A	A_{dB}	A	A_{dB}
1.00	0	31.62	30
1.41	3	100.00	40
2.00	6	10^3	60
3.16	10	10^4	80
5.62	15	10^5	100
10.00	20	10^6	120

Standard Form for Bode Plots

- Consider a transfer function of the form:

$$\mathcal{H}(s) = \frac{\mathcal{K}(s+z_1)}{s(s+p_1)}$$

where $\mathcal{K}$, z_1 and p_1 are positive real numbers.

- The frequency response is obtained as:

$$\mathcal{H}(j\omega) = \frac{\mathcal{K}(j\omega+z_1)}{j\omega(j\omega+p_1)}$$

- The standard form Bode plots is:

$$\mathcal{H}(j\omega) = \underbrace{\frac{\mathcal{K}z_1}{p_1}}_{\mathcal{K}_0}\frac{(1+j(\omega/z_1))}{j\omega(1+j(\omega/p_1))}$$

Amplitude and Phase in Standard Form

- Consider the standard form:

$$\mathcal{H}(j\omega) = \mathcal{K}_0\frac{(1+j(\omega/z_1))}{j\omega(1+j(\omega/p_1))}$$

- The amplitude is obtained as:

$$|\mathcal{H}(j\omega)| = \mathcal{K}_0\frac{|1+j(\omega/z_1)|}{\omega|1+j(\omega/p_1)|} = \mathcal{K}_0\frac{\sqrt{1+(\omega/z_1)^2}}{\omega\sqrt{1+(\omega/p_1)^2}}$$

- The phase is obtained as:

$$\angle\mathcal{H}(j\omega) = \underbrace{\angle(1+j(\omega/z_1))}_{tan^{-1}(\omega/z_1)} - \underbrace{\angle(1+j(\omega/p_1))}_{tan^{-1}(\omega/p_1)} - \underbrace{\angle(j\omega)}_{90^\circ}$$

Amplitude in Decibels

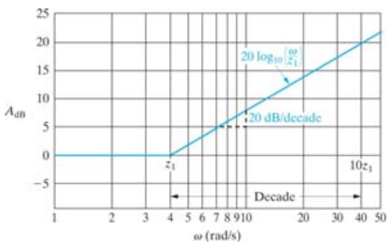
- The amplitude is obtained in the dB scale as:

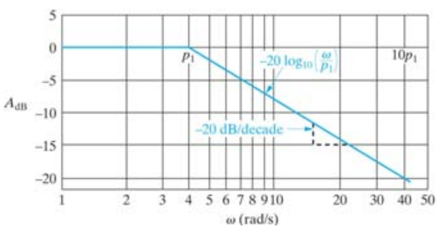
$$\begin{aligned}\mathcal{A}_{\text{dB}} &= 20\,\mathcal{log}_{10}\left|\mathcal{H}(j\omega)\right| \\ &= 20\,\mathcal{log}_{10}\,\mathcal{K}_0 + 20\,\mathcal{log}_{10}\sqrt{1+(\omega/z_1)^2} \\ &\quad -20\,\mathcal{log}_{10}\,\omega - 20\,\mathcal{log}_{10}\sqrt{1+(\omega/p_1)^2}\end{aligned}$$

- Other factors in the numerator/denominator lead to additional terms to be added/subtracted.
- Straight-line plots are obtained by adding the contribution of each term, with the approximation:

$$20\,\mathcal{log}_{10}\sqrt{1+(\omega/\omega_c)^2} \approx \begin{cases} 0 & ,\ \omega \le \omega_c \\ 20\,\mathcal{log}_{10}(\omega/\omega_c) & ,\ \omega > \omega_c \end{cases}$$

Straight-Line Approximations





- Above the **corner frequency** ω_c, the approximation is a straight line on a log frequency scale:

$$\underbrace{20\,\mathcal{log}_{10}(\omega/\omega_c)}_{y} = \underbrace{20}_{a}\,\underbrace{\mathcal{log}_{10}\,\omega}_{x} - \underbrace{20\,\mathcal{log}_{10}\,\omega_c}_{b}$$

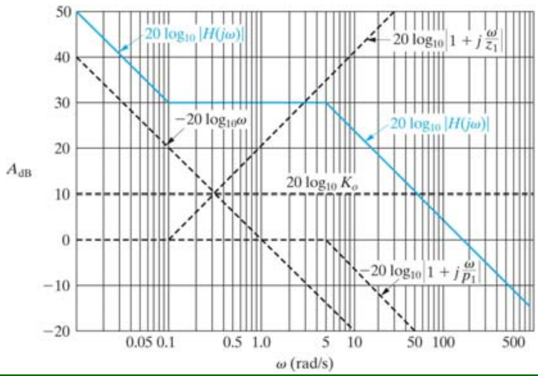
- The slope is expressed as **20 dB per decade**:

$$\omega{:}\,\omega_0 \to 10\omega_0 \Rightarrow x{:}\,x_0 = \mathcal{log}_{10}\,\omega_0 \to x_0+1 \Rightarrow y{:}\,y_0 = ax_0 - b \to y_0 + 20$$

Straight-Line Amplitude Plot

- Complete plot is obtained by adding the contribution of each term on the graph.

