Chapter 5 Signal Space Analysis

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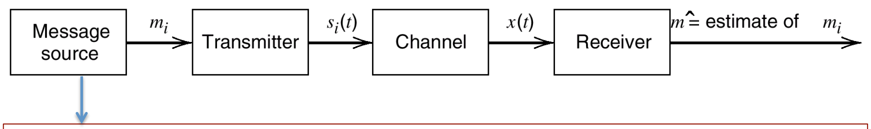
Objective

- *Geometric representation of signals with finite energy, which provides a mathematically elegant and highly insightful tool for the study of data transmission.*
- *Maximum likelihood procedure for the detection of a signal in AWGN channel.*
- *Derivation of the correlation receiver that is equivalent to the matched filter receiver discussed in the previous chapter.*
- *Probability of symbol error and the union bound for its approximate calculation.*

Content

5.1 Introduction



- A *message source* emits one *symbol* every T seconds, with the symbols belonging to an alphabet of *M* symbols denoted by $m_1, m_2, \ldots, m_M$
- *A priori* probabilities $p_1, p_2, \ldots, p_M$ specify the message source output probabilities.
- If the M symbols of the alphabet are *equally likely,* we may express the probability that symbol *m,* is emitted by the source as:

$$p_i = P(m_i) = \frac{1}{M} \text{ for } i = 1, 2, \ldots, M$$

5.1 Introduction

- The transmitter takes the message source output m, and codes it into a *distinct* signal $s_i(t)$ suitable for transmission over the channel.
- The signal $s_i(t)$ occupies the full duration T allotted to symbol m.
- Most important, $s_t(t)$ is a real-valued ***energy signal*** (i.e., a signal with finite energy), as shown by:

$$E_i = \int_0^T s_i^2(t)\,dt, \quad i = 1, \; 2, \; \ldots, \; M$$

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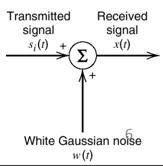
5.1 Introduction

The channel is assumed to have two characteristics:

1. The channel is *linear,* with a bandwidth that is wide enough to accommodate the transmission of signal $s_i(t)$ with negligible or no distortion.
2. The channel noise, *w(t),* is the sample function of a *zero-mean white Gaussian noise process.*

We refer to such a channel as an ***additive white Gaussian noise*** **(AWGN)** ***channel***. Accordingly, we may express the received signal *x(t)* as

$$x(t) = s_i(t) + w(t), \quad \begin{cases} 0 \le t \le T \\ i = 1, 2, \ldots, M \end{cases}$$



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5.1 Introduction

- The receiver has the task of observing the received signal *x(t)* for a duration of T seconds and making a best *estimate* of the transmitted signal $s_i(t)$ or, equivalently, the symbol m_i.
- However, owing to the presence of channel noise, this decision-making process is statistical in nature, with the result that the receiver will make occasional errors.
- The requirement is therefore to design the receiver so as to minimize the ***average probability of symbol error,*** defined as:

$$P_e = \sum_{i=1}^{M} p_i P\left(\hat{m} \neq m_i \middle| m_i\right)$$, where p_i is the priori probability

$P\left(\hat{m} \neq m_i \middle| m_i\right)$ Is the conditional probability

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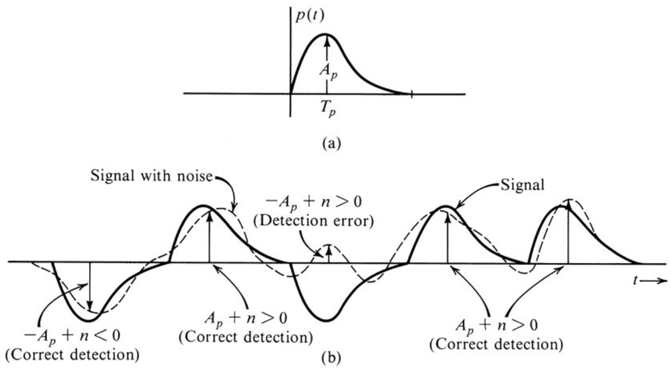
Detection Errors Example



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5.2 Geometric Representation of Signals

The essence of *geometric representation of signals* is to represent any set of M energy signals $\{s_i(t)\}$ as linear combinations of *N orthonormal basis functions,* where N ≤ M.

That is to say, given a set of real-valued energy signals $s_1(t), s_2(t), \ldots, s_M(t)$, each of duration *T* seconds, we write

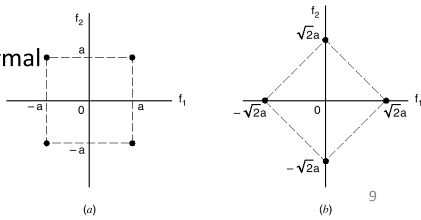
$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t), \qquad \begin{cases} 0 \le t \le T \\ i = 1, 2, \ldots, M \end{cases}$$

Where the coefficients of the expansion are defined by:

$$s_{ij} = \int_0^T s_i(t)\phi_j(t)\,dt, \qquad \begin{cases} i = 1, 2, \ldots, M \\ j = 1, 2, \ldots, N \end{cases}$$

The real-valued basis function are orthonormal

$$\int_0^T \phi_i(t)\phi_j(t)\,dt = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \ne j \end{cases}$$



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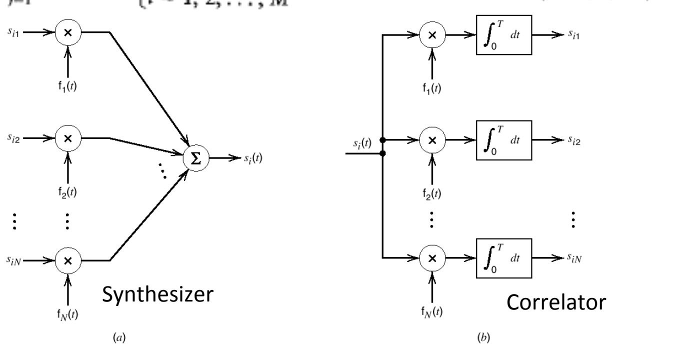
5.2 Geometric Representation of Signals

- The set of coefficients may naturally be viewed as an *N-dimensional vector,* denoted by $\mathbf{s_i}$. The important point to note here is that the vector $\mathbf{s_i}$ bears a *one-to-one* relationship with the transmitted signal $s_i(t)$:

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t), \qquad \begin{cases} 0 \le t \le T \\ i = 1, 2, \ldots, M \end{cases} \qquad s_{ij} = \int_0^T s_i(t)\phi_j(t)\,dt, \qquad \begin{cases} i = 1, 2, \ldots, M \\ j = 1, 2, \ldots, N \end{cases}$$



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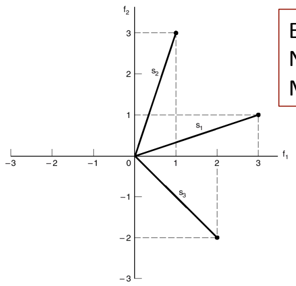
5.2 Geometric Representation of Signals

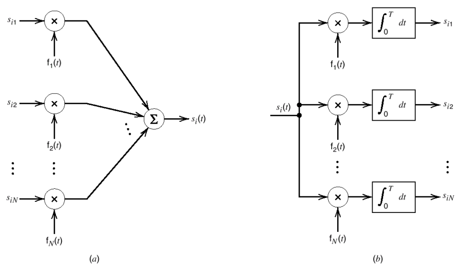
- **Signal Vector:** We may state that each signal is completely determined by the *vector* of its coefficients

$$\mathbf{s}_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix}, \qquad i = 1, 2, \ldots, M$$

- **Signal Space:** The N-Dimensional Euclidean space is called the signal space



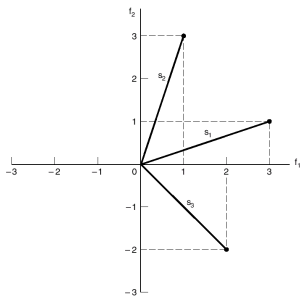
Example
N=2
M=3



5.2 Geometric Representation of Signals

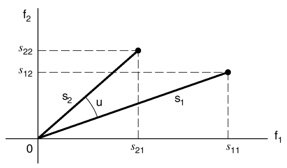
- **Length**: In an N-dimensional Euclidean space, it is customary to denote the length (also called the ***absolute value*** or ***norm***) of a signal vector $\mathbf{s_i}$ by the symbol $\|\mathbf{s_i}\|$
- **Squared-Length**: The squared-length of any signal vector $\mathbf{s_i}$ is defined to be the ***inner product*** or ***dot product*** of $\mathbf{s_i}$,with itself, as shown by:

$$\|\mathbf{s}_i\|^2 = \mathbf{s}_i^T\mathbf{s}_i = \sum_{j=1}^{N} s_{ij}^2, \qquad i = 1, 2, \ldots, M$$



- The ***inner product*** of the signals $s_i(t)$ and $s_k(t)$ over the interval [0, T] is defined as:

$$\int_0^T s_i(t)s_k(t)\,dt = \mathbf{s}_i^T\mathbf{s}_k$$



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5.2 Geometric Representation of Signals

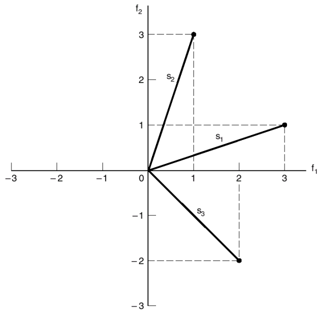
- **Squared Euclidean Distance**:

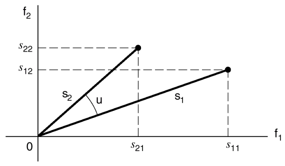
$$\| \mathbf{s}_i - \mathbf{s}_k \|^2 = \sum_{j=1}^{N} (s_{ij} - s_{kj})^2$$
$$= \int_0^T (s_i(t) - s_k(t))^2 dt$$

- **Angle θ_{ik}** between two signal vectors $\mathbf{s}_i$ and $\mathbf{s}_k$

$$\cos \theta_{ik} = \frac{\mathbf{s}_i^T \mathbf{s}_k}{\| \mathbf{s}_i \| \, \| \mathbf{s}_k \|}$$

- The two vectors $\mathbf{s}_i$ and $\mathbf{s}_k$ are ***orthogonal*** or ***perpendicular*** to each other if their inner product $\mathbf{s}_i^T \mathbf{s}_k$ is zero, in which case θ_{ik} = 90 degrees.





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5.2 Geometric Representation of Signals

There is an interesting relationship between the energy content of a signal and its representation as a vector.

$$E_i = \sum_{j=1}^{N} s_{ij}^2$$
$$= \| \mathbf{s}_i \|^2$$

Signal energy is equal to its inner product or squared-length

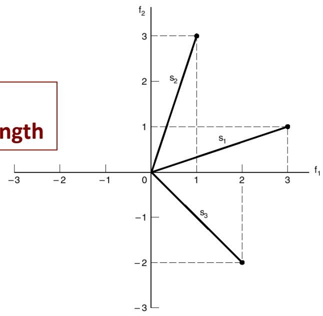
Proof:

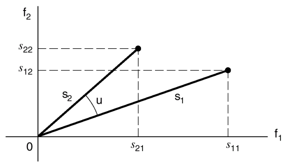
$$E_i = \int_0^T s_i^2(t)\, dt$$

$$E_i = \int_0^T \left[\sum_{j=1}^{N} s_{ij} \phi_j(t) \right] \left[\sum_{k=1}^{N} s_{ik} \phi_k(t) \right] dt$$

$$E_i = \sum_{j=1}^{N} \sum_{k=1}^{N} s_{ij} s_{ik} \int_0^T \phi_j(t) \phi_k(t) dt$$

$$E_i = \sum_{j=1}^{N} s_{ij}^2$$
$$= \| \mathbf{s}_i \|^2$$





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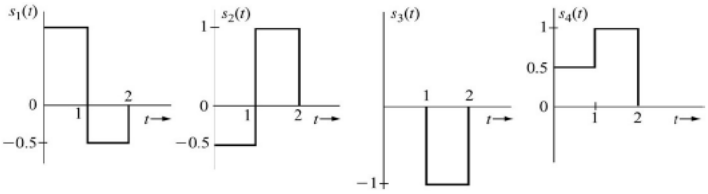
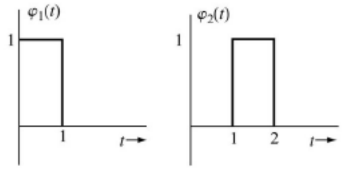
5.2 Example 1/3

