- The Previous example is an example of Linear Block codes. The Parity bits are linear combination of the message. Therefore, we can represent the encoder by a Linear system described by Matrices.

- <u>4.2.2</u> Representing Linear Codes in a Vector Space.

The 2-D code in Example 4.2.2 can be described using vector space.

$$\text{codeword} \rightarrow \bar{C} = \bar{m} \begin{bmatrix} 1&1&1&1&0&0&0&0&0&0&0&0&0& &0&0&0 \\ 1&0&1&0&1&1&0&0&0&0&0&0&0& &0&0&0 \\ 1&0&1&0&0&0&1&1&0&0&0&0&0& &0&0&0 \\ 1&1&0&0&0&0&0&0&1&1&0&0&0& &0&0&0 \\ 1&0&0&0&1&0&0&0&1&0&1&0&0&\cdots&0&0&0 \\ 1&0&0&0&0&0&1&0&1&0&0&1&0& &0&0&0 \\ 1&1&0&0&0&0&0&0&0&0&0&0&1& &1&0&0 \\ 1&0&0&0&1&0&0&0&0&0&0&0&1& &0&1&0 \\ 1&0&0&0&0&0&1&0&0&0&0&0&1& &0&0&1 \end{bmatrix}_{9 \times 16}$$

$$\boxed{\bar{C} = \bar{m}\, G}$$

$\bar{C}$: is 1X16 code word

$\bar{m}$: is 1X9 message bits

G : 9X16 Generator Matrix

~~At~~ At the receiver, we need to find the syndrome bits.

syndrome vector $\bar{S} = [S_0 \, S_1 \cdots S_6]$

$\bar{S} = \bar{V} \, H^T$; where superscript T stands for matrix transpose.

— The matrix H is called the parity check matrix and in the above example, it has size 7×16.

$$
\begin{array}{c}
\begin{array}{cccccccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \end{array} \\
H = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}_{7 \times 16}
\end{array}
$$

notice that if $\underline{k}$ is the length of the message bits, $\underline{n}$ is the length of codeword,

then $G_{k \times n}$ and $H_{\underbrace{(n-k)}_{\text{redundant bits}} \times n}$

<u>Do Example 4.2.2</u>

4.3 Linear Block Codes

- An (n,k) linear block codes encodes k-bit message vectors into n-bit code vectors. The code rate is $R = \frac{k}{n}$.
- In linear codes, the sum of any two codewords is a codeword $\Rightarrow \bar{c}_1 + \bar{c}_2 \in C$
- Zero vector $(\bar{0})$ must be a codeword.

- We can define our codewords using a generator matrix G of size $k \times n$.

$$\bar{c} = \bar{m}\, G \quad ; \text{where } \begin{array}{l} \bar{c} \text{ is } 1\times n \text{ codeword} \\ \bar{m} \text{ is } 1\times m \text{ message} \end{array}$$

Let the rows of G be $\bar{g}_0, \bar{g}_1, \cdots, \bar{g}_{k-1}$

then
$$\bar{C} = \bar{m} \begin{bmatrix} \bar{g}_0 \\ \bar{g}_1 \\ \vdots \\ \bar{g}_{k-1} \end{bmatrix} = m_0 \bar{g}_0 + m_1 \bar{g}_1 + \cdots + m_{k-1} \bar{g}_{k-1}$$

- The number of codewords $|C| = 2^k$, since there are 2^k distinct messages.

- The set of vectors $\{\bar{g}_i\}$ are linearly independent since we must have a set of unique codewords.
  - linearly independent vectors mean that no vector $\bar{g}_i$ can be expressed as a linear combination of the other vectors.
  - These vectors are called basis vectors of the vector space C.
  - The dimension of this vector space is the number of basis vectors which are k.
  - $\bar{g}_i \in C \Rightarrow$ the rows of G are all legal codewords.

4.3.2

- Hamming weight of a vector is the number of nonzero elements in that vector

  example $\bar{c} = [1001101] \Rightarrow w_H(\bar{c}) = 4$

- Hamming distance between two vectors is equal to the Hamming wieght of their sum.

$$d_H(\bar{c}_1, \bar{c}_2) = w_H(\bar{c}_1 + \bar{c}_2)$$

- The minimum Hamming distance of a linear block code is equal to the minimum Hamming weight of the nonzero code vectors.

- Since each $\bar{g}_i \in C$, we must have $w_H(\bar{g}_i) \geq d_{min}$.

  this is a necessary condition but not sufficient.

Therefore, if ~~the~~ Hamming weight of one of the rows of G is less than d_{min}, $\Rightarrow$ d_{min} is not correct or G is not correct.

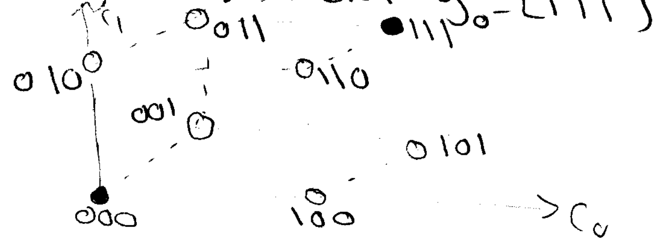
4.4 Decoding Linear Block codes

At the receiver,

$$\bar{r} = \bar{c} + \bar{e} \quad ; \quad \bar{r} \in V$$

and $\bar{e}$ is the error vector of n bits.

