Operations on Multiple Random Variables

5.1 Expectation

Let X and Y be two random variables with joint pdf $f_{XY}(x,y)$

then

$$\boxed{\bar{g} = E[\underbrace{g(X,Y)}_{\text{Any function of } X \text{ and } Y.}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)\, dx\, dy}$$

In general,

$$\bar{g} = E[g(X_1,\dots,X_N)]$$
$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(x_1,\dots,x_N)\, f_{X_1,\dots,X_N}(x_1,\dots,x_N)\, dx_1\cdots dx_N$$

Example 5.1-1

Let $g(X_1,\dots,X_N) = \sum_{i=1}^{N} \alpha_i X_i$

$$\Rightarrow E[g(X_1,\dots,X_N)]$$
$$= E\left[\sum_{i=1}^{N} \alpha_i X_i\right]$$
$$= \sum_{i=1}^{N} \alpha_i E[X_i]$$

Thus, the expectation of a sum of random variables is the sum of the expectations.

Joint Moments about the origin

$$m_{nk} = E[X^n Y^k]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{XY}(x,y)\, dx\, dy$$

Note that:

$$m_{n0} = E[X^n]$$
$$m_{0k} = E[Y^k]$$

The sum $n+k$ is called the order of the moments

* First order moments

$$m_{10} = E[X] = \bar{X}$$
$$m_{01} = E[Y] = \bar{Y}$$

* Second order moments

$$m_{20} = E[X^2] = \overline{X^2}$$
$$m_{02} = E[Y^2] = \overline{Y^2}$$

$\boxed{\text{*correlation}} \longrightarrow m_{11} = E[XY] = R_{XY}$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\, dx\, dy$$

- If $R_{XY} = E[X]E[Y]$

  $\Rightarrow$ X and Y are uncorrelated

- If X and Y are independent, ~~that is a suffic~~ $\Rightarrow$ X and Y are uncorrelated
- However, the opposit is not always true.

If $R_{xy} = 0$

$\Rightarrow$ X and Y are orthogonal.

Example 5.1-2

Let X be a R.V with $\bar{X} = 3$ and $\sigma_X^2 = 2$

$\Rightarrow E[X^2] = \sigma_X^2 + \bar{X}^2 = 11$

Let $Y = -6X + 22$

thus, $E[Y] = E[-6X+22]$
$= -6\bar{X} + 22 = 4$

Correlation $R_{xy} = m_{11} = E[XY]$
$= E[-6X^2 + 22X]$
$= -6E[X^2] + 22\bar{X}$
$= 0$

$\therefore$ X and Y are orthogonal.

Note that $R_{xy} \neq E[X]E[Y]$

$\Rightarrow$ X and Y are correlated.

In General,

the $(n_1+n_2+\cdots+n_N)$ order joint moment is

$$m_{n_1 n_2 \cdots n_N} = E[X_1^{n_1} X_2^{n_2} \cdots X_N^{n_N}]$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1^{n_1} \cdots x_N^{n_N} f_{X_1,\dots,X_N}(x_1,\dots,x_N)\, dx_1 \cdots dx_N$$

Joint Central Moments

$$\mu_{nk} = E[(X-\bar{X})^n (Y-\bar{Y})^k]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\bar{X})^n (y-\bar{Y})^k \cdot f_{XY}(x,y)\,dx\,dy$$

Second order Moments

$$\mu_{20} = E[(X-\bar{X})^2] = \sigma_X^2$$
$$\mu_{02} = E[(Y-\bar{Y})^2] = \sigma_Y^2$$

* Covariance

$$C_{XY} = \mu_{11} = E[(X-\bar{X})(Y-\bar{Y})]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\bar{X})(y-\bar{Y}) f_{XY}(x,y)\,dx\,dy$$
$$= R_{XY} - \bar{X}\,\bar{Y}$$
$$C_{XY} = R_{XY} - E[X]E[Y]$$

- If X and Y are independent or uncorrelated

$$\Rightarrow R_{XY} = E[X]E[Y]$$
$$\Rightarrow \boxed{C_{XY} = 0}$$

- If X and Y are orthogonal

$$\Rightarrow C_{XY} = -E[X]E[Y]$$

* Correlation Coefficient

$$\rho = \frac{C_{XY}}{\sigma_X \sigma_Y}$$
$$= E\left[\frac{(X-\bar{X})}{\sigma_X}\frac{(Y-\bar{Y})}{\sigma_Y}\right]$$