- Find a set of orthonormal basis function for the following



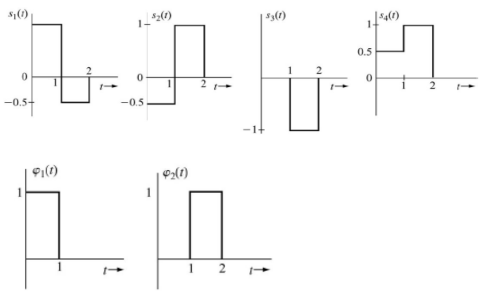
Solution:

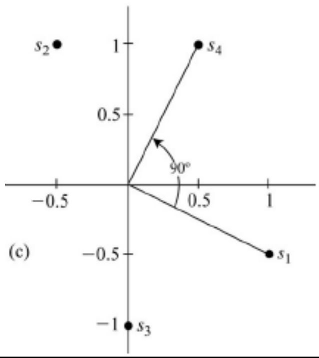


5.2 Example 2/3

- Find the signal vector of the four signals
  $s_1 = (1 , -0.5)$, $s_2 = (-0.5, 1)$, $s_3 = (0, -1)$, $s_4 = (0.5, 1)$

- Represent these signals geometrically in the vector space





5.2 Example 3/3

- Find the energy of signals $s_1(t)$ and $s_4(t)$

$$E_1 = \|s_1\|^2 = 1^2 + (-0.5)^2 = 1.25$$

$$E_4 = \|s_4\|^2 = (0.5)^2 + 1^2 = 1.25$$

- Find the Squared Euclidean Distance between $s_1(t)$ and $s_4(t)$

$$\begin{aligned} d_{14}^2 &= \|s_1 - s_4\|^2 \\ &= (1-0.5)^2 + (-0.5-1)^2 \\ &= 0.25 + 2.25 = 2.5 \end{aligned}$$

- Find the angle between $s_1(t)$ and $s_4(t)$

$$\cos\theta_{14} = \frac{S_1^T S_4}{\|S_1\|\|S_4\|} = 0 \quad \Rightarrow \theta_{14} = 90^o$$ Thus, $s_1(t)$ and $s_2(t)$ are orthogonal

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Gram-Schmidt Orthogonalization Procedure

Gram-Schmidt orthogonalization procedure provides a *complete orthonormal set of basis functions.*

- Suppose we have a set of M energy signals denoted by $s_1(t)$, $s_2(t), \ldots, s_M(t)$.
- Starting with $s_1(t)$ chosen from this set arbitrarily, the first basis function is defined by:

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

- Where E_1 is the energy of the signal $s_1(t)$. Then, clearly, we have

$$\begin{aligned} s_1(t) &= \sqrt{E_1}\phi_1(t) \\ &= s_{11}\phi_1(t) \end{aligned}$$

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Gram-Schmidt Orthogonalization Procedure

- Next, using the signal $s_2(t)$, we define the coefficient s_{21} as

$$s_{21} = \int_0^T s_2(t)\phi_1(t)dt$$

The projection of s_2(t) into the basis Φ_1(t)

- We may thus introduce a new intermediate function

$$g_2(t) = s_2(t) - s_{21}\phi_1(t)$$

Subtract the contribution of the first basis from s_2(t)

- Note that g_2(t) is orthogonal to $\Phi_1(t)$
- Now, we are ready to define the second basis function as:

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)dt}} = \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_2 - s_{21}^2}}$$

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Gram-Schmidt Orthogonalization Procedure

- Continuing in this fashion, we may in general define

$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij}\phi_j(t)$$

- Where $s_{ij} = \int_0^T s_i(t)\phi_j(t)dt, \quad j = 1, 2, \ldots, i-1$

- The basis function are

$$\phi_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T g_i^2(t)dt}}, \quad i = 1, 2, \ldots, N$$

- The dimension N is less than or equal to the number of given signals, M, depending on whether the signals are linearly independent or not.

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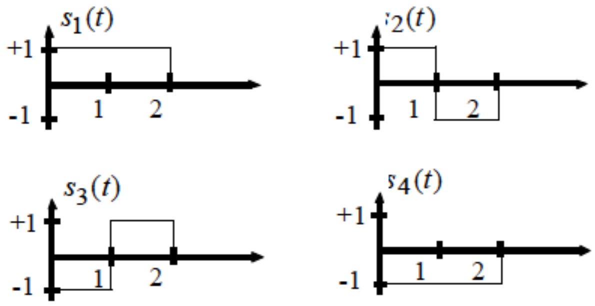
Gram-Schmidt Orthogonalization Procedure: Example



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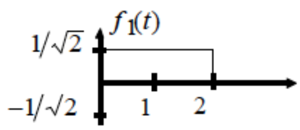
Example: Step 1

$$E_1 = \int_{-\infty}^{\infty} |s_1(t)|^2 dt = 2$$

- $f_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{s_1(t)}{\sqrt{2}}$



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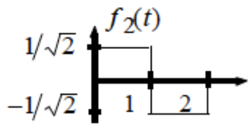
Example: Step 2

$$c_{12} = \int_{-\infty}^{\infty} f_1(t) s_2(t) dt = 0$$

$$f_2'(t) = s_2(t) - c_{12} f_1(t) = s_2(t)$$

$$E_2 = \int_{-\infty}^{\infty} |s_2(t)|^2 dt = 2$$

- $f_2(t) = \frac{s_2(t)}{\sqrt{E_2}} = \frac{s_2(t)}{\sqrt{2}}$



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Example: Step 3

$$c_{13} = \int_{-\infty}^{\infty} f_1(t) s_3(t) dt = 0$$

$$c_{23} = \int_{-\infty}^{\infty} f_2(t) s_3(t) dt = -\sqrt{2}$$

$$f_3'(t) = s_3(t) - c_{13} f_1(t) - c_{23} f_2(t)$$
$$= s_3(t) + \sqrt{2} f_2(t) = 0$$

- No new basis function

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Example: Step 4

$$c_{14} = \int_{-\infty}^{\infty} f_1(t) s_4(t) dt = -\sqrt{2}$$

$$c_{24} = \int_{-\infty}^{\infty} f_2(t) s_4(t) dt = 0$$

$$f_4'(t) = s_4(t) - c_{14} f_1(t) - c_{24} f_2(t)$$
$$= s_4(t) + \sqrt{2} f_1(t) = 0$$

