Ch. 14 Filters

– What is a Filter?

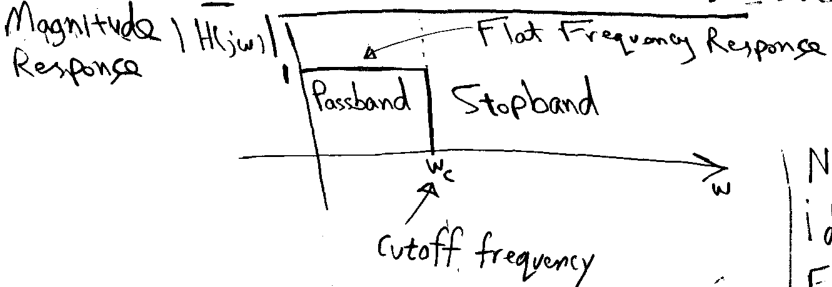
Filters are frequency-Selective Circuits. They can pass some frequencies in a certain band and can stop others.

– Discussion: – think about applications for Filters.
– what Filters are used for?

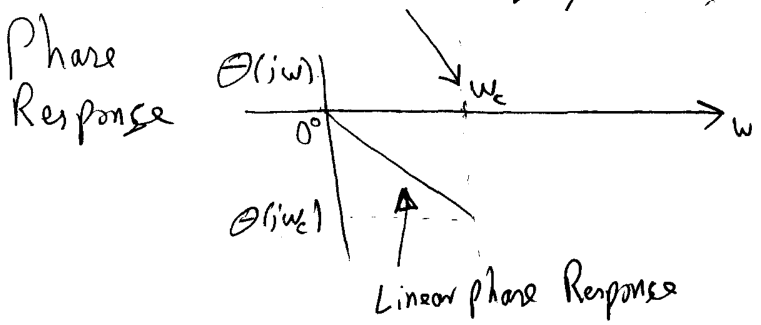
14.1 There are four basic Types of Filters

$$H(s) = \frac{V_o}{V_i}$$

1 Low Pass Filters [Ideal]



Note: ideal Filters

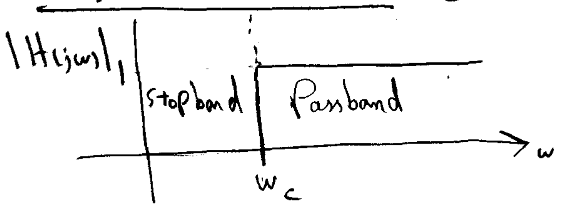


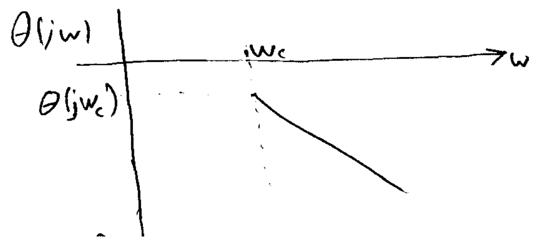
Def:

Passband: the band of frequencies that pass through the circuit.

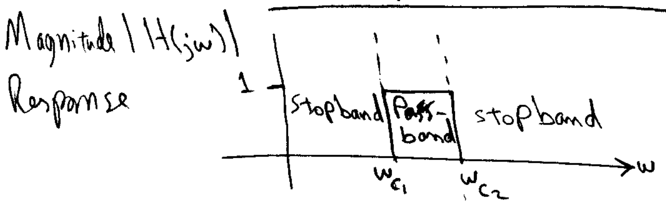
Stopband: the band of frequencies that attenuate and get blocked by the circuit

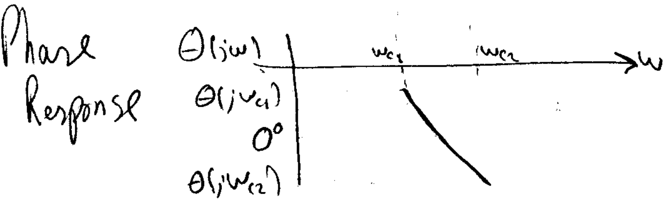
2 High Pass Filters [Ideal]



