Resonance Circuits

Parallel

objective

* To study the frequency Domain characteristics of RLC circuits.

Fig 10.7

Def:

* Resonance Frequency:
 The frequency that makes the imaginary part of the ~~circuit~~ admittance or impedance equal to zero.

— The admittance seen by the current source in the above Figure is:

$$Y = Y_R + Y_L + Y_C = \frac{1}{R} + \frac{1}{jwL} + jwC$$

$$= \frac{1}{R} + j\left(wC - \frac{1}{wL}\right)$$

— From the above definision

$$w_r C - \frac{1}{w_r L} = 0$$

$$\Rightarrow w_r C = \frac{1}{w_r L} \Rightarrow \boxed{w_r = \frac{1}{\sqrt{LC}}}$$

Resonance Frq. for RLC Parallel.

$\Rightarrow$ The Voltage V at Resonance is

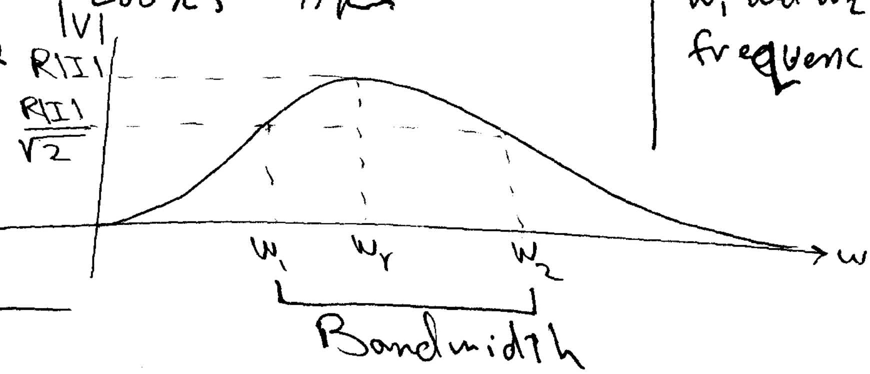
$$|V| = \frac{|I|}{|Y|} = R|I|$$

$\Rightarrow$ the Parallel LC combination "tank circuit" is like an open circuit.

Resonance Circuit

— The voltage magnitude in the frequency domain looks like

Amplitude Responce in Freq. Domain $\Rightarrow$



ω_1 and ω_2 are the half power frequencies defined at $\frac{|V|}{\sqrt{2}}$

***Def**: Bandwidth*

The Bandwidth "BW" is defined as the difference between the half power frequencies:

$$BW = \omega_2 - \omega_1$$

Note: The smaller the bandwidth, the sharper the responce.

***Def**: Quality Factor*

The quality Factor "Q" is defined at the resonance Frequency to be:

$$Q = 2\pi \left(\frac{\text{maximum energy stored}}{\text{total energy lost in a Period}} \right)$$

$$= 2\pi \frac{[W_C(t) + W_L(t)]_{max}}{P_R \, T}$$

$W_C(t) = \frac{1}{2} C v^2(t)$ ≡ energy stored in the capacitor

$W_L(t) = \frac{1}{2} L i^2(t)$ = Energy stored in the inductor

Resonance Circuit

Assume that the current source is

$$i(t) = I\cos\omega_r t, \quad \text{where } \omega_r = \frac{1}{\sqrt{LC}}$$

At Resonance, $Y = \frac{1}{R}$

$$\Rightarrow v(t) = R\,i(t) = RI\cos\omega_r t.$$

— Energy stored in the capacitor

$$w_C(t) = \frac{1}{2}Cv^2(t) = \frac{1}{2}C\left(RI\cos\omega_r t\right)^2 = \frac{1}{2}CR^2I^2\omega_r t$$

— Energy stored in the inductor.

$$w_L(t) = \frac{1}{2}L\,i_L^2(t).$$

* Find $i_L(t)$ First.

For the parallel RLC circuit,

$$\bar{I}_L = \frac{\bar{V}}{j\omega L} = \frac{RI\angle 0^\circ}{\omega_r L\angle 90^\circ} = \frac{RI}{\omega_r L}\angle -90^\circ$$

$$\Rightarrow i_L(t) = \frac{RI}{\omega_r L}\cos(\omega_r t - 90^\circ) = \frac{RI}{\omega_r L}\sin\omega_r t$$

Thus,

$$w_L(t) = \frac{1}{2}L\,i_L^2(t) = \frac{1}{2}\frac{R^2I^2}{\omega_r^2 L}\sin^2\omega_r t = \frac{1}{2}CR^2I^2\sin^2\omega_r t$$

∴ Total Energy stored is

$$w_C(t) + w_L(t) = \frac{1}{2}CR^2I^2\left(\cos^2\omega_r t + \sin^2\omega_r t\right) = \frac{1}{2}CR^2I^2$$

— Energy LOST in one Period T.

the power absorbed by a resistor is $P_R = \frac{I^2R}{2}$

$$\Rightarrow P_R T = \frac{1}{2}I^2RT = \frac{1}{2}I^2R\left(\frac{2\pi}{\omega_r}\right) = \frac{\pi I^2R}{\omega_r}$$

$\Rightarrow$ Quality Factor of the Parallel RLC is

$$Q = \frac{2\pi\left(\frac{1}{2} C R^2 I^2\right)}{\pi I^2 R / w_r} = w_r R C$$

since $w_r C = \frac{1}{w_r L}$

$$\Rightarrow Q = w_r R C = \frac{R}{w_r L} = R\sqrt{\frac{C}{L}}$$

— Half Power Frequencies

$$w_{1,2} = \mp \frac{w_r}{2Q} + w_r\sqrt{\left(\frac{1}{2Q}\right)^2 + 1}$$

— Bandwidth

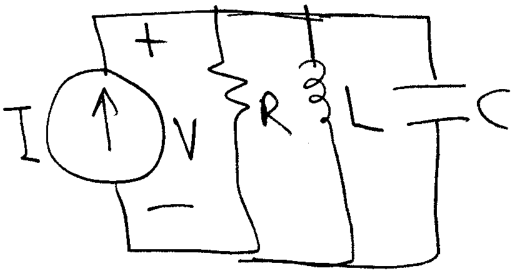
$$BW = w_2 - w_1 = \frac{w_r}{Q} = \frac{1}{RC}$$

— Selectivity $\equiv \boxed{Q = \frac{w_r}{BW}}$

Resonance Circuit

Summary

* Parallel

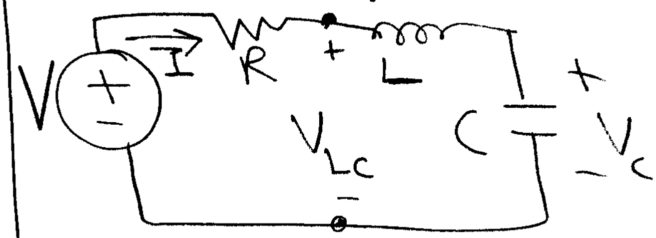


- Resonance Freq. $\omega_r = \frac{1}{\sqrt{LC}}$
- Quality Factor $Q = \omega_r RC = R\sqrt{\frac{C}{L}}$
- Half Power Freq.