Note that $-1 \le \rho \le 1$

Example 5.1-3

Let $X = \sum_{i=1}^{N} \alpha_i X_i$

$$\bar{X} = E[X] = \sum_{i=1}^{N} \alpha_i E[X_i]$$
$$= \sum_{i=1}^{N} \alpha_i \bar{X}_i$$

Thus, $X - \bar{X} = \sum_{i=1}^{N} \alpha_i (X_i - \bar{X}_i)$

$$\sigma_X^2 = E[(X-\bar{X})^2]$$
$$= E\left[\sum_{i=1}^{N} \alpha_i (X_i - \bar{X}_i) \sum_{j=1}^{N} \alpha_j (X_j - \bar{X}_j)\right]$$
$$= \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \alpha_j E[(X_i - \bar{X}_i)(X_j - \bar{X}_j)]$$
$$\boxed{\sigma_X^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \alpha_j C_{X_i X_j}}$$

For the special case of uncorrelated R.Vs.

$$\Rightarrow C_{X_i X_j} = \begin{cases} 0 & i \ne j \\ \sigma_{X_i}^2 & i = j \end{cases}$$
$$\boxed{\Rightarrow \sigma_X^2 = \sum_{i=1}^{N} \alpha_i^2 \sigma_{X_i}^2}$$

5.2 Joint Characteristic Functions.

$$\Phi_{X,Y}(w_1,w_2) = E\left[e^{jw_1X + jw_2X}\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\, e^{jw_1x + jw_2y}\, dx\,dy$$

$\Rightarrow$ two dimensional Fourier Transform

$$f_{XY}(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_{XY}(w_1,w_2)\, e^{-jw_1X - jw_2Y}\, dw_1\,dw_2$$

— Marginal characteristic functions

$$\Phi_X(w_1) = \Phi_{XY}(w_1, 0)$$

$$\Phi_Y(w_2) = \Phi_{XY}(0, w_2)$$

- Joint moments m_{nk}

$$m_{nk} = (-j)^{n+k} \left.\frac{\partial^{n+k}\Phi_{XY}(w_1,w_2)}{\partial w_1^n \partial w_2^k}\right|_{w_1=0, w_2=0}$$

Example

$$\Phi_{XY}(w_1,w_2) = \exp(-2w_1^2 - 8w_2^2)$$

$$\bar{X} = E\{X\} = m_{10} = -j \left.\frac{\partial \Phi_{XY}(w_1,w_2)}{\partial w_1}\right|_{w_1=0, w_2=0}$$

$$= -j(-4w_1)\exp(-2w_1^2 - 8w_2^2)\Big|_{w_1=0, w_2=0}$$

$$= 0$$

$$\bar{Y} = E\{Y\} = m_{01} = -j(-16w_2)\exp(-2w_1^2 - 8w_2^2)\Big|_{w_1=0, w_2=0}$$

$$= 0$$

Also, $R_{XY} = E\{XY\} = m_{11}$

$$= (-j)^2 \frac{\partial^2}{\partial w_1 \partial w_2}\left[\exp(-2w_1^2 - 8w_2^2)\right]\Big|_{w_1=0, w_2=0}$$

$$= -(-4w_1)(-16w_2)\exp(-2w_1^2 - 8w_2^2)\Big|_{w_1=0, w_2=0}$$

$$= 0$$

Since the means are zero

$\Rightarrow C_{XY} = R_{XY} = 0$

$\Rightarrow$ X and Y are uncorrelated.

~~Example~~

In General,

$$\Phi_{X_1,\dots,X_N}(w_1,\dots,w_N)$$
$$= E[e^{jw_1X_1+\cdots+jw_NX_N}]$$

Example, $Y = X_1 + X_2 + \cdots + X_N$

be the sum of N statistically independent random variables X_i, $i=1,\dots,N$.

$$\Rightarrow f_{X_1X_2\cdots X_N}(x_1,x_2,\dots,x_N) = \prod_{i=1}^{N} f_{X_i}(x_i)$$

Thus, $\Phi_{X_1\dots X_N}(w_1,\dots,w_N) =$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\prod_{i=1}^{N} f_{X_i}(x_i)\right]\exp\left[j\sum_{i=1}^{N} w_i x_i\right]dx_1\cdots dx_N$$

$$= \prod_{i=1}^{N}\int_{-\infty}^{\infty} f_{X_i}(x_i)\, e^{jw_ix_i}\, dx_i = \prod_{i=1}^{N}\Phi_{X_i}(w_i)$$