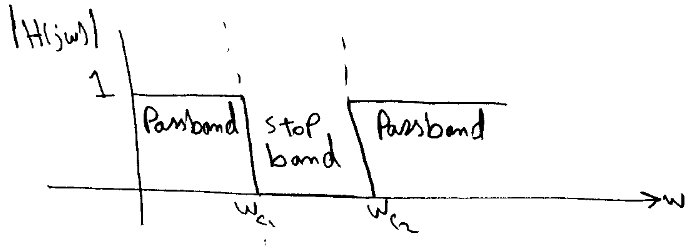


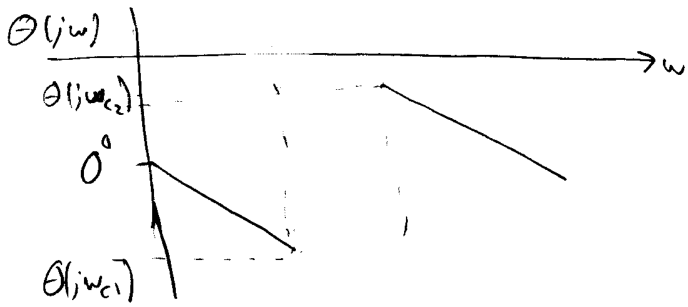
3. Bandpass filter [Ideal]





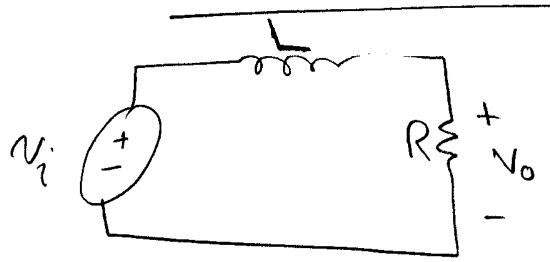
4. Bandreject Filter [Ideal]





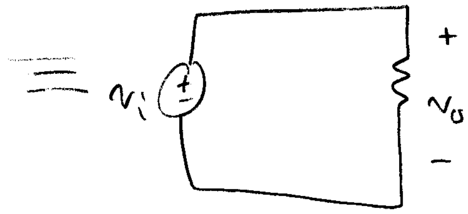
14.2 Low-Pass Filters

Series RL Circuit:



The impedance of the inductor is $j\omega L$.

at $\omega = 0$



$$v_o = v_i$$

$$\Rightarrow |H(j\omega)|\Big|_{\omega=0} = \frac{v_o}{v_i} = 1$$

$$\theta(j\omega)\Big|_{\omega=0} = 0 \Rightarrow \text{No phase shift}$$

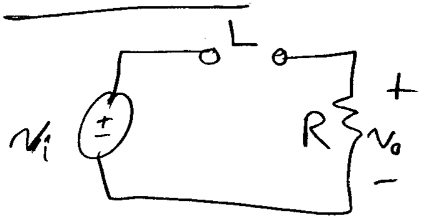
increasing ω Low frequency $\Rightarrow$ inductor impedance increases $j\omega L$ $\Rightarrow$ voltage drop $\Rightarrow v_o < v_i$ Passband

in addition, Phase shift increases

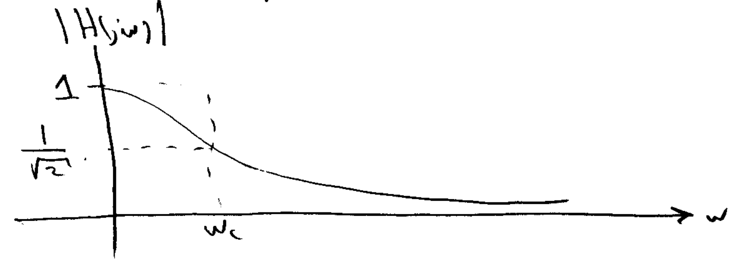
At High frequency $\Rightarrow$ there will be high voltage drop across the inductor $\Rightarrow v_o \ll v_i$ Stop band