$$\omega_{1,2} = \mp \frac{\omega_r}{2Q} + \omega_r \sqrt{\left(\frac{1}{2Q}\right)^2 + 1}$$

- Bandwidth, $BW = \frac{\omega_r}{Q} = \frac{1}{RC}$

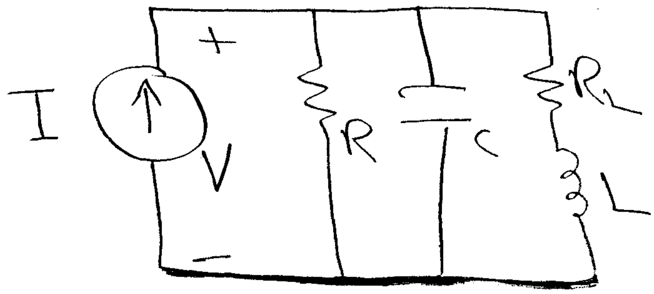
* Series



- $\omega_r = \frac{1}{\sqrt{LC}}$
- $Q = \frac{\omega_r L}{R} = \frac{\sqrt{L/C}}{R}$
- $\omega_{1,2} = \mp \frac{\omega_r}{2Q} + \omega_r \sqrt{\left(\frac{1}{2Q}\right)^2 + 1}$
- $BW = \frac{\omega_r}{Q} = \frac{R}{L}$

Resonance Circuit

* Series-Parallel RLC



* First Find the Admittance seen by the current Sourse.

$$Y = \frac{I}{V} = \frac{1}{R} + jwc + \frac{1}{R_L + jwL} = \frac{1}{R} + jwc + \frac{R_L - jwL}{R_L^2 + w^2L^2}$$

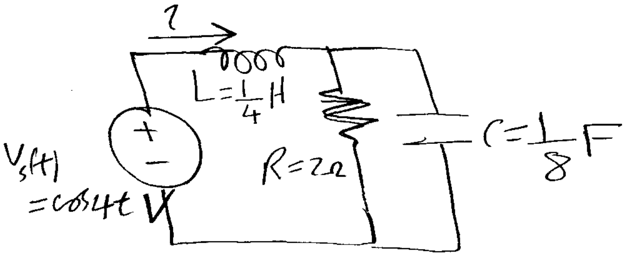
$$= \frac{1}{R} + \frac{R_L}{R_L^2 + w^2L^2} + j\left(wC - \frac{wL}{R_L^2 + w^2L^2}\right)$$

Imaginary Part zero $\Longrightarrow$ $w_r C - \dfrac{w_r L}{R_L^2 + w_r^2 L^2} = 0$

$$\Longrightarrow w_r = \sqrt{\frac{1}{LC} - \frac{R_L^2}{L^2}} \ ; \quad \frac{1}{LC} > \frac{R_L^2}{L^2}$$

Resonance Circuit

Example



(a) Find Resonance Freq.

(b) Find Quality Factor.

$\boxed{a}$ The impedence seen by the Source is

$$\cancel{\bar{Z} = \frac{jw}{4} + 2 \| 8/}$$

$$\bar{Z} = jwL + \frac{R\left(\frac{1}{jwc}\right)}{R + \frac{1}{jwc}}$$

$$= jwL + \frac{R}{1+jwcR} \cdot \frac{1-jwcR}{1-jwcR}$$

$$= jwL + \frac{R - jwcR^2}{1 + w^2c^2R^2}$$

- Subsitute

$$\Rightarrow \bar{Z} = \frac{jw}{4} + \frac{2 - jw\left(\frac{1}{8}\right)(4)}{1 + w^2\left(\frac{1}{64}\right)(4)}$$

$$= \frac{jw}{4} + \left(\frac{2 - jw\frac{1}{2}}{1 + w^2\frac{1}{16}} \cdot \frac{16}{16}\right)$$

$$= \frac{jw}{4} + \frac{32 - j8w}{16 + w^2}$$

$$= \frac{32}{16+w^2} + \frac{jw}{4}\left(1 - \frac{32}{16+w^2}\right)$$

$$= \frac{32}{\ldots} + \frac{jw(w^2 - 16)}{\ldots}$$

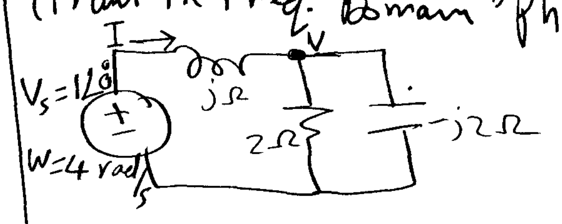
imaginary Part = 0

$$\Rightarrow \boxed{w_r = 4 \text{ rad/s}} \Leftarrow \text{Resonance Freq.}$$

$\boxed{b}$ to Find the quality Factor, we need to do the followings.

1. Set the input freq. at w_r.
2. find $V(t) \equiv$ voltage across capc.
   $i(t) \equiv$ current inside induc.
3. Find $w_c(t) = \frac{1}{2} c V^2(t)$
   $w_L(t) = \frac{1}{2} L i^2(t)$
   $P_R = \frac{1}{2} \frac{|V|^2}{R}$ or $\frac{1}{2}|I|^2 R$
4. Find
$$Q = \frac{2\pi \left[w_c(t) + w_L(t)\right]_{max}}{P_R T}$$

To Do that, we draw the Circuit in Freq. Domain "Phasor"