$\therefore$ $\Phi_Y(w) = E[e^{jwY}]$

$$= E\left[\exp\left(j\sum_{i=1}^{N} wX_i\right)\right]$$

$$= \Phi_{X_1,\dots,X_N}(w,\dots,w) = \prod_{i=1}^{N}\Phi_{X_i}(w)$$

$\therefore$ the pdf of Y

$$\underbrace{f_Y(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\prod_{i=1}^{N}\Phi_{X_i}(w)\right]e^{-jwy}\,dw}$$

If X_i are identically distributed

$\Rightarrow$ have the same $\Phi_{X_i}(w)$

$$\Rightarrow f_Y(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\Phi_X(w)]^N e^{-jwy}\,dw$$

5.3 Jointly Gaussian R.V.

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \cdot \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{(x-\bar{X})^2}{\sigma_x^2} - \frac{2\rho(x-\bar{X})(y-\bar{Y})}{\sigma_x\sigma_y} + \frac{(y-\bar{Y})^2}{\sigma_y^2}\right]\right\}$$

$\bar{X} = E(X)$

$\bar{Y} = E\{Y\}$

$\sigma_x^2 = E[(X-\bar{X})^2]$

$\sigma_y^2 = E\{(Y-\bar{Y})^2\}$

$$\rho = E\left[\frac{(X-\bar{X})(Y-\bar{Y})}{\sigma_x\sigma_y}\right]$$

correlation coefficient.

- If $\rho = 0 \Rightarrow X$ and Y are uncorrelated

$\Rightarrow f_{XY}(x,y) = f_X(x) f_Y(y)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{(x-\bar{X})^2}{2\sigma_x^2}\right)$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} \exp\left(-\frac{(y-\bar{Y})^2}{2\sigma_y^2}\right)$$

"Any uncorrelated gaussian R.V.s are also statistically independent"

Theorem Let X and Y be two correlated guassian R.V. and Let $\theta = \frac{1}{2}\tan^{-1}\left[\frac{2\rho\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2}\right]$

$\Rightarrow$ For the following Linear transformations

$Y_1 = X\cos(\theta) + Y\sin(\theta)$

$Y_2 = -X\sin(\theta) + Y\cos(\theta)$

$\Rightarrow Y_1$ and Y_2 are two statistically independent R.V.

5.4 Transformation of Multiple R.V.

case 1:

One Function

Let $Y = g(X_1, X_2, \ldots, X_N)$

$$\Rightarrow F_Y(y) = P\{Y \le y\}$$

$$= P\{g(X_1, X_2, \ldots, X_N) \le y\}$$

$$= \int \cdots \int_{\{g(x_1, x_2, \ldots, x_N) \le y\}} f_{X_1, X_2, \ldots, X_N}(x_1, x_2, \ldots, x_N)\, dx_1 dx_2 \cdots dx_N$$

pdf

$$\Rightarrow f_Y(y) = \frac{dF_Y(y)}{dy}$$

$$= \frac{d}{dy} \int \cdots \int_{\{g(x_1, \ldots, x_N) \le y\}} f_{X_1 X_2 \ldots X_N}(x_1, x_2, \ldots, x_N)\, dx_1 dx_2 \cdots dx_N$$