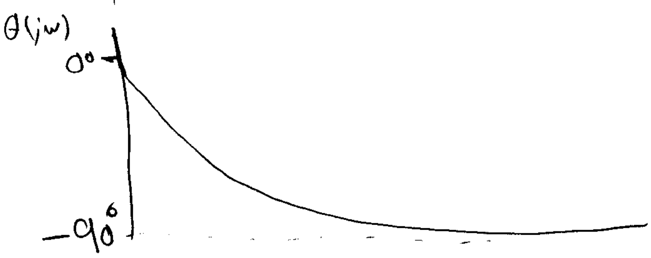
in addition $\Rightarrow$ more phase shift

At $\omega = \infty$



$\Rightarrow$ The inductor impedance will be open $\Rightarrow v_o = 0$





Low-Pass Filters

Cutoff frequency [the Half Power Frequency]

The definition for cutoff frequency widely used ~~by~~ is the frequency for which the transfer function magnitude is decreased by the factor $\frac{1}{\sqrt{2}}$ from its maximum value

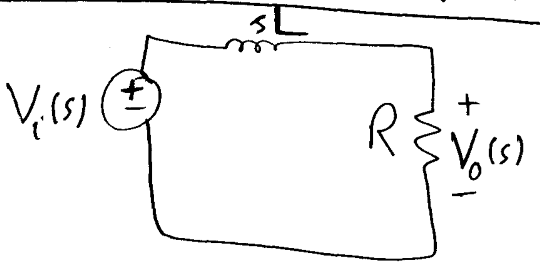
$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} H_{max}$$

Thus, the output voltage in the passband ~~band~~ is at least 70.7% of the maximum possible amplitude.

In addition, the average power delivered to a load is at least 50% of the maximum average power. , in passband,

$\Longrightarrow$ It is called the half power frequency.

Transfer Function of the Series RL Circuit



$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{R}{sL+R} = \frac{R/L}{s+R/L}$$

$$H(j\omega) = \frac{R/L}{j\omega + R/L} \cancel{\ne R/L}$$

$$\therefore \quad |H(j\omega)| = \frac{R/L}{\sqrt{\omega^2 + (R/L)^2}} \equiv \text{Magnitude Response}$$

and $\theta(j\omega) = -\tan^{-1}\left(\frac{\omega L}{R}\right) \equiv$ Phase Response

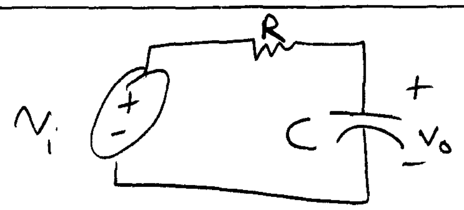
Low-pass Filters

At the cutoff frequency, w_c, is

$$|H(jw_c)| = \frac{1}{\sqrt{2}}|1| = \frac{R/L}{\sqrt{w_c^2 + (R/L)^2}}$$

$$\Rightarrow \boxed{w_c = \frac{R}{L}}$$

A Series RC Circuit



Transfer Function

$$H(s) = \frac{1/RC}{s + \frac{1}{RC}}$$

$$|H(jw)| = \frac{1/RC}{\sqrt{w^2 + \left(\frac{1}{RC}\right)^2}}$$

cutoff freq.

$$w_c = \frac{1}{RC}$$

Qualitative Analysis

To have a quick understanding of the circuit characteristics, examine it by inspection.

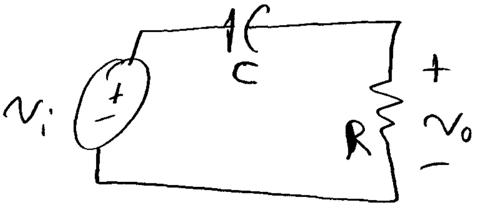
1. at zero frequency ($w=0$)
   $\Rightarrow$ capacitor's impedance is infinite
   $\Rightarrow v_o = v_i$
2. Increasing the frequencies
   $\Rightarrow$ the impedance of the capacitor decreases
   $\Rightarrow$ voltage divides
   $\Rightarrow v_o < v_i$
3. Infinite frequency ($w = \infty$)
   $\Rightarrow$ the impedance of the capacitor is zero
   $\Rightarrow$ short circuit
   $\Rightarrow v_o = 0$