By KCL at node $\underline{V}$

$$\frac{\vec{V} - 1}{j} + \frac{\vec{V}}{2} + \frac{\vec{V}}{-j2} = 0$$

$$\Rightarrow \vec{V} = \sqrt{2} \angle -45^\circ$$

Resonance Circuit

$$\vec{I} = \frac{1-\vec{V}}{j} = \frac{1-2/(1+j)}{j}$$

$$= \frac{(-1+j)/(1+j)}{j} = \frac{-1+j}{-1+j} = 1\angle 0^\circ$$

$\therefore$

$v(t) = \sqrt{2}\cos(4t-45^\circ)$

$i(t) = \cos 4t$

— Energy at Capacitor

$$w_C(t) = \frac{1}{2}Cv^2(t) = \frac{1}{8}\cos^2(4t-45^\circ)$$

— Energy at Inductor

$$w_L(t) = \frac{1}{2}Li^2(t) = \frac{1}{8}\cos^2 4t$$

$$\Rightarrow w_L(t) + w_C(t) = \frac{1}{8}\cos^2 4t + \frac{1}{8}\cos^2(4t-45^\circ)$$

identity: $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$$\Rightarrow w_L(t) + w_C(t) = \frac{1}{16}[1+\cos 8t] + \frac{1}{16}[1+\cos(8t-90^\circ)]$$

$$= \frac{1}{16}[2+\underbrace{\cos 8t+\sin 8t}] = \frac{1}{16}[2+\underbrace{\sqrt{2}\cos(8t-45^\circ)}]$$

identity: $\cos(x-y) = \cos x\cos y + \sin x \sin y$

$\therefore$ Maximum Energy Stored is

$$[w_L(t)+w_C(t)]_{max} = \frac{1}{16}\{2+\sqrt{2}\} \ J.$$

— The Power dissipated by the resistor is

$$P_R = \frac{1}{2}\frac{|V|^2}{R} = \frac{1}{2}\left(\frac{2}{2}\right) = \frac{1}{2}W$$

$$\Rightarrow P_R T = P_R\left(\frac{2\pi}{w_r}\right) = \frac{1}{2}\left(\frac{2\pi}{4}\right) = \frac{\pi}{4} \ J$$

So

$$Q = 2\pi\left[\frac{1/16\,(2+\sqrt{2})}{\pi/4}\right]$$

$$= 1 + \frac{1}{\sqrt{2}} = 1.707.$$

Resonance Circuit

— Series Inductive Reactance

assume $\vec{I} = I\angle 0°$

$$\Rightarrow w_L(t) = \frac{1}{2} L i^2(t) = \frac{1}{2} L I^2 \cos^2 \omega t$$

$$\{w_L(t)\}_{max} = \frac{1}{2} L I^2$$

Since $X_s = \omega L \Rightarrow L = \frac{X_s}{\omega}$

$$\Rightarrow \{w_L(t)\}_{max} = \frac{X_s}{2\omega} I^2$$

— The energy LOST in the Resistor per period is

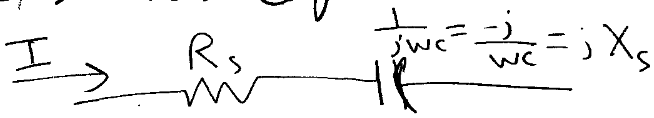
$$P_s T = \frac{1}{2} I^2 R_s \left(\frac{2\pi}{\omega}\right)$$

$$= \frac{\pi I^2 R_s}{\omega}$$

$\Rightarrow$ The Quality factor

$$\boxed{Q = \frac{2\pi \left[X_s I^2 / 2\omega\right]}{\pi I^2 R_s / \omega} = \frac{X_s}{R_s}}$$

— Series Capacitive Reactance



following the same approach,

$$Q = \frac{|X_s|}{R_s}$$

— In General.

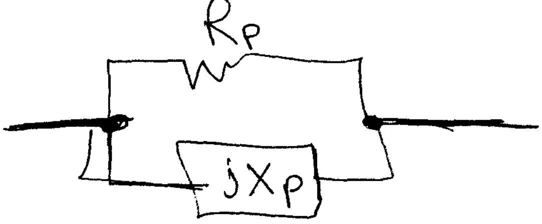
Series

For a resistance in series with a reactance X_s, the Quality Factor is

$$\boxed{Q = \frac{|X_s|}{R_s}}$$

Resonance Circuit

Parallel



$$Q = \frac{R_p}{|X_p|}$$

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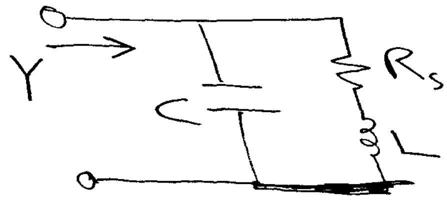
* **High-Q resonant Circuit.**

$$\vec{Y} = j\omega C + \frac{1}{R_s + j\omega L}$$



The admittance is:

$$Y = j\omega C + \frac{1}{R_s + j\omega L}$$

Note: when the series reactance, L and R_s, has large Q,
$\Rightarrow$ High-Q coil

$\therefore \frac{X_L}{R_s} \gg 1 \Rightarrow \frac{\omega L}{R_s} \gg 1$
$\Rightarrow \omega L \gg R_s$

$\therefore$

$$Y \approx j\omega C + \frac{1}{j\omega L}$$

$\therefore$ The Resonance Freq. is approximated by

$$\omega_r \approx \frac{1}{\sqrt{LC}}$$

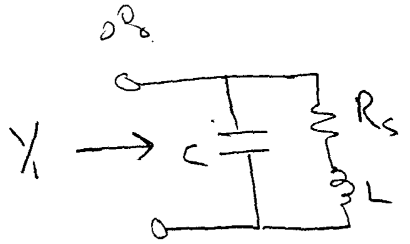
* **Approximate Equivalent Circuit for High-Q resonant circuit**

The impedance Z is:

$$Z = \frac{(R_s + j\omega L)(\frac{1}{j\omega C})}{R_s + j\omega L + \frac{1}{j\omega C}} \approx \frac{(j\omega L)(\frac{1}{j\omega C})}{R_s + j\omega L + \frac{1}{j\omega C}} = \frac{\frac{L}{C}}{R_s + j\omega L + \frac{1}{j\omega C}}$$

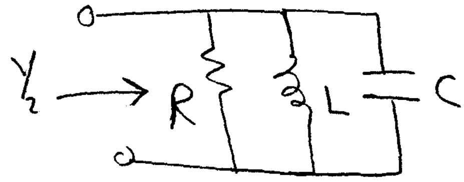
$$\Longrightarrow Y \approx \frac{C}{L}\left(R_s + j\omega L + \frac{1}{j\omega C}\right) = \frac{R_s C}{L} + j\omega C + \frac{1}{j\omega L}$$

Resonance Circuit



Approximation $\approx$

Equivalent Circuit



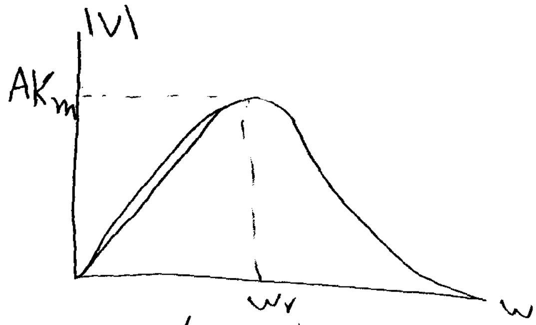
where $R = \dfrac{L}{R_s C}$

See Example 10.5

* Scaling Laws

① Magnitude Scaling

Magnitude Scale by K_m $\Longrightarrow$



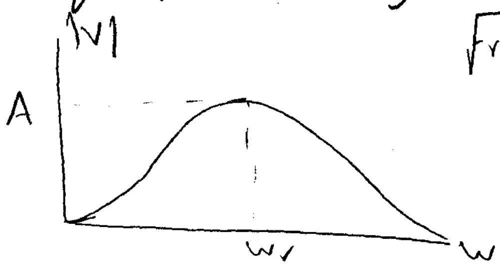
* scale Amplitude only
* Freq. Axis is the same.