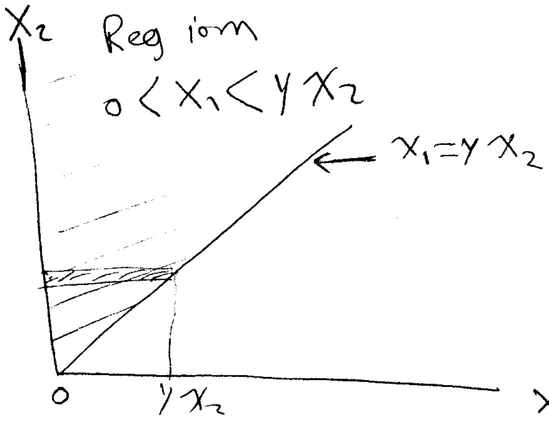
Example 5-4-1

Let $Y = \frac{X_1}{X_2}$ where X_1 and X_2 are positive

the event $\{Y = \frac{X_1}{X_2} \le y\}$

$$\Rightarrow 0 < \frac{X_1}{X_2} \le Y$$

$$0 < X_1 \le Y X_2$$



$$\& \; F_Y(y) = P\left\{\frac{X_1}{X_2} \le y\right\}$$

$$= \int_0^\infty \int_0^{y x_2} f_{X_1 X_2}(x_1, x_2)\, dx_1\, dx_2$$

$\therefore$ the pdf is

$$f_Y(y) = \frac{dF_Y(y)}{dy} \quad \text{Using Leibniz's Rule.}$$

$$= \int_0^\infty x_2 f_{X_1 X_2}(y x_2, x_2)\, dx_2$$

Recall Leibniz's Rule

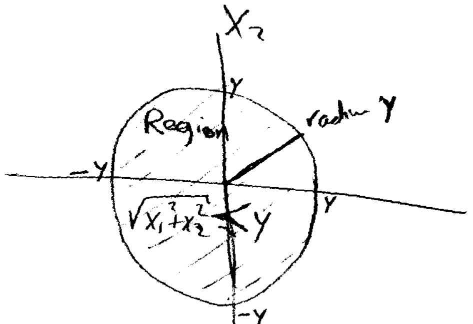
$$G(u) = \int_{\alpha(u)}^{\beta(u)} H(x, u)\, dx$$

$$\frac{dG(u)}{du} = H(\beta(u), u) \frac{d\beta(u)}{du} - H(\alpha(u), u) \frac{d\alpha(u)}{du} + \int_{\alpha(u)}^{\beta(u)} \frac{\partial H(x, u)}{\partial u}\, dx$$

Example 5.4-2

Let $Y = (X_1^2 + X_2^2)^{1/2}$

$$F_Y(y) = P\{Y = (X_1^2 + X_2^2)^{1/2} \le y\}$$



$$F_Y(y) = \int_{x_2=-y}^{y} \underbrace{\int_{x_1=-(y^2-x_2^2)^{1/2}}^{(y^2-x_2^2)^{1/2}} f_{X_1X_2}(x_1, x_2)\, dx_1}_{I(y, x_2)}\, dx_2$$

where $I(y, x_2) = \int_{-(y^2-x_2^2)^{1/2}}^{(y^2-x_2^2)^{1/2}} f_{X_1X_2}(x_1, x_2)\, dx_1$

Using Leibniz's Rule,

$$f_Y(y) = \underbrace{\cancel{I}(y, y)}_{\to 0} + \underbrace{I(\cancel{y, -y})}_{\to 0} + \int_{-y}^{y} \underbrace{\frac{\partial I(y, x_2)}{\partial y}}_{\text{Apply Leibniz Rule again.}} dx_2$$

$$f_Y(y) = \int_{-y}^{y} \left\{ f_{X_1X_2}[(y^2-x_2^2)^{1/2}, x_2] \frac{y}{(y^2-x_2^2)^{1/2}} + f_{X_1X_2}[-(y^2-x_2^2)^{1/2}, x_2] \frac{y}{(y^2-x_2^2)^{1/2}} + \int_{-(y^2-x_2^2)^{1/2}}^{(y^2-x_2^2)^{1/2}} \underbrace{\frac{\partial f_{X_1X_2}(x_1, x_2)}{\partial y}}_{\to 0} dx_1 \right\} dx_2$$

$$\therefore f_Y(y) = \int_{-y}^{y} \left\{ f_{X_1X_2}[(y^2-x_2^2)^{1/2}, x_2] + f_{X_1X_2}[-(y^2-x_2^2)^{1/2}, x_2] \right\} * \frac{y}{(y^2-x_2^2)^{1/2}}\, dx_2$$

If X_1 and X_2 are gaussian, what will be ~~$f_Y(y)$~~ Y?