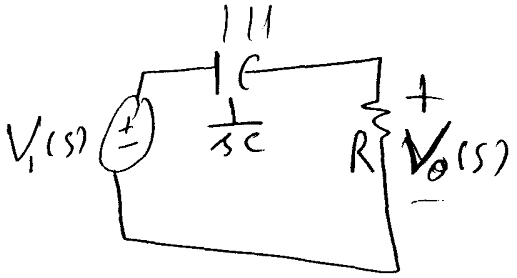
In General, the transfer function of a low-pass filter is

$$H(s) = \frac{w_c}{s + w_c}$$

14.3 High-Pass Filters

The Series RC Circuit





$$H(s) = \frac{R}{\frac{1}{sC} + R} = \frac{s}{s + 1/RC}$$

$$H(j\omega) = \frac{j\omega}{j\omega + 1/RC}$$

$$\Rightarrow |H(j\omega)| = \frac{\omega}{\sqrt{\omega^2 + (1/RC)^2}}$$

$$\theta(j\omega) = 90^\circ - \tan^{-1}\omega RC$$

the cutoff Freq.

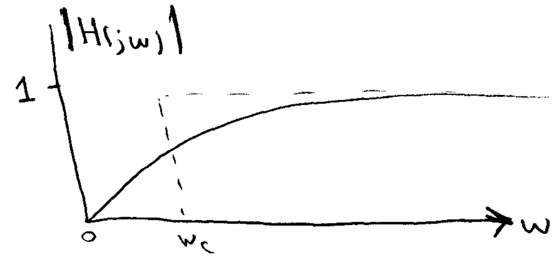
$$\omega_c = \frac{1}{RC}$$

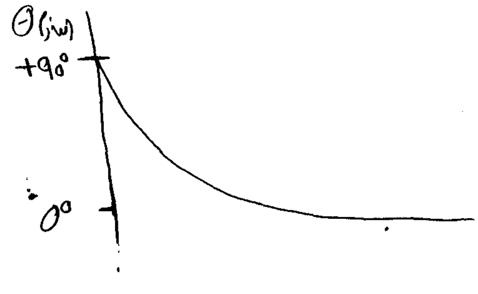
Qualitative Analysis

① at $\omega = 0$ $\Rightarrow$ Capacitor behaves like open circuit
$\Rightarrow$ No current flow
$\Rightarrow$ $v_o = 0$

② Increasing ω $\Rightarrow$ capacitor's impedance decreases
$\Rightarrow$ voltage is divided
$\Rightarrow$ v_o increases

③ at $\omega = \infty$ $\Rightarrow$ capacitor is short
$\Rightarrow$ $v_o = v_i$



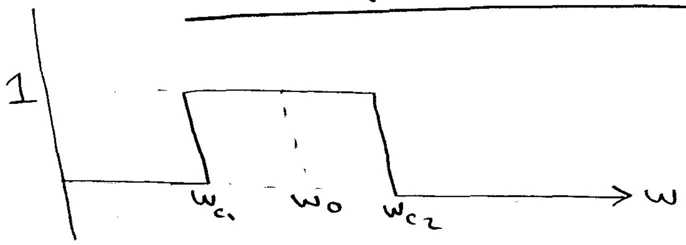


See Example 14.3 For Series RL High-Pass Filter

In General, The transfer Function for a high-Pass Filter is

$$H(s) = \frac{s}{s + \omega_c}$$

14.4 Bandpass Filters



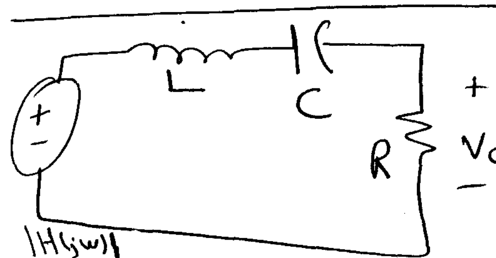
Def:

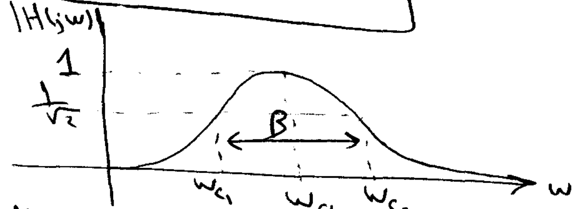
$\omega_0 \equiv$ Center frequency or resonant frequency

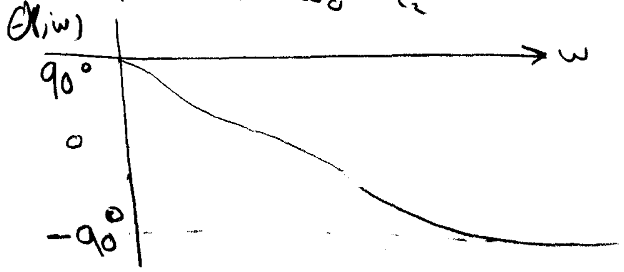
Bandwidth $(\beta) = \omega_{c_2} - \omega_{c_1}$

Quality Factor $Q = \dfrac{\omega_0}{\beta}$

Series RLC Circuit







Qualitative Analysis

① at $\omega = 0$ $\Rightarrow$ Inductor is short and Capacitor is open

$\Rightarrow$ No current flow

$\Rightarrow$ $v_o = 0$

② Increasing ω

- The impedance of the inductor is positive $(+j\omega L)$ and the impedance of the capacitor is negative $\left(\frac{-j}{\omega C}\right)$
- There will be a voltage drop across the inductor and capacitor and $v_o < v_i$
- At some frequency (ω_0), the impedance of the inductor will cancel the impedance of the capacitor.

  Therefore, $v_o = v_i$.

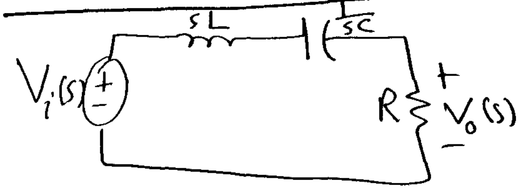
  This is at the resonant frequency.
- Increasing ω further, voltage drop is created across the inductor and capacitor and $v_o < v_i$ again.

③ at $\omega = \infty$

- The inductor is open and the capacitor is short.

$\Rightarrow$ $v_o = 0$

Quantitative Analysis



$$H(s) = \frac{R}{R + sL + \frac{1}{sC}}$$

$$= \frac{\frac{R}{L}s}{s^2 + \left(\frac{R}{L}\right)s + \frac{1}{LC}}$$

$$|H(j\omega)| = \frac{\omega\left(\frac{R}{L}\right)}{\sqrt{\left(\frac{1}{LC} - \omega^2\right)^2 + \left[\omega\left(\frac{R}{L}\right)\right]^2}}$$

$$\theta(j\omega) = 90° - \tan^{-1}\left[\omega\left(\frac{R}{L}\right) \right.$$

Bandpass Filters

Center Frequency

the center frequency is the resonant frequency that makes the imaginary part zero.

$$j\omega_0 L + \frac{1}{j\omega_0 C} = 0$$

$$\Rightarrow \omega_0 = \sqrt{\frac{1}{LC}}$$

Cutoff Frequency

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} |H_{max}|$$

$$\frac{\omega_c\left(\frac{R}{L}\right)}{\sqrt{\left\{\frac{1}{LC} - \omega_c^2\right\}^2 + \left(\omega_c \frac{R}{L}\right)^2}} = \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{\frac{\left[\frac{1}{LC} - \omega_c^2\right]^2}{\omega_c^2\left(\frac{R}{L}\right)^2} + 1}} = \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{\left[\left(\omega_c \frac{L}{R}\right) - \left(\frac{1}{\omega_c RC}\right)\right]^2 + 1}} = \frac{1}{\sqrt{2}}$$