$\mathcal{K}_0 = \sqrt{10}$, $z_1 = 0.1$ rad/sec, $p_1 = 5$ rad/sec



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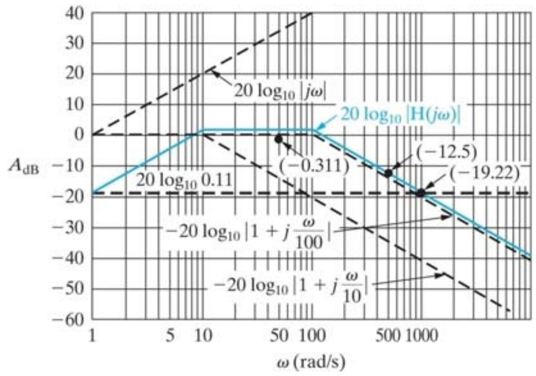
Bode Amplitude Plot: Example

- Sketch the straight-line amplitude plot for:

$$\mathcal{H}(s) = \frac{110s}{s^2 + 110s + 1000}$$



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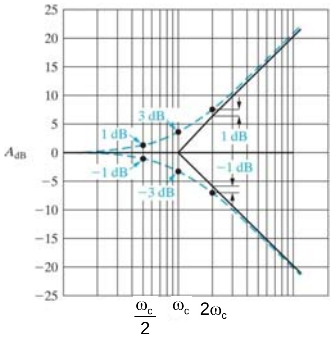
More Accurate Amplitude Plots



20 dB per decade
≈ 6 dB per **octave**

- When the poles and zeros are distant enough from each other, the amplitude can be refined around ω_c's:

	$\omega = 0.5\omega_c$	$\omega = \omega_c$	$\omega = 2\omega_c$
$20\log_{10}\sqrt{1+(\omega/\omega_c)^2}$	≈ 1 dB	≈ 3 dB	≈ 7 dB

Computation of the Phase Angle

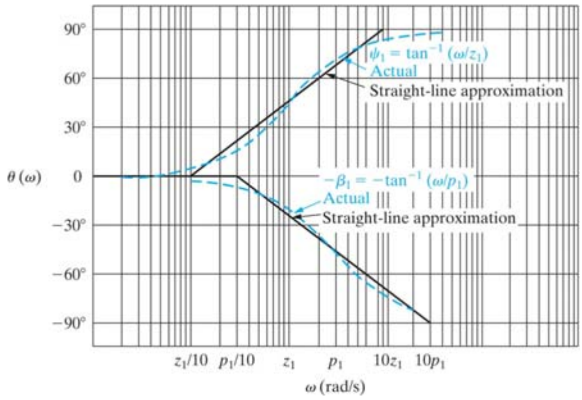
- The phase angle is simply computed as:

$$\theta(\omega) = \angle\mathcal{H}(j\omega) = \underbrace{\angle(1+j\omega/z_1)}_{\tan^{-1}(\omega/z_1)} - \underbrace{\angle(1+j\omega/p_1)}_{\tan^{-1}(\omega/p_1)} - 90^\circ$$

- Again, other factors in the numerator/denominator lead to additional terms to be added/subtracted.
- Straight-line plots are obtained by adding the contribution of each term, with the approximation:

$$\angle(1+j\omega/\omega_c) \approx \begin{cases} 0^\circ & , \omega < 0.1\omega_c \\ 0.5 + 0.5\log_{10}(\omega/\omega_c) & , 0.1\omega_c \le \omega \le 10\omega_c \\ 90^\circ & , \omega > 10\omega_c \end{cases}$$

Approximate versus Actual Phase Angle

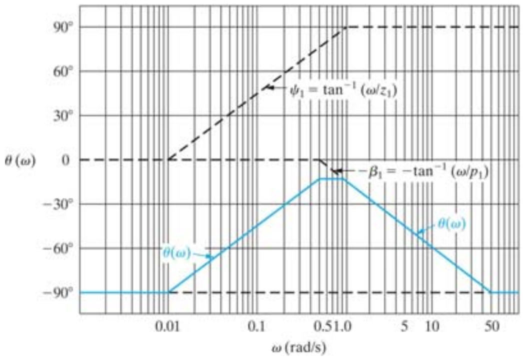


- The maximum deviation between the approximate and actual variation of the phase angle is 6°.

Straight-Line Phase Angle Plot

- Complete plot is obtained by adding the contribution of each term on the graph.

$\mathcal{K}_0 = \sqrt{10}$, $z_1 = 0.1$ rad/sec, $p_1 = 5$ rad/sec



Bode Phase Angle Plot: Example

- Sketch the straight-line phase angle plot for:

$$\mathcal{H}(s) = \frac{110s}{s^2 + 110s + 1000} = \frac{110s}{(s+10)(s+100)}$$