To Magnitude scale the response of a Parallel RLC circuit by the factor K_m,

$$\Longrightarrow \quad R \to K_m R \quad L \to K_m L \quad C \to \frac{C}{K_m}$$

Note: The resonance freq. and the bandwidth still the same.

② Frequency Scaling



Freq. Scale by K_f $\Longrightarrow$

Resonance Circuit

Handout P.12

To frequency Scale a parallel RLC circuit by the factor K_f,

$$R \rightarrow R \qquad L \rightarrow \frac{L}{K_f} \qquad C \rightarrow \frac{C}{K_f}$$

Note:

$w_{r_2} = K_f w_{r_1}$

$BW_2 = K_f BW_1$

See Examples 10.6 and 10.7

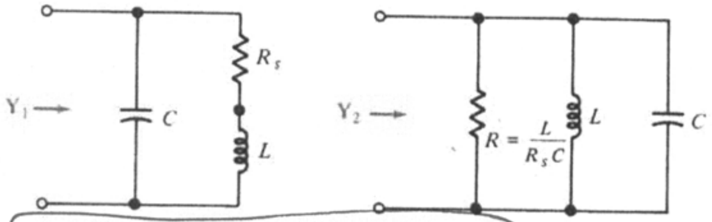


Fig. 10.27 Approximate equivalent circuits.

which is the expression for a parallel RLC circuit, where $R = L/R_sC$. Hence, for a high-Q coil, the two admittances shown in Fig. 10.27 are approximately equal.

Do in class

EXAMPLE 10.5

The practical tank circuit shown in Fig. 10.28 is to be in resonance at 1 MHz, that is, $\omega_r = 2\pi \times 10^6$ rad/s. Assuming a high-Q coil, $\omega_r \approx 1/\sqrt{LC}$, from which

$$L \approx \frac{1}{\omega_r^2 C} = \frac{1}{(4\pi^2 \times 10^{12})(500 \times 10^{-12})} = 50.7\ \mu\text{H}$$

The quality factor of the series reactance at the resonance frequency is

$$\frac{X_s}{10} = \frac{\omega_r L}{10} = \frac{318}{10} = 31.8$$

which is much greater than 1, as was assumed. Since

$$\frac{L}{R_s C} = 10{,}132 \approx 10{,}000$$

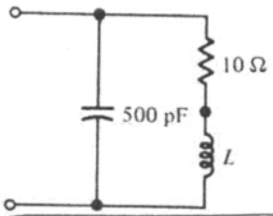


Fig. 10.28 Practical tank circuit.

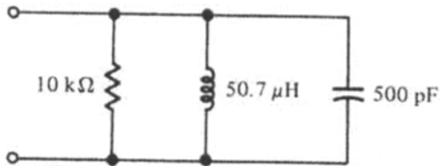


Fig. 10.29 Equivalent of practical tank circuit.

then the tank circuit given in Fig. 10.28 can be approximated by the parallel RLC circuit shown in Fig. 10.29. The quality factor of this parallel RLC circuit is

$$Q = \omega_r RC = (2\pi \times 10^6)(10 \times 10^3)(500 \times 10^{-12}) = 31.4$$

Scaling

In the last example discussed, the element values given were practical in nature. This, however, has not been the trend previously, nor for the most part will it be in the remainder of this text. The reason for this is the simplification of computations. However, by a process known as "scaling," results obtained for circuits having simplified, nonrealistic element values can be extended easily to practical situations.

There are two ways to scale a circuit—**magnitude** or **impedance scaling** and **frequency scaling**. To magnitude-scale by the factor K_m, just multiply the impedance of each element by the real, positive number K_m. A resistor of R ohms is scaled to a resistor of $R' = K_m R$ ohms. An L-henry inductor of impedance $j\omega L$ ohms is scaled to an impedance of $j\omega K_m L$ ohms—that is, an $L' = K_m L$-henry inductor. A C-farad capacitor of impedance $1/j\omega C$ ohms is scaled to an impedance of $K_m/j\omega C = 1/j\omega(C/K_m)$ ohms—that is, a $C' = C/K_m$-farad capacitor. In summary, to scale an impedance by the factor K_m, scale R, L, and C as follows:

$$\boxed{R \rightarrow K_m R \qquad L \rightarrow K_m L \qquad C \rightarrow \frac{C}{K_m}} \longrightarrow \text{Magnitude Scaling Law.}$$

EXAMPLE 10.6

The parallel RLC circuit shown in Fig. 10.30 has the admittance

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{1}{R} + \frac{1}{j\omega L} + j\omega C = \frac{1}{2} + \frac{1}{j\omega} + \frac{j\omega}{25}$$

and the resonance frequency

$$\omega_r = \frac{1}{\sqrt{LC}} = 5 \text{ rad/s}$$

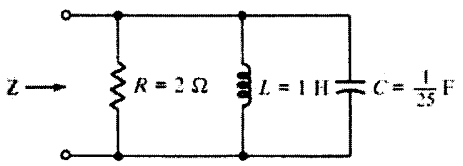


Fig. 10.30 Parallel RLC impedance.

Scaling this circuit by $K_m = 5000$, we get $R' = 2K_m = 10$ kΩ, $L' = 1K_m = 5$ kH, $C' = 1/25K_m = (1/125{,}000)$ F $= 8\ \mu$F and an admittance $\mathbf{Y}'$ given by

$$\mathbf{Y}' = \frac{1}{K_m R} + \frac{1}{j\omega K_m L} + \frac{j\omega C}{K_m} = \frac{1}{10{,}000} + \frac{1}{j\omega 5000} + \frac{j\omega}{125{,}000}$$

and the resulting resonance frequency is

$$\omega_r' = \frac{1}{\sqrt{K_m L(C/K_m)}} = \frac{1}{\sqrt{LC}} = 5\ \text{rad/s}$$

That is, the resonance frequency is not affected by magnitude scaling. In actuality, the shape and the frequency axis of the amplitude response of an impedance are not affected by magnitude scaling, but values on the vertical axis are multiplied by K_m.