Consider the (3,1) repetition code, this is a 1-dimensional code with a sigle basis vector $\bar{g}_0 = [1\,1\,1]$.



So, we have only two valide codewords (000, 111),
but the received vector $\bar{r}$, could be any one of these.

Def: Hamming Sphere

Hamming sphere of radius t is the set of all possible vectors $\bar{r}$ that are at a Hamming distance less than or equal to $\underline{t}$ from a valid codeword.

$$V(n,t) = \sum_{j=0}^{t} \binom{n}{j}$$

Def: Hamming Bound

$$n - k = r \geq \log_2 \left(V(n,t) \right)$$

this is a lower limit on the number of redundant bits.

Def: Perfect codes

A code that satisfies the Hamming bound with equality is called a "perfect code".

Minimum Distance Decoder

The received vector $\bar{v} = \bar{c} + \bar{e}$, the decoder selects the codeword $\bar{c}$ that minimizes the Hamming distant $d_H(\bar{v}, \bar{c})$.

Complete Decoders

Decoders that selects $\bar{c}$ which produce the minimum $d_H(\bar{v}, \bar{c})$

Bounded-distance Decoders

If $d_H(\bar{v}, \bar{c}) \leq t \Rightarrow$ select min $d_H(\bar{v}, \bar{c})$

if $d_H(\bar{v}, \bar{c}) > t \Rightarrow$ Declare a decoder failure, "Error Detection"

Standard array decoders

The simplest, but most expensive strategy for implementing error correction is to simply look up the codeword ($\bar{c}$) in a decoding table that contains all possible $\bar{v}$. The table size grows exponentially with codeword length.

The lookup table will have 2^k columns and $2^{n-k} = 2^r$ rows.

See Example 4.4.1 in Page 137.

Syndrome Decoders and the Parity-Check Theorem.

- The standard-array method becomes impractical with increasing block length.
- Syndrome decoders are more efficient and mostly used with linear Block codes.

- The rows of the code's generator matrix G are basis vectors in the code space C. It is always possible to find basis vectors that give us a systematic code.

$$\Rightarrow \quad G = [P_{k\times r} \mid I_{k\times k}]_{k\times n} \qquad r = n-k$$

where $I_{k\times k}$ is the $k\times k$ identity Matrix and $P_{k\times r}$ is a $k\times k$ parity-bit generator.

The systematic code will be in this Form

$$\bar{c} = [\underbrace{c_0\, c_1 \cdots c_{r-1}}_{\text{parity bits}}\ \underbrace{m_0\, m_1 \cdots m_{k-1}}_{\text{Information bits}}]$$

- At the receiver, the parity-check matrix is

$$H = [I_{r\times r} \mid -P^T]_{r\times n}$$

$\Rightarrow$ For any code vector,

$$\bar{c}H^T = \bar{m}GH^T = \bar{m}\,[P_{k\times r} \mid I_{k\times k}]\begin{bmatrix} I_{r\times r} \\ \hline -P_{k\times r} \end{bmatrix}$$

$$= \bar{m}\,[P_{k\times r} - P_{k\times r}] = \bar{0}$$

∴ Multiplying valid code vectors with H^T result zero vector $\bar{0}$.

$$\boxed{\bar{c}H^T = \bar{0}}$$

— The syndrom vector is $\bar{S} = \{S_0 S_1 \cdots S_{r-1}\}$

$$\bar{S} = \bar{r} H^T$$
$$= (\bar{c} + \bar{e}) H^T = \bar{c} H^T + \bar{e} H^T$$
$$= \bar{0} + \bar{e} H^T = \bar{e} H^T$$

so $\boxed{\bar{S} = \bar{e} H^T}$ a function only of the error vector.

$\Rightarrow$ $\bar{S}$ is zero, if and only if $\bar{e}$ is a valid code vector.

So $\bar{S}$ is zero, { if there are no errors $\Rightarrow \bar{e} = \bar{0}$
or if the errors make $\bar{e} = \bar{c} \Rightarrow$ valid codeword

otherwise, $\bar{S} \neq \bar{0}$ and we have a detectable error. So, the decoder will either detect it or correct it.

— Also, notice that

$GH^T = 0 \Rightarrow$ the rows of H is orthogonal to the rows of G.

$\Rightarrow$ the rows of H must be also linearly independent

— The possible syndrome vectors are in one-to-one correspondence with the error patterns in the standard array.
Therefore, a particular syndrome uniquely identifies a particular error pattern.

— Thus, instead of storing the standard array with 2^k columns and 2^r rows,
It is sufficient to store in a table the $2^r - 1$ nonzero error patterns corresponding to the $2^r - 1$ possible nonzero syndromes.

— Error Correction using the syndrome table.

For complete Decoders {
- If the syndrome is zero $\Rightarrow$ no error.
- If the syndrome is nonzero $\Rightarrow$ look up the error pattern for this syndrome
$\Rightarrow$ Add it to the received vector

For Bounded-distance decoders

- We store only those error patterns which satisfy our distance criterion.
- If we get a nonzero syndrome for which there is no table entry, we declare a decoder failure.