- No new basis function. Procedure Complete

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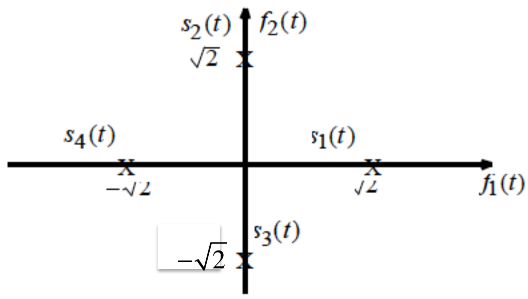
Signal Constellation Diagram



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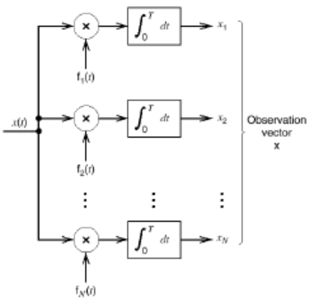
5.3 Conversion of the Continuous AWGN Channel into a Vector Channel

- Suppose that the input to the bank of N product integrators or correlators is the received signal *x(t)* defined as:

$$x(t) = s_i(t) + w(t), \qquad \begin{cases} 0 \le t \le T \\ i = 1, 2, \ldots, M \end{cases}$$

- where *w(t)* is a sample function of a white Gaussian noise process W(t) of zero mean and power spectral density $N_0/2$.
- the output of correlator j is the sample value of a random variable X_j



$$x_j = \int_0^T x(t)\phi_j(t)dt = s_{ij} + w_j, \qquad j = 1, 2, \ldots, N$$

where $s_{ij} = \int_0^T s_i(t)\phi_j(t)dt$ and $w_j = \int_0^T w(t)\phi_j(t)dt$

deterministic

Random Variable

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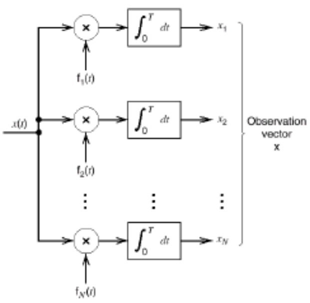
5.3 Conversion of the Continuous AWGN Channel into a Vector Channel

- Each correlator output X_j is a Gaussian random variable with mean s_{ij} and variance $N_o/2$. *(see the proof in section 5.3 in textbook)*

$$f_{X_j}(x_j|m_i) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(x_j - s_{ij})^2\right], \qquad \begin{matrix} j = 1, 2, \ldots, N \\ i = 1, 2, \ldots, M \end{matrix}$$

- Also the correlator output X_j are mutually uncorrelated and therefore they are statistically independent.
- Thus, the joint conditional pdf of the observation vector **X** of length N is:



$$f_{\mathbf{X}}(\mathbf{x}|m_i) = (\pi N_0)^{-N/2} \exp\left[-\frac{1}{N_0}\sum_{j=1}^{N}(x_j - s_{ij})^2\right], \qquad i = 1, 2, \ldots, M$$

5.4 Likelihood Functions

- At the receiver, we are given the observation vector **x** and the requirement is to estimate the message symbol m; that is responsible for generating x.
- We introduce the ***likelihood function***, denoted by *L(m$_i$)*

$$L(m_i) = f_{\mathbf{X}}(\mathbf{x}|m_i), \qquad i = 1, 2, \ldots, M$$

- In practice, we find it more convenient to work with the ***log-likelihood function***, denoted by *l(m$_i$)*

$$l(m_i) = \log L(m_i), \qquad i = 1, 2, \ldots, M$$

- For the observation vector x over AWGN channels, the log-likelihood functions are:

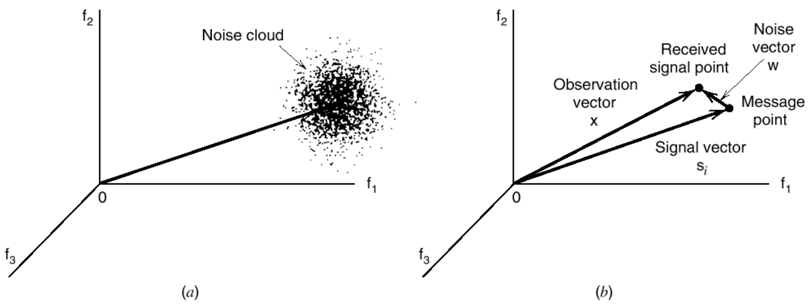
$$l(m_i) = -\frac{1}{N_0}\sum_{j=1}^{N}(x_j - s_{ij})^2, \qquad i = 1, 2, \ldots, M$$

Squared Euclidean Distance

5.5 Coherent Detection of Signals in Noise

Signal Detection Problem:

Given the observation vector **x**, perform a mapping from **x** to an estimate $\hat{m}$ of the transmitted symbol, m_i, in a way that would minimize the probability of error in the decision-making process.



5.5 Maximum a posteriori probability rule

Probability of error

- Suppose that, given the observation vector x, we make the decision $\hat{m} = m_i$. The probability of error in this decision, which we denote by $P_e(m_i|\mathbf{x})$, is simply

$$P_e(m_i|\mathbf{x}) = P(m_i \text{ not sent } |\mathbf{x})$$
$$= 1 - P(m_i \text{ sent } |\mathbf{x})$$

Optimum decision rule

The maximum a posteriori probability (MAP) rule is:

$$\text{Set } \hat{m} = m_i \text{ if}$$
$$P(m_i \text{ sent} \mid \mathbf{x}) \geq P(m_k \text{ sent} \mid \mathbf{x}) \qquad \text{for all } k \neq i$$

Using Bayes' rule, the MAP rule becomes:

$$\text{Set } \hat{m} = m_i \text{ if}$$
$$\frac{p_k f_{\mathbf{X}}(\mathbf{x}|m_k)}{f_{\mathbf{X}}(\mathbf{x})} \text{ is maximum for } k = i$$

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5.5 MAP rule

- **The MAP rule is:**

$$\text{Set } \hat{m} = m_i \text{ if}$$
$$\frac{p_k f_{\mathbf{X}}(\mathbf{x}|m_k)}{f_{\mathbf{X}}(\mathbf{x})} \text{ is maximum for } k = i$$

- **Where**
  - whete p_k is the *a priori* probability of transmitting symbol m_k
  - $f_x(\mathbf{x}\ \mathrm{I}\ m_k)$ is the conditional probability density function of the random observation vectot **X** given the transmission of symbol m_k
  - and $f_x(\mathbf{x})$ is the unconditional probability density function of **X**.
- **Note that**
  - The denominator term $f_x(\mathbf{x})$ is independent of the transmitted symbol.
  - The *a priori* probability $p_k = p_i$ when all the source symbols are transmitted with equal probability.
  - The conditional probability density function $f_x(\mathbf{x}|\ m_k)$ bears a one-to-one relationship to the log-likelihood function $l(m_k)$.