To frequency-scale by the factor K_f, since the impedance of a resistor is not frequency dependent, a resistor of R ohms is left as is. However, for an L-henry inductor of impedance $j\omega L$, at frequency $\omega = \omega_0$ the impedance is $j\omega_0 L$. What inductor L'' will have such an impedance at the scaled frequency $K_f\omega_0$? Since

$$j\omega_0 L = j(K_f\omega_0)L'' \qquad \Rightarrow \qquad L'' = \frac{L}{K_f}$$

to frequency-scale an inductor L by K_f, divide L by K_f.

For a C-farad capacitor of impedance $1/j\omega C$, at frequency $\omega = \omega_0$ the impedance is $1/j\omega_0 C$. The capacitor C'' that has such an impedance at the scaled frequency $K_f\omega_0$ is determined from

$$\frac{1}{j\omega_0 C} = \frac{1}{j(K_f\omega_0)C''} \qquad \Rightarrow \qquad C'' = \frac{C}{K_f}$$

That is, to frequency-scale a capacitor C by K_f, divide C by K_f. In summary, to frequency-scale by the factor K_f, scale R, L, and C as follows:

$$\boxed{R \to R \qquad L \to \frac{L}{K_f} \qquad C \to \frac{C}{K_f}} \longrightarrow \text{Freq. Scaling.}$$

EXAMPLE 10.7

To frequency-scale, by $K_f = 10^5$, the previously magnitude-scaled parallel RLC circuit given in the preceding example, we get $R'' = R' = 10\ \text{k}\Omega$, $L'' = L'/K_f = 5000/100{,}000 = 50\ \text{mH}$, $C'' = C'/K_f = 1/(125{,}000)(100{,}000) = 80\ \text{pF}$ and admittance $\mathbf{Y}''$ given by

$$\mathbf{Y}'' = \frac{1}{10^4} + \frac{1}{j\omega 50(10^{-3})} + j\omega 80(10^{-12})$$

and a resonance frequency

$$\omega_r'' = \frac{1}{\sqrt{(50)(10^{-3})(80)(10^{-12})}} = 5(10^5)\ \text{rad/s} = 10^5\omega_r = K_f\omega_r$$

$\boxed{10.4}$ Complex Freq. "S-Domain"

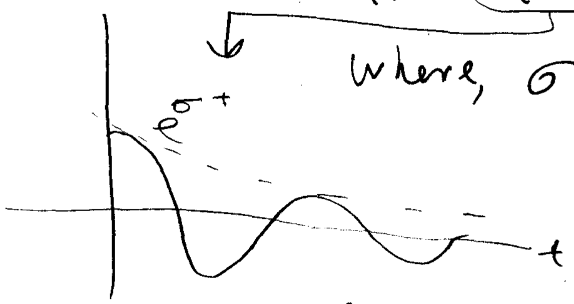
Background

- Real sinusoids $x(t) = A\cos(\omega t + \theta)$
- Complex sinusoids "Phasors"

$$x(t) = Ae^{j(\omega t+\theta)} = A\cos(\omega t+\theta) + jA\sin(\omega t + \theta)$$

Now $\Longrightarrow$ - Damped complex sinusoids "Complex Freq"

$$x(t) = \underbrace{Ae^{\sigma t}\cos(\omega t+\theta)} + jAe^{\sigma t}\sin(\omega t+\theta)$$



where, $\sigma \equiv$ damping factor nepers/second "neper frequency"

$$\therefore\ x(t) = Ae^{\sigma t}e^{j(\omega t+\theta)}$$

Let $s = \sigma + j\omega \equiv$ $\underline{s}$ is called Complex Frequency.

$$\therefore\ x(t) = Ae^{j\theta}e^{(\sigma+j\omega)t}$$

$$= Ae^{j\theta}e^{st} = \bar{V}_{in}e^{st}$$

Complex Freq.

Impedance and Admittance

~~Resistor~~

~~$Z_R =$~~

— Impedance

— Resistor $\boxed{Z_R = R}$

— Inductor $\boxed{Z_L = Ls}$

— Capacitor $\boxed{Z_C = \frac{1}{Cs}}$

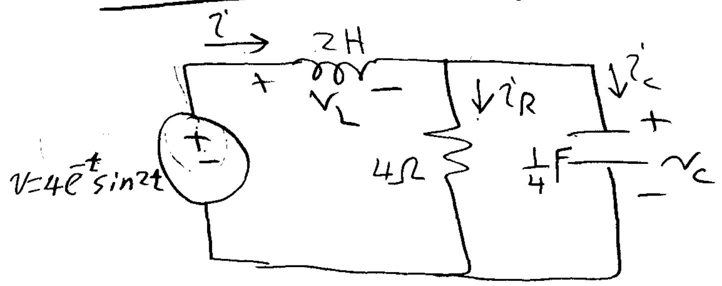
— Admittance

~~Z_R~~ $Y_R = \frac{1}{R}$

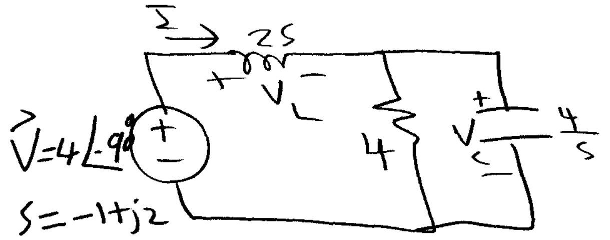
$Y_L = \frac{1}{Ls}$

$Y_C = Cs$

Example 10.9



S- Transform ⇓



* The Parallel RC has impedance

$$Z_{RC} = \frac{4\left(\frac{4}{s}\right)}{4 + 4/s} = \frac{4}{s+1}$$

⇒ By voltage division

$$\vec{V}_c = \frac{Z_{RC}}{Z_{RC} + Z_L}\vec{V}$$

$$= \frac{\frac{4}{s+1}}{\frac{4}{s+1} + 2s}\vec{V} = \frac{2}{2 + s^2 + s}\vec{V}$$

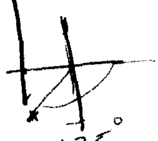
$$\vec{V}_c = \frac{2}{s^2 + s + 2}\vec{V}$$

since $\vec{V} = 4\angle -90°$ and $s = -1 + 2j$

$$\Rightarrow V_c = \frac{4\angle -90° \,(2)}{(-1+2j)^2 + (-1+2j) + (2)}$$

$$= \frac{8\angle -90°}{1 - 4j - 4 - 1 + 2j + 2}$$

$$= \frac{8\angle -90°}{2 - 2j} = \frac{4\angle -90°}{-1 - j}$$

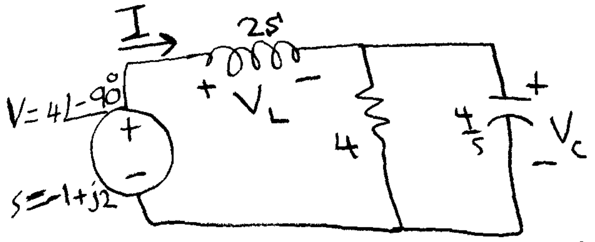


Complex Freq.