$$\therefore \sqrt{\left[\frac{\omega_c L}{R} - \frac{1}{\omega_c RC}\right]^2 + 1} = \sqrt{2}$$

$$\left[\frac{\omega_c L}{R} - \frac{1}{\omega_c RC}\right]^2 + 1 = 2$$

$$\left[\frac{\omega_c L}{R} - \frac{1}{\omega_c RC}\right]^2 = 1$$

$$\Rightarrow \frac{\omega_c L}{R} - \frac{1}{\omega_c RC} = \pm 1$$

$$\boxed{\omega_c^2 L \pm \omega_c R - \frac{1}{C} = 0}$$

The two frequencies $\{\omega_{c_1}$ and $\omega_{c_2}\}$ must be positive

$$\Rightarrow \omega_{c_1} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)}$$

$$\omega_{c_2} = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)}$$

- what is the releation between the center freq. and cutoff freq.?

$$\boxed{\omega_0 = \sqrt{\omega_{c_1} \cdot \omega_{c_2}}}$$

Verify.

- Bandwidth

$$\beta = \omega_{c_2} - \omega_{c_1} = \frac{R}{L}$$

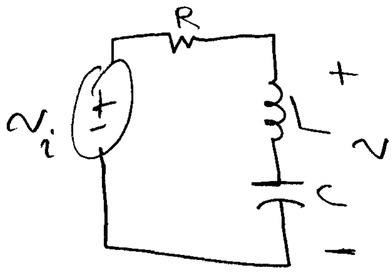
- Quality Factor

$$Q = \frac{\omega_0}{\beta} = \frac{\sqrt{1/LC}}{R/L} = \sqrt{\frac{L}{CR^2}}$$

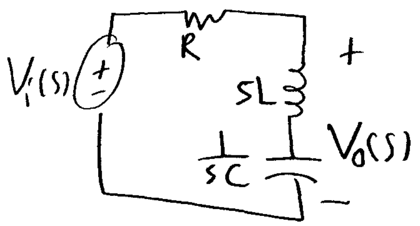
- Note:
  The transfer function of a bandpass Filter is:

$$H(s) = \frac{\beta s}{s^2 + \beta s + \omega_0^2} = \frac{R/L\, s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

14.5 Bandreject Filters



Quantitative Analysis



$$H(s) = \frac{sL + \frac{1}{sC}}{R + sL + \frac{1}{sC}}$$

$$= \frac{s^2 + \frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$= \frac{s^2 + w_0^2}{s^2 + \beta s + w_0^2}$$

$$|H(jw)| = \frac{\left|\frac{1}{LC} - w^2\right|}{\sqrt{\left(\frac{1}{LC} - w^2\right)^2 + \left(\frac{wR}{L}\right)^2}}$$

$$\theta(jw) = -\tan^{-1}\left(\frac{\frac{wR}{L}}{\frac{1}{LC} - w^2}\right)$$

- Centre Freq. $w_0 = \sqrt{\frac{1}{LC}}$
- Cutoff Freq. $w_{c1} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$

  $w_{c2} = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$
- Bandwidth $B = \frac{R}{L}$
- Quality Factor $Q = \sqrt{L}$

Qualitative Analysis

① at $w = 0$
- Inductor is short
- Capacitor is open
- No voltage drop across the Resistor

$\Rightarrow v_o = v_i$

② Increasing w
- Inductor has positive impedance
- Capacitor has negative impedance
- voltage divides between the Resistor and the inductor and capacitor
- $v_o < v_i$

③ At Resonance (w_0)
- when $w = w_0$, the inductor's impedance will cancel the capacitor's impedance

$\Rightarrow$ Both will act as short circuit with zero impedance

$\Rightarrow v_o = 0$

④ Increasing w

$v_o < v_i$ same as part ②

⑤ $w = \infty$
- Inductor is open
- Capacitor is short
- No voltage drop across the Resistor