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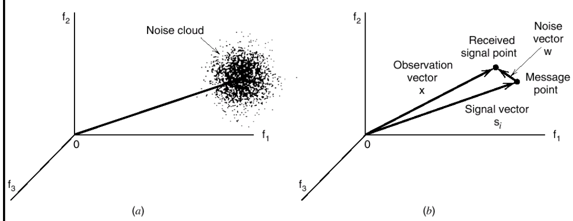
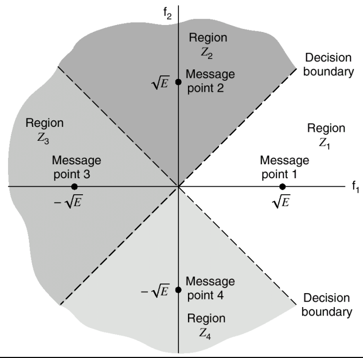
5.5 Maximum likelihood (ML) rule

- Thus, for equally probable symbols, the MAP rule becomes equivalent to the Maximum likelihood (ML) rule such as:

$$\text{Set } \hat{m} = m_i \text{ if}$$
$$l(m_k) \text{ is maximum for } k = i$$

Where $l(m_k)$ is the log-likelihood function



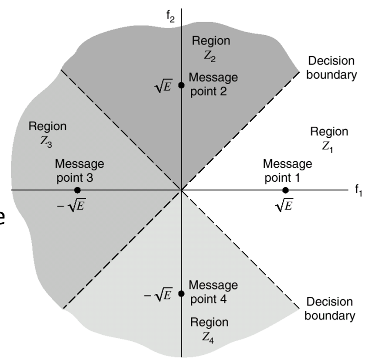


5.5 Graphical Interpretation of MLD rule

- Let Z denote the N-dimensional space of all possible obsetvation vectors x.
- We refer to this space as the *observation space.*
- Because we have assumed that the decision rule must say $\hat{m} = m_i$ where i = 1,2,..., M, the total observation space Z is correspondingly partitioned into *M-decision regions,* denoted by Z_1, Z_2,.. . , Z_M .
- Accordingly, we may restate the ML decision rule of as follows:

Observation vector x lies in region Z_i if

$l(m_k)$ is maximum for $k = i$



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5.5 MLD rule for AWGN channels

- Recall that for AWGN channels,

$$l(m_i) = -\frac{1}{N_0}\sum_{j=1}^{N}(x_j - s_{ij})^2, \quad i = 1, 2, \ldots, M$$

- Note that *$l(m_k)$* attains its maximum value when the summation term is minimized.
- Therefore, the MLD rule for AWGN channels is to **minimize the squared-Euclidian distance**

Observation vector $\mathbf{x}$ lies in region Z_i if $\sum_{j=1}^{N}(x_j - s_{kj})^2$ is minimum for $k = i$ Where $\sum_{j=1}^{N}(x_j - s_{kj})^2 = \|\mathbf{x} - \mathbf{s}_k\|^2$

Observation vector $\mathbf{x}$ lies in region Z_i if the Euclidean distance $\|\mathbf{x} - \mathbf{s}_k\|$ is minimum for $k = i$

For equally likely signals, the maximum likelihood decision rule is simply to choose the message point closest to the received signal point

5.5 MLD rule for AWGN channels

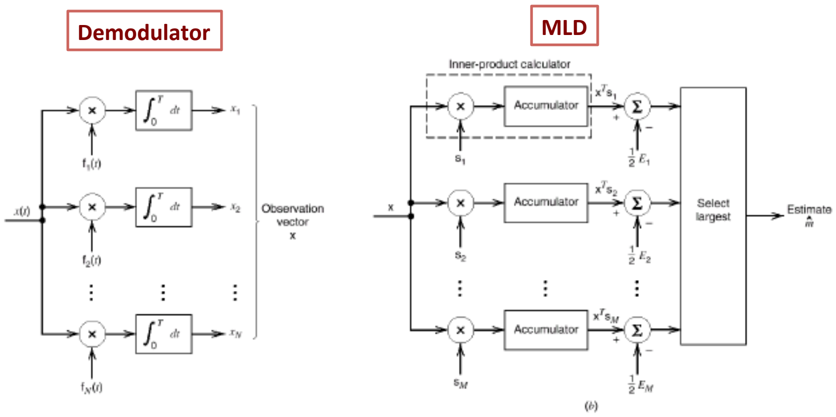
- The squared Euclidean distance could be expanded as:

$$\sum_{j=1}^{N}(x_j - s_{kj})^2 = \sum_{j=1}^{N} x_j^2 - 2\sum_{j=1}^{N} x_j s_{kj} + \sum_{j=1}^{N} s_{kj}^2$$

  - The first summation term of this expansion is independent of the index *k* and may therefore be ignored.
  - The second summation term is the inner product of the observation vector x and signal vector s_k.
  - The third summation term is the energy of the transmitted signal $s_k(t)$
- Therefore, the MLD rule becomes

Observation vector $\mathbf{x}$ lies in region Z_i if $\sum_{j=1}^{N} x_j s_{kj} - \frac{1}{2}E_k$ is maximum for $k = i$

5.6 Correlation Receiver

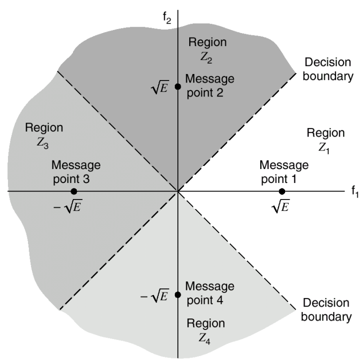


5.7 Probability of Error

- Suppose also that symbol m_i is transmitted, an error occurs whenever the received signal point does not fall inside region Z_i
- Averaging over all possible transmitted symbols, we readily see that the *average probability of symbol error, P_e* is