$\therefore$ in time domain,

$$v_c(t) = 2\sqrt{2}\,e^{-t}\cos(2t+45^\circ)\ V$$

10.5 Poles and Zeros



$$\underset{\text{output}}{\vec{V}_c} = \frac{2\,\vec{V}_{\text{input}}}{s^2+s+2}$$

Define: Voltage transfer function

$$H_c(s) = \frac{\vec{V}_c}{\vec{V}} = \frac{\text{Output}}{\text{Input}}$$

$$= \frac{2}{s^2+s+2}$$

factor the denominator,

$$H_c(s) = \frac{2}{\left(s+\frac{1}{2}-j\frac{\sqrt{7}}{2}\right)\left(s+\frac{1}{2}+j\frac{\sqrt{7}}{2}\right)}$$

How? for $ax^2+bx+c=0$

Remember that $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\therefore$ $s = -\frac{1}{2} \pm j\frac{\sqrt{7}}{2}$

Roots

Define: Poles

Poles are the roots of the denominator. At these values, the transfer function becomes infinite.

The poles of $H_c(s)$ are

$$s = -\frac{1}{2} \pm j\frac{\sqrt{7}}{2}$$

Define: Zeros

Zeros are the ~~values~~ roots of the numerator. At these values, the transfer function becomes zero.

for $H_c(s)$ in our examples there are no zeros.

- Let us find the voltage transfer function

$$H_L(s) = \frac{\vec{V}_L}{\vec{V}}$$

$$\vec{V}_L = \frac{2s\vec{V}}{2s + \frac{4}{s+1}} = \frac{2s(s+1)\vec{V}}{2s(s+1)+4}$$

$$\therefore H_L(s) = \frac{\vec{V}_L}{\vec{V}} = \frac{s(s+1)}{s^2+s+2}$$

The poles of $H_L(s)$ are:

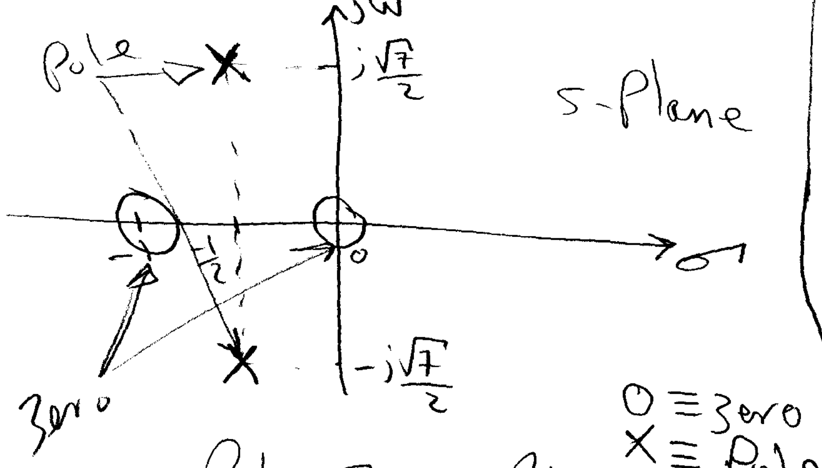
$$s = -\frac{1}{2} \pm j\frac{\sqrt{7}}{2}$$

The zeros of $H_L(s)$ are:

$s=0$ and $s=-1$

Define: S-Plane

Since $s=\sigma+j\omega$ is a complex number, we can plot the point s in the S-plane.



Pole-Zero Plot

- Let us consider the output to be the inductor current. The transfer function in this case is:

$$H_I(s) = \frac{I}{V}$$ ⇒ which is the admittance seen by the voltage source

- Find the impedance first.

$$Z = Z_L + Z_{RC} = 2s + \frac{4}{s+1}$$
$$= \frac{2(s^2+s+2)}{s+1}$$

$$\therefore H_I(s) = \frac{1}{Z} = \frac{\frac{1}{2}(s+1)}{s^2+s+2}$$

Zeros at $s=-1$

Poles: Same as $H_L(s)$ and $H_C(s)$

Is it always true?

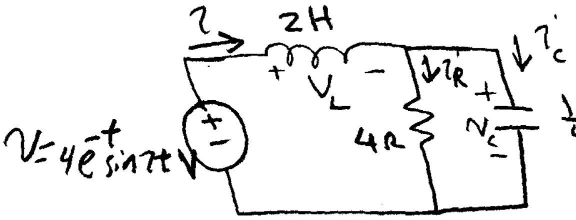
Yes, all transfer functions of the same circuit will have the same poles giving that one portion of the circuit is not physically separated from the next.

Define: Natural frequencies

the poles of a circuit are called natural frequencies and they determine the natural response.

Complex Freq.

Relation between the s-Domain Analysis and the natural and step Response.?



- Find the differential Equation of the voltage across the capacitor.

$$i = i_R + i_C = \frac{v_c}{4} + \frac{1}{4}\frac{dv_c}{dt}$$

and

$$v_L = v - v_c = 2\,\frac{di}{dt}$$

$$\Rightarrow \frac{d^2 v_c(t)}{dt} + \frac{dv_c(t)}{dt} + 2v_c(t) = 2v(t)$$

- Natural Response

when $v(t) = 0 \Rightarrow$ Natural Response

$$\frac{d^2 v_c(t)}{dt} + \frac{d v_c(t)}{dt} + 2v_c(t) = 0$$

The solution will be in the form of $A e^{st}$.

the characteristic equation is

$$s^2 + s + 2 = 0$$

$$\Rightarrow s = -\frac{1}{2} \pm j\frac{\sqrt{7}}{2}$$

thus, it is a fact that the roots of the characteristic equation is the poles of the s-domain transfer function.

