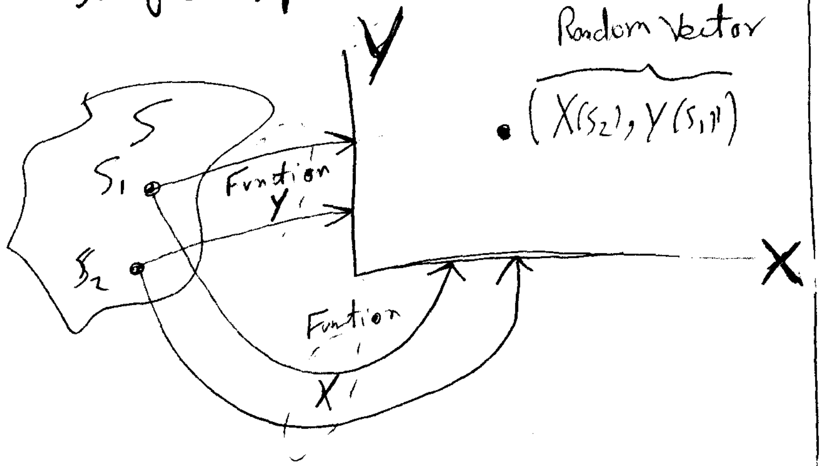
Ch. 4 Multiple Random Variables

4.1 Random Vector

$$\vec{X} = (X_1, X_2, \ldots, X_N)$$

is an N-Dimensional random vector Defined on an N-Dimensional joint Sample space



4.2 Joint Distribution.

the probability of the joint event