Case 2:
Multiple Functions

Assum we have N transformations

$$Y_1 = T_1(X_1, X_2, \ldots, X_N)$$
$$Y_2 = T_2(X_1, X_2, \ldots, X_N)$$
$$\vdots$$
$$Y_N = T_N(X_1, X_2, \ldots, X_N)$$

and let the inverse function be

$$X_j = T_j^{-1}(Y_1, Y_2, \ldots, Y_N), \quad j = 1, 2, \ldots, N$$

& The joint pdf of Y is

$$f_{Y_1Y_2\cdots Y_N}(y_1, y_2, \ldots, y_N)$$
$$= f_{X_1X_2\cdots X_N}(x_1 = T_1^{-1}, \ldots, x_N = T_N^{-1})\,|J|$$

where $|J|$ is called the Jacobian and it is the determinant of

$$J = \begin{vmatrix} \frac{\partial T_1^{-1}}{\partial y_1} & \cdots & \frac{\partial T_1^{-1}}{\partial y_N} \\ \vdots & & \\ \frac{\partial T_N^{-1}}{\partial y_1} & \cdots & \frac{\partial T_N^{-1}}{\partial y_N} \end{vmatrix}$$

Example

$$Y_1 = T_1(X_1, X_2) = aX_1 + bX_2$$
$$Y_2 = T_2(X_1, X_2) = cX_1 + dX_2$$

$$\Rightarrow X_1 = T_1^{-1}(Y_1, Y_2) = \frac{(dY_1 - bY_2)}{(ad - bc)}$$

$$X_2 = T_2^{-1}(Y_1, Y_2) = \frac{-cY_1 + aY_2}{ad - bc}$$

$$\Rightarrow J = \begin{vmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{vmatrix} = \frac{1}{ad - bc}$$

$\therefore$

$$f_{Y_1Y_2}(y_1, y_2) = \frac{f_{X_1X_2}\left(\frac{dy_1 - by_2}{ad-bc}, \frac{-cy_1 + ay_2}{ad-bc}\right)}{|ad - bc|}$$

5.7 Sampling and Some Limit Theorems

Assume the $\hat{X}_n$, $n = 1, 2, \dots, N$ are N samples of identically distributed R.V X_n

$$\boxed{\Rightarrow \text{Sample mean} \quad \hat{\bar{X}}_N = \frac{1}{N}\sum_{n=1}^{N} X_n}$$

- This is an estimator of the mean value of N measurments.

- If $E[\hat{\bar{X}}_N] = \bar{X}$

$\Rightarrow$ The estimator is "Unbiased"

for the sample mean,

$$E[\hat{\bar{X}}_N] = E\left[\frac{1}{N}\sum_{n=1}^{N} X_n\right]$$
$$= \frac{1}{N}\sum_{n=1}^{N} E[X_n] = \bar{X}$$

- The variance of the Sample mean estimator is:

$$E[(\hat{\bar{X}}_N - \bar{X})^2] = \sigma_{\hat{X}_N}^2$$
$$= E[\hat{\bar{X}}_N^2 - 2\bar{X}\hat{\bar{X}}_N + \bar{X}^2]$$
$$= E[\hat{\bar{X}}_N^2] - 2\bar{X}E[\hat{\bar{X}}_N] + \bar{X}^2$$
$$= E[\hat{\bar{X}}_N^2] - \bar{X}^2$$
$$= E\left[\frac{1}{N}\sum_{n=1}^{N} X_n \frac{1}{N}\sum_{m=1}^{N} X_m\right] - \bar{X}^2$$
$$= \frac{1}{N^2}\sum_{n=1}^{N}\sum_{m=1}^{N} E[X_n X_m] - \bar{X}^2$$

Assuming independance

$$\Rightarrow E[X_n X_m] = \begin{cases} E[X^2] & \text{when } n=m \\ E[X_n]E[X_m] = \bar{X}^2 & \text{when } n \neq m \end{cases}$$

$$\Rightarrow \sigma_{\hat{X}_N}^2 = \frac{1}{N^2}\left[N E(X^2) + (N^2 - N)\bar{X}^2\right] - \bar{X}^2$$
$$= \frac{1}{N}\left[E(X^2) - \bar{X}^2\right]$$
$$= \frac{\sigma_X^2}{N}$$

Since $\sigma_{\hat{X}_N}^2 \to 0$ as $N \to \infty$

$\Rightarrow$ This estimator is "Consistent"

— Second moment estimator.

$$\hat{X^2_N} = \frac{1}{N}\sum_{n=1}^{N} X_n^2$$

Also called Power estimator ~~of~~ a random voltage.

— Variance Estimator

$$\hat{\sigma_X^2} = \frac{1}{N-1}\sum_{n=1}^{N}\left(X_n - \hat{X}_N\right)^2$$

<u>why (N-1)?</u>

to make it unbiased Estimator.

<u>See problem 5-7-3</u>