∴ The poles will determine the type of the response.

case 1 Complex poles $\Rightarrow$ underdamped.

$$v_c(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where $s = -\alpha \pm j\omega_d$

Neper Freq (α), Damped Freq (ω_d)

$$v_c(t) = B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t$$

case 2 Real poles $\Rightarrow$ overdamped.

$$v_c(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

case 3 Double pole "$s_1 = s_2 = -\alpha$"

$\Rightarrow$ critically damped

$$v_c(t) = A_1 e^{-\alpha t} + A_2 t e^{-\alpha t}$$

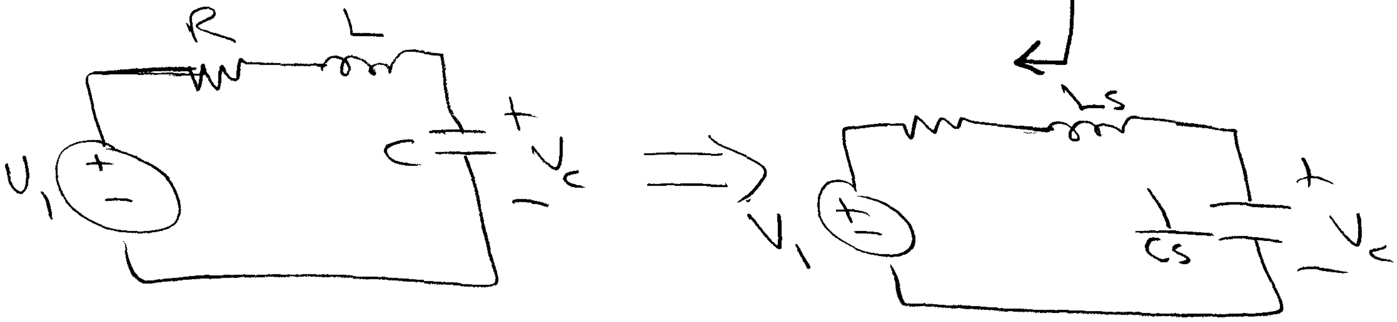
Complex Freq

Pole Locations

* The Natural response of a circuit depends on the pole locations in the s-plane

Demo: 1- http://www.jhu.edu/~signals/explore/index.html

2- In Matlab; rlcdemo



$$H(s) = \frac{V_c}{V_1} = \frac{\frac{1}{Cs}}{R + Ls + \frac{1}{Cs}} = \frac{1/LC}{s^2 + (\frac{R}{L}) + \frac{1}{LC}}$$

$$= \frac{\omega_n^2}{s^2 + 2\alpha s + \omega_n^2} \quad \Bigg| \text{ where } \alpha = \frac{R}{2L}, \quad \omega_n = \frac{1}{\sqrt{LC}}$$

We can factor the denominator of $H(s)$ as:

$$H(s) = \frac{\omega_n^2}{(s - s_1)(s - s_2)} \; ; \text{ where } s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_n^2}$$

assume that the poles are complex

$\Rightarrow s_{1,2} = -\alpha \pm j\omega_d$

where $\omega_d = \sqrt{\omega_n^2 - \alpha^2}$; notice that $\omega_n^2 = \alpha^2 + \omega_d^2$

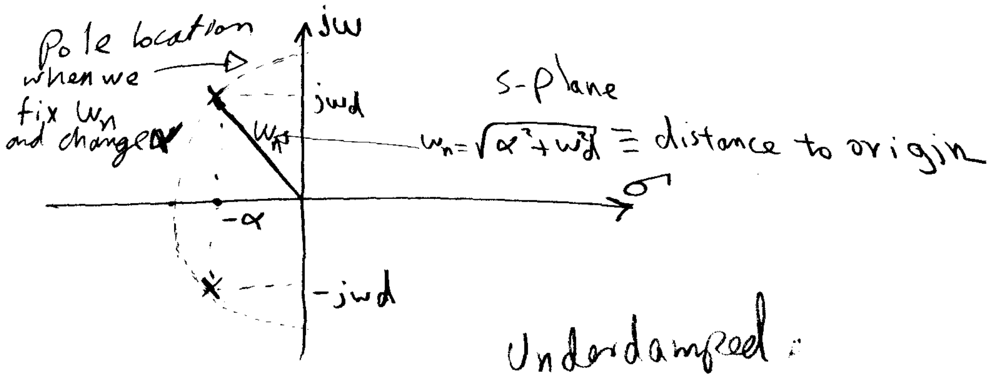
$\Rightarrow \omega_n = \sqrt{\alpha^2 + \omega_d^2}$

Complex Freq

Handout P.7

Poles are

$s = -\alpha \pm j\omega_d$



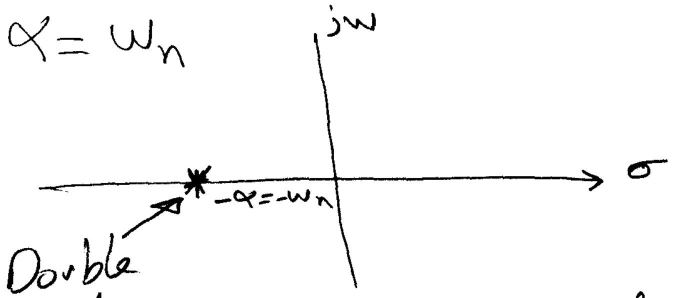
Underdamped:

- $\omega_n > \alpha$

— now increase α until

$\alpha = \omega_n$

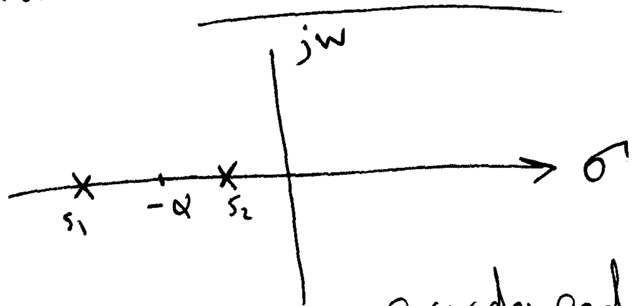
$\Rightarrow$



Critically damped

— now increase α to be $> \omega_n$

$\Rightarrow$



Overdamped.

Complex Freq

In General

$$H(s) = \frac{\overset{\text{gain}}{K}\,\overbrace{(s-z_1)(s-z_2)\cdots(s-z_m)}^{\text{zeros}}}{\underbrace{(s-P_1)(s-P_2)\cdots(s-P_n)}_{\text{poles}}}$$

Note: complex zeros or poles occur in Conjugate pairs.

The natural Response has the form:

$$V_n(t) = A_1 e^{P_1 t} + A_2 e^{P_2 t} + \cdots + A_n e^{P_n t}$$

Poles determine the Natural Response.