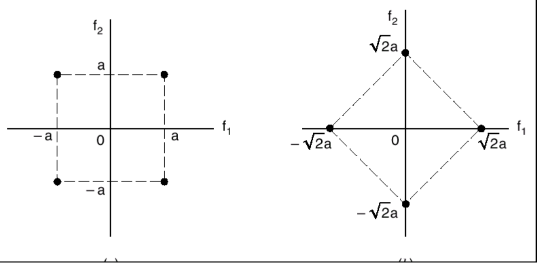
$$
\begin{aligned}
P_e &= \sum_{i=1}^{M} p_i \, P(\mathbf{x} \text{ does not lie in } Z_i | m_i \text{ sent}) \\
&= \frac{1}{M} \sum_{i=1}^{M} P(\mathbf{x} \text{ does not lie in } Z_i | m_i \text{ sent}) \\
&= 1 - \frac{1}{M} \sum_{i=1}^{M} P(\mathbf{x} \text{ lies in } Z_i | m_i \text{ sent})
\end{aligned}
$$

$$P_e = 1 - \frac{1}{M} \sum_{i=1}^{M} \int_{Z_i} f_{\mathbf{X}}(\mathbf{x} | m_i) \, d\mathbf{x}$$



5.7 Invariance of the Probability of Error to Rotation and Translation

- Changes in the orientation of the signal constellation with respect to both the coordinate axes and origin of the signal space do *not* affect the probability of symbol error P_e
- This result is a consequence of two facts
  - In maximum likelihood detection, the probability of symbol error P_e depends solely on the relative Euclidean distances between the message points in the constellation.
  - The additive white Gaussian noise is *spherically symmetric* in all directions in the signal space.



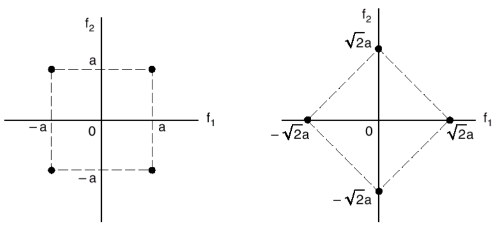
5.7 Invariance of the Probability of Error to Rotation and Translation

- Suppose all the message points in a signal constellation are translated by a constant vector amount **a**

$$\mathbf{s}_{i,\text{translate}} = \mathbf{s}_i - \mathbf{a}, \qquad i = 1, 2, \ldots, M$$

- The observation vector is correspondingly translated by the same vector amount $\mathbf{x}_{\text{translate}} = \mathbf{x} - \mathbf{a}$
- Then, $\|\mathbf{x}_{\text{translate}} - \mathbf{s}_{i,\text{translate}}\| = \|\mathbf{x} - \mathbf{s}_i\| \quad \text{for all } i$

If a signal constellation is translated by a constant vector amount, then the probability of symbol error P_e incurred in maximum likelihood signal detection over an AWGN channel is completely unchanged.



(a) (b)

5.7 Minimum Energy Signals

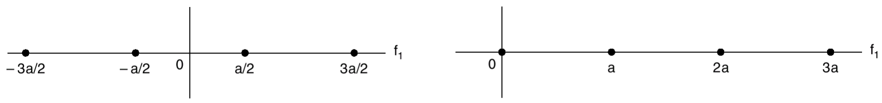
Given a signal constellation $\{\mathbf{s}_i\}_{i=1}^{M}$, the corresponding signal constellation with minimum average energy is obtained by subtracting from each signal vector $\mathbf{s}_i$ in the given constellation an amount equal to the constant vector E[**s**],

Where $E[\mathbf{s}] = \sum_{i=1}^{M} \mathbf{s}_i p_i$

Thus the minimum translate vector is $\mathbf{a}_{min} = E[\mathbf{s}]$

and the minimum energy of the translated signal constellation is

$$\mathscr{E}_{\text{translate,min}} = \mathscr{E} - \| \mathbf{a}_{\text{min}} \|^2$$



(*a*) (*b*)

5.7 Minimum Energy Signals

Proof:

The average energy of this signal constellation translated by vector amount **a** is:

$$\mathscr{E}_{\text{translate}} = \sum_{i=1}^{M} \| \mathbf{s}_i - \mathbf{a} \|^2 p_i$$

The squared Euclidean distance between $\mathbf{s}_i$ and **a** is expanded as:

$$\| \mathbf{s}_i - \mathbf{a} \|^2 = \| \mathbf{s}_i \|^2 - 2\mathbf{a}^T\mathbf{s}_i + \| \mathbf{a} \|^2$$

Therefore

$$\mathscr{E}_{\text{translate}} = \sum_{i=1}^{M} \| \mathbf{s}_i \|^2 p_i - 2 \sum_{i=1}^{M} \mathbf{a}^T \mathbf{s}_i p_i + \| \mathbf{a} \|^2 \sum_{i=1}^{M} p_i$$
$$= \mathscr{E} - 2\mathbf{a}^T E[\mathbf{s}] + \| \mathbf{a} \|^2$$

Where $E[\mathbf{s}] = \sum_{i=1}^{M} \mathbf{s}_i p_i$

Differentiating the above Equation with respect to the vector **a** and then setting the result equal to zero, the minimizing translate is: $\mathbf{a}_{min} = E[\mathbf{s}]$

and the minimum energy is $\mathscr{E}_{\text{translate,min}} = \mathscr{E} - \| \mathbf{a}_{\text{min}} \|^2$

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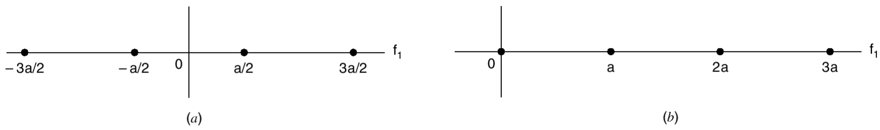
Example

Assuming equally likely signals, Find the Average energy of the following signal constellations



For (a)

$$E_a = \frac{1}{4}\left(2\left(\frac{\alpha^2}{4}\right) + 2\left(\frac{9\alpha^2}{4}\right)\right)$$
$$= \frac{5}{4}\alpha^2$$

For (b)

$$E_b = \frac{1}{4}\left(\alpha^2 + 4\alpha^2 + 9\alpha^2\right)$$
$$= \frac{14}{4}\alpha^2$$

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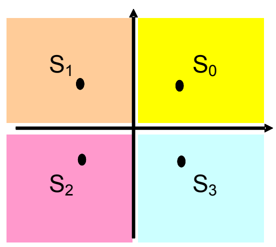
Pairwise Error Probability

For AWGN channels and equally likely signals, the pairwise error probability of two signals s_i and s_k depends on the Euclidean distance between the two signals:

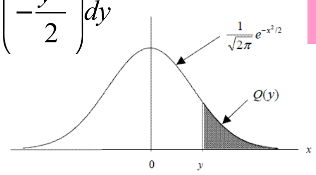
$$\Pr\{s_i \to s_k\} = \frac{1}{2} erfc\left(\frac{\|s_i - s_k\|}{2\sqrt{N_0}}\right) = Q\left(\frac{\|s_i - s_k\|}{\sqrt{2N_0}}\right)$$



Where Q(.) is the Gaussian Q function.

