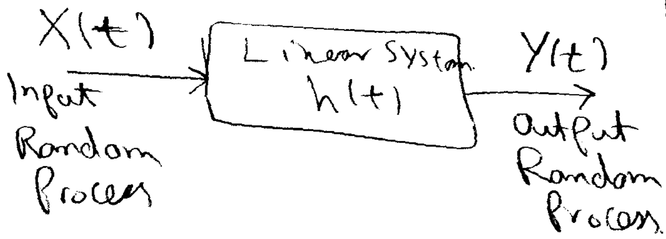
8.2 Random Signal Response of Linear Systems

$$Y(t) = \int_{-\infty}^{\infty} h(\eta) X(t-\eta)\, d\eta$$



Mean

$$E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(\eta) X(t-\eta)\, d\eta\right]$$

$$= \int_{-\infty}^{\infty} h(\eta) E[X(t-\eta)]\, d\eta$$

$$= \bar{X} \int_{-\infty}^{\infty} h(\eta)\, d\eta = \bar{Y}$$

∴ $\bar{Y}$ Equals to the mean of $X(t)$ times the area of the impulse Response.

Power of $Y(t)$ "Mean-Squared Value"

$$E[Y^2(t)] = E\left[\int_{-\infty}^{\infty} h(\eta_1) X(t-\eta_1)\, d\eta_1 \cdot \int_{-\infty}^{\infty} h(\eta_2) X(t-\eta_2)\, d\eta_2\right]$$

$$\overline{Y^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t-\eta_1) X(t-\eta_2)]\, h(\eta_1) h(\eta_2)\, d\eta_1 d\eta_2$$

$$\overline{Y^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\eta_1-\eta_2) h(\eta_1) h(\eta_2)\, d\eta_1 d\eta_2$$

Example

Let $R_{XX}(\eta_1-\eta_2) = \frac{N_0}{2}\delta(\eta_1-\eta_2)$

$\Rightarrow$

$$\overline{Y^2} = E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{N_0}{2}\delta(\eta_1-\eta_2) h(\eta_1)\, d\eta_1 \cdot h(\eta_2)\, d\eta_2$$

$$= \frac{N_0}{2}\int_{-\infty}^{\infty} h^2(\eta_2)\, d\eta_2$$

Autocorrelation Function

$$R_{YY}(t, t+\tau) = E[Y(t)Y(t+\tau)]$$

$$= E\left[\int_{-\infty}^{\infty} h(\eta_1) X(t-\eta_1) d\eta_1 \int_{-\infty}^{\infty} h(\eta_2) X(t+\tau-\eta_2) d\eta_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t-\eta_1)X(t+\tau-\eta_2)] h(\eta_1) h(\eta_2) d\eta_1 d\eta_2$$

$$\Rightarrow R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\eta_1-\eta_2) h(\eta_1) h(\eta_2) d\eta_1 d\eta_2$$

$$= R_{XX}(\tau) * h(-\tau) * h(\tau)$$

Note that $\overline{Y^2} = E[Y^2] = R_{YY}(0)$

Cross-Correlation Functions of Input and Output

$$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)]$$

$$= E\left[X(t)\int_{-\infty}^{\infty} h(\eta) X(t+\tau-\eta) d\eta\right]$$

$$= \int_{-\infty}^{\infty} E[X(t)X(t+\tau-\eta)] h(\eta) d\eta$$

$$\Rightarrow R_{XY}(\tau) = \int_{-\infty}^{\infty} R_{XX}(\tau-\eta) h(\eta) d\eta$$

$$= R_{XX}(\tau) * h(\tau)$$

Also,

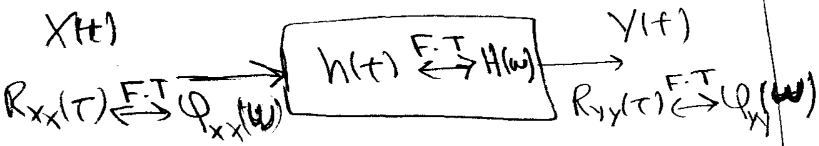
$$R_{YX}(\tau) = R_{XX}(\tau) * h(-\tau)$$

Also, Note that

$$R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau)$$

and $R_{YY}(\tau) = R_{YX}(\tau) * h(\tau)$

$\boxed{8.4}$ Power Density Spectrum



The output Power Density Spectrum is

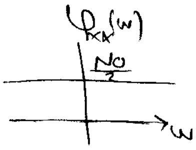
$$\Phi_{YY}(\omega) = \Phi_{XX}(\omega)\,|H(\omega)|^2$$

Thus, the average power of the output is

$$P_{YY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{XX}(\omega)\,|H(\omega)|^2\,d\omega$$

Example

Let $\Phi_{XX}(\omega) = \frac{N_0}{2}$



Let $H(\omega) = \dfrac{1}{1 + j\frac{\omega L}{R}}$

$\Rightarrow |H(\omega)|^2 = \dfrac{1}{1 + \left(\frac{\omega L}{R}\right)^2}$

$\Rightarrow \Phi_{YY}(\omega) = \Phi_{XX}(\omega)|H(\omega)|^2 = \dfrac{\frac{N_0}{2}}{1 + \left(\frac{\omega L}{R}\right)^2}$

Average power

$$P_{YY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\frac{N_0}{2}}{1 + \left(\frac{\omega L}{R}\right)^2}\,d\omega$$

$$= \frac{N_0 R}{4L}$$

Cross-Power Density Spectrum

$$\Phi_{XY}(\omega) = \Phi_{XX}(\omega)\,H(\omega)$$

$$\Phi_{YX}(\omega) = \Phi_{XX}(\omega)\,H(-\omega)$$