$\Rightarrow v_o = v_i$

Bode Diagrams

E.1, E.2, and E.4 only [Real Poles and Zeros]

$$H(s) = \frac{K(s+z_1)}{s(s+p_1)}$$

$$H(j\omega) = \frac{K(j\omega+z_1)}{j\omega(j\omega+p_1)}$$

$$H(j\omega) = \frac{Kz_1(1+j\omega/z_1)}{p_1(j\omega)(1+j\omega/p_1)}$$

Let $K_0 = \frac{Kz_1}{p_1}$

$$H(j\omega) = \frac{K_0\,|1+j\omega/z_1|\angle\psi_1}{|\omega|\angle 90^\circ\,|1+j\omega/p_1|\angle\beta_1}$$

$$= \frac{K_0|1+j\omega/z_1|}{|\omega|\,|1+j\frac{\omega}{p_1}|}\angle(\psi_1-90^\circ-\beta_1)$$

Amplitude $|H(j\omega)| = \frac{K_0\,|1+j\omega/z_1|}{\omega\,|1+j\omega/p_1|}$

Phase

$$\theta(\omega) = \psi_1 - 90^\circ - \beta_1$$

Note that $\psi_1 = \tan^{-1}\omega/z_1$

$\beta_1 = \tan^{-1}\omega/p_1$

E.2 Straight Line Amplitude Plots

- In logarithmic value (decibel (dB))

$$A_{dB} = 20\log_{10}|H(j\omega)|$$

- why?

← Because addition and subtraction are easier than multiplication and division.

$$A_{dB} = 20\log_{10}\frac{K_0\,|1+j\omega/z_1|}{\omega\,|1+j\frac{\omega}{p_1}|}$$

$$= 20\log_{10}K_0 + 20\log_{10}|1+j\omega/z_1| - 20\log_{10}\omega - 20\log_{10}|1+j\omega/p_1|$$

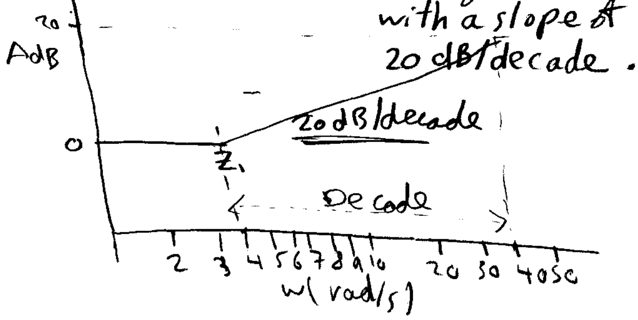
Methodology: Plot each term alone and then add them together to get A_{dB}.

Term 1: $20\log_{10}K_0$ is just a Horizontal Line.

Term 2: $20\log_{10}|1+j\omega/z_1|$

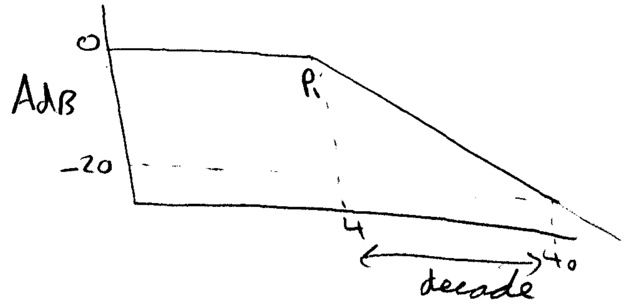
when $\omega < z_1 \Rightarrow$ Term2 $= 0$

when $\omega \geq z_1 \Rightarrow$ Term2 is a Straight Line with a slope of 20 dB/decade.



Term 3: $-20\log_{10}\omega$ is a straight line with -20 dB/decade.

Term 4: $-20\log_{10}|1+j\omega/p_1|$



Bode Diagrams

~~N~~Now add up all Plots to get the Amplitude Plot.

Example Let $K_0 = \sqrt{10}$, $Z_1 = 0.1$ rad/s, $P_1 = 5$ rad/s

$$\Rightarrow A_{dB} = 20\log_{10}\sqrt{10} + 20\log_{10}\left|1 + j\frac{\omega}{0.1}\right| - 20\log_{10}\omega - 20\log_{10}\left|1 + j\frac{\omega}{5}\right|$$