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{y^2}{2}\right) dy$$

$$Q(x) = \frac{1}{2} erfc\left(\frac{x}{\sqrt{2}}\right)$$



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The Q-function in Matlab

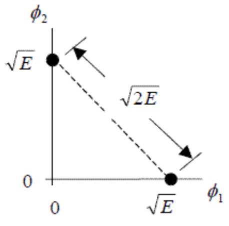
```
function out=q(x)

%Q Function (Gaussian Q-function)
%  Area under the tail of a Gaussian pdf with
%  mean zero and variance 1 from x to inf.
%
%  See also: ERF, ERFC, QINV

out=0.5*erfc(x/sqrt(2));
```


Pairwise Error Probability: Example FSK

- The signal constellation for binary FSK is:



E is the average signal Energy

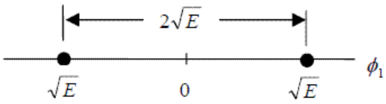
The Euclidean distance between the two signals is:

$$d_{12} = \left\| s_1 - s_2 \right\| = \sqrt{2E}$$

$$\Pr\left\{ s_i \to s_j \right\} = Q\left(\sqrt{\frac{E}{N_0}} \right) = \frac{1}{2} erfc\left(\sqrt{\frac{E}{2N_0}} \right)$$

Pairwise Error Probability: Example Binary PSK

- The signal constellation for binary PSK is:



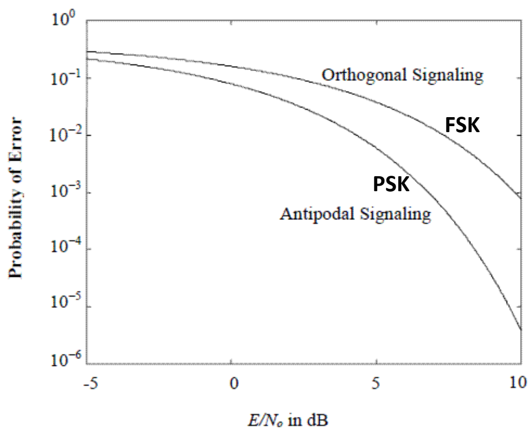
E is the average signal Energy

The Euclidean distance between the two signals is:

$$\|s_1 - s_2\| = 2\sqrt{E}$$

$$\Pr\{s_i \to s_j\} = Q\left(\sqrt{\frac{2E}{N_0}}\right) = \frac{1}{2} erfc\left(\sqrt{\frac{E}{N_0}}\right)$$

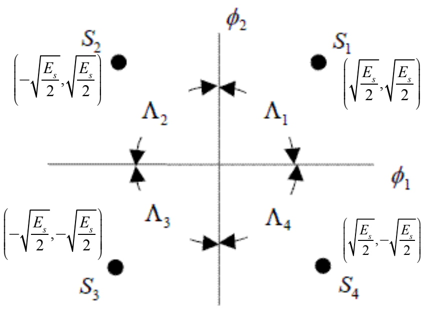
Error Probability for Binary FSK and PSK



Signal to noise ration (SNR)

Decision Regions

- Minimum distance detection rule:



The average symbol energy E_s is defined as:

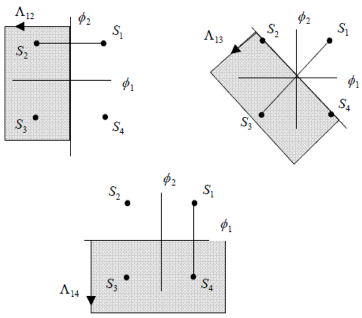
$$E_s = \frac{\sum_{i=1}^{M} |S_i|^2}{M}$$

Where M is the signal set size.

Union Bound

- Assume that the signal set size is M, for equally probable transmission, the probability of error is:

$$P_e \le \sum_{j=2}^{M} \Pr\{E|s_i\}$$



- For example, QPSK:

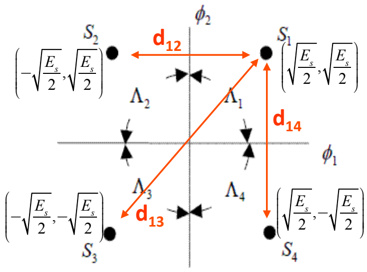
$$P_e \le \Pr\{s_1 \to s_2\} + \Pr\{s_1 \to s_3\} + \Pr\{s_1 \to s_4\}$$

Union Bound: QPSK example

- The Euclidean Distances are:

$$d_{12} = d_{14} = 2\left(\sqrt{\frac{E_s}{2}}\right) = \sqrt{2E_s}$$

$$d_{13} = 2\sqrt{E_s}$$



- The symbol error rate for QPSK is:

$$P_e \le 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) + Q\left(\sqrt{\frac{2E_s}{N_0}}\right) = erfc\left(\sqrt{\frac{E_s}{2N_0}}\right) + \frac{1}{2} erfc\left(\sqrt{\frac{E_s}{N_0}}\right)$$

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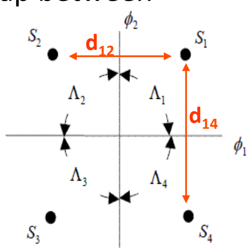
Tight Union Bound: QPSK example

- To get a tighter union bound, reduce overlap between decision regions.

$$d_{12} = d_{14} = 2\left(\sqrt{\frac{E_s}{2}}\right) = \sqrt{2E_s}$$



- The symbol error rate for QPSK is:

$$P_e \le \Pr\{s_1 \to s_2\} + \Pr\{s_1 \to s_4\}$$

$$P_e \le 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) = erfc\left(\sqrt{\frac{E_s}{2N_0}}\right)$$