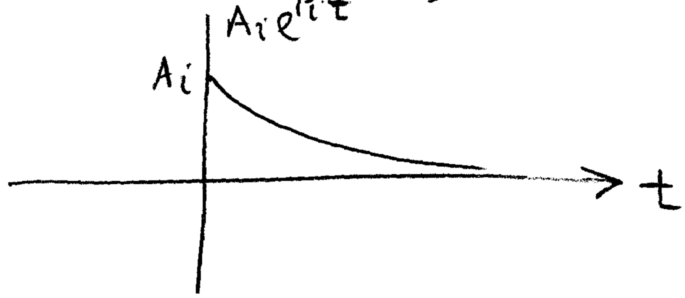
Case 1: Real Poles

a. if $P_i = 0$

$\Rightarrow A_i e^0 = A_i$ ← constant

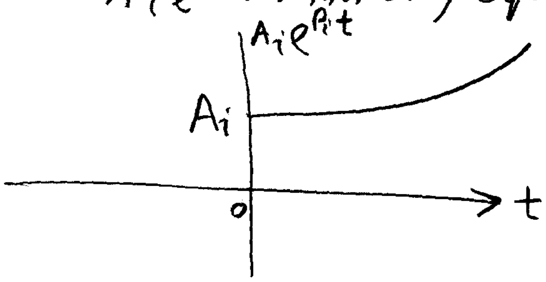
b. if $P_i < 0$

$\Rightarrow A_i e^{P_i t}$ is decaying exponential



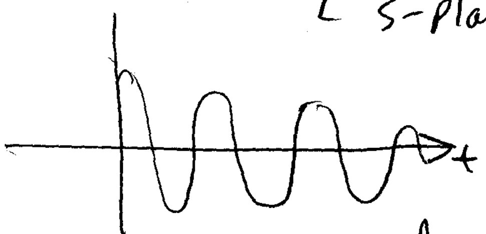
c. if $P_i > 0$

$\Rightarrow A_i e^{P_i t}$ is increasing exponential



Case 2: Complex Poles $s = -\alpha \pm j\omega_d$

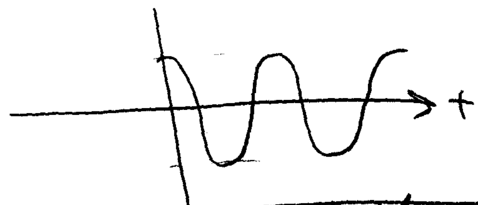
if $\alpha < 0$ [Left half of the S-plane]



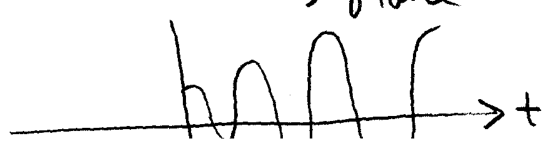
Damped Sinusiod

Stable

if $\alpha = 0 \Rightarrow$ Sinusoid



if $\alpha > 0$ [right half of the S-plane]



Graphical Determination of Frequency Responses

- We saw how the location of poles affect the natural response. "Time Domain"
- In this section, we study the effect of poles and zeros on the Frequency Response. "Frequency Domain"
- Given the transfer function

$$H(s) = \frac{K(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

The Frequency Response is at $s=j\omega$

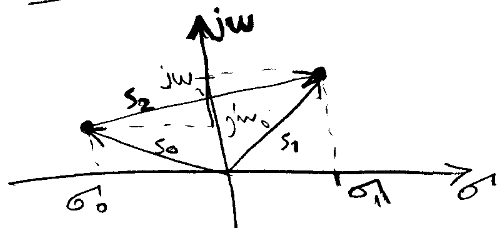
$$\Rightarrow H(j\omega) = \frac{K(j\omega-z_1)(j\omega-z_2)\cdots(j\omega-z_m)}{(j\omega-p_1)(j\omega-p_2)\cdots(j\omega-p_n)}$$

Notice that,

* $(j\omega-p_i)$ is the vector from pole p_i to the point $j\omega$. $\xrightarrow{\text{Phasor Form}} D_i \angle \theta_i$

* $(j\omega-z_i)$ is the vector from zero z_i to the point $j\omega$ $\xrightarrow{\text{Phasor Form}} N_i \angle \phi_i$

Theory: Vector Addition



$\vec{S}_1 = \vec{S}_2 + \vec{S}_0 \Rightarrow \vec{S}_2 = \vec{S}_1 - \vec{S}_0$

from $\vec{S}_0$ to $\vec{S}_1$

Thus, we have:

$$H(j\omega) = \frac{K(N_1\angle\phi_1)(N_2\angle\phi_2)\cdots(N_m\angle\phi_m)}{(D_1\angle\theta_1)(N_2\angle\theta_2)\cdots(D_n\angle\theta_n)}$$

$$= \frac{K N_1 N_2 \cdots N_m}{D_1 D_2 \cdots D_n} \angle(\phi_1+\phi_2+\cdots+\phi_m-\theta_1-\theta_2-\cdots-\theta_n)$$

Thus,

$$\text{Magnitude } |H(j\omega)| = \frac{K N_1 N_2 \cdots N_m}{D_1 D_2 \cdots D_n}$$

Complex Freq.

Example 10.11

$$H(s) = \frac{4(s+2)}{s^2+2s+5}$$

$$= \frac{4(s+2)}{(s+1-j2)(s+1+j2)}$$

Zeros at $z_1 = -2$

Poles at $P_1 = -1+j2$ and $P_2 = -1-j2$

Frequency Response:

Set $s = jw$. Thus,

$$H(jw) = \frac{4(jw+2)}{(jw+1-j2)(jw+1+j2)}$$

$$= \frac{4(2+jw)}{(5-w)^2+j2w}$$

(a) at $w = 1$ rad/s

- Draw vectors from all Poles and zeros to $jw = j1$
- The vector from P_1 to $j1$ is $(j1-P_1) = \sqrt{2}\angle -45°$
- vector from P_2 to $j1$ is $(j1-P_2) = \sqrt{10}\angle 71.6°$
- vector from z_1 to $j1$ is $(j1-z_1) = \sqrt{5}\angle 26.6°$

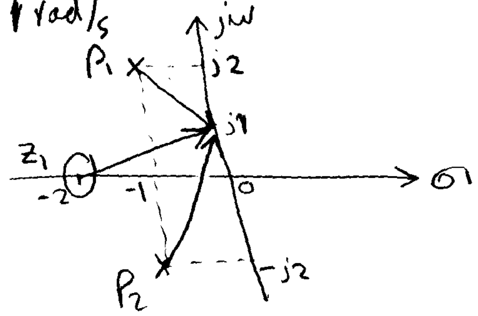
or

$$H(j1) = \frac{4(\sqrt{5}\angle 26.6°)}{(\sqrt{2}\angle -45°)(\sqrt{10}\angle 71.6°)}$$

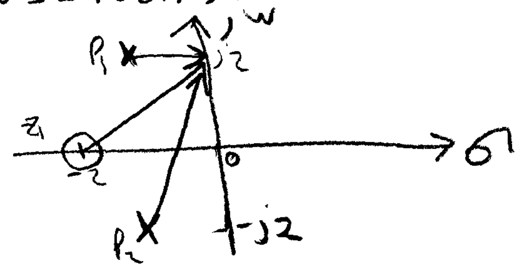
$$= 2\angle 0°$$

P.10

(a) $w = 1$ rad/s



(b) $w = 2$ rad/s



Answer $H(j2) = \frac{4(2\sqrt{17}\angle 45°)}{(1\angle 0°)(\sqrt{17}\angle 76°)}$

$= 2.7\angle -31$