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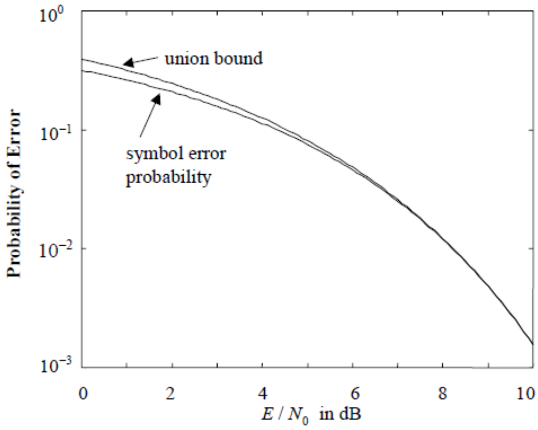
QPSK Symbol error probability and union bound



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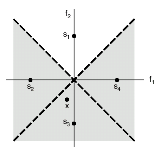
Union Bound: Circularly Symmetric

The probability of symbol error, averaged over all the M symbols, is overbounded as follows:

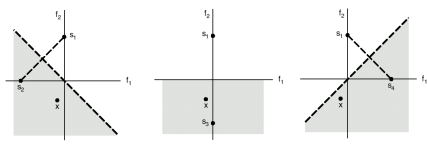
$$P_e = \sum_{i=1}^{M} p_i P_e(m_i)$$
$$\leq \frac{1}{2} \sum_{i=1}^{M} \sum_{\substack{k=1 \\ k \neq i}}^{M} p_i \operatorname{erfc}\left(\frac{d_{ik}}{2\sqrt{N_0}}\right)$$

For *circularly symmetric* constellations about the origin, such as QSPK

$$P_e \leq \frac{1}{2} \sum_{\substack{k=1 \\ k \neq i}}^{M} \operatorname{erfc}\left(\frac{d_{ik}}{2\sqrt{N_0}}\right) \text{ for all } i$$



(a)



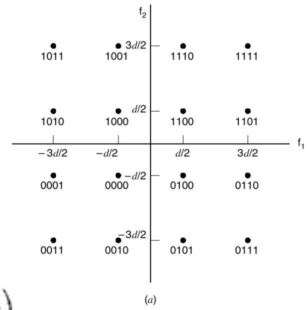
(b)

Union Bound: Rectangular Constellations

For rectangular constellations, such as 16QAM, the error rate will be dominated by the minimum distance.

$$d_{min} = \min_{k \neq i} d_{ik} \quad \text{for all } i \text{ and } k$$

Thus $$\text{erfc}\left(\frac{d_{ik}}{2\sqrt{N_0}}\right) \leq \text{erfc}\left(\frac{d_{min}}{2\sqrt{N_0}}\right) \text{ for all } i \text{ and } k$$



And the average probability of symbol error will be:

$$P_e \leq \frac{(M-1)}{2} \text{erfc}\left(\frac{d_{min}}{2\sqrt{N_0}}\right)$$

Since erfc is bounded by $$\text{erfc}\left(\frac{d_{min}}{2\sqrt{N_0}}\right) \leq \frac{1}{\sqrt{\pi}} \exp\left(-\frac{d_{min}^2}{4N_0}\right)$$

Then $$P_e \leq \frac{(M-1)}{2\sqrt{\pi}} \exp\left(-\frac{d_{min}^2}{4N_0}\right)$$

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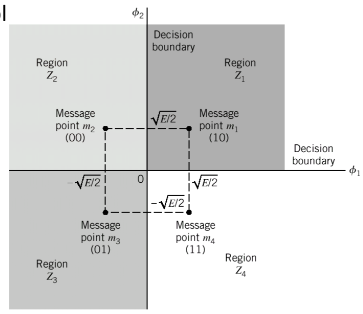
Bit versus symbol error probability

Case 1: Gray Code

In the first case, we assume that it is possible to perform the mapping from binary to M-ary symbols in such a way that the two binary M-tuples corresponding to any pair of adjacent symbols in the M-ary modulation scheme differ in only one bit position.

Moreover, given a symbol error, the most probable number of bit errors is one. subject to the aforementioned mapping constraint. Since there are $\log_2 M$ bits per symbol it follows that the average probability of symbol error is related to the bit error rate as follows:

$$\begin{aligned} P_e &= P\left(\bigcup_{i=1}^{\log_2 M} \{i\text{th bit is in error}\}\right) \\ &\leq \sum_{i=1}^{\log_2 M} P(i\text{th bit is in error}) \\ &= \log_2 M \cdot (\text{BER}) \end{aligned}$$



Bit versus symbol error probability

Case 1: Gray Code

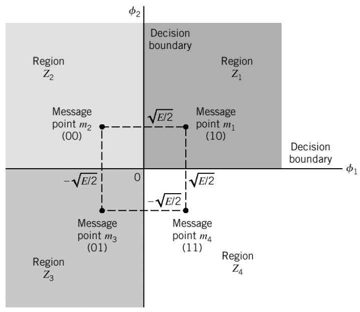
We also note that

$$P_e \geq P(i\text{th bit is in error}) = \text{BER}$$

It follows therefore that the bit error rate is bounded as follows:

$$\frac{P_e}{\log_2 M} \leq \text{BER} \leq P_e$$

$$P_e = P\left(\bigcup_{i=1}^{\log_2 M} \{i\text{th bit is in error}\}\right) \leq \sum_{i=1}^{\log_2 M} P(i\text{th bit is in error}) = \log_2 M \cdot (\text{BER})$$



Bit versus symbol error probability

Case 2

Let M = 2^K, where K is an integer. We assume that all symbol errors are equally likely and occur with probability

$$\frac{P_e}{M-1} = \frac{P_e}{2^K - 1}$$ where P_e is the average probability of symbol error

What is the probability that the i^{th} bit in a symbol is in error?

there are 2^{K-1} cases of symbol error in which this particular
are 2^{K-1} cases in which it is not changed.

Hence, the bit error rate is

$$\text{BER} = \left(\frac{2^{K-1}}{2^K - 1}\right)P_e = \left(\frac{M/2}{M-1}\right)P_e$$

Note that for large M, the bit error rate approaches the limiting value of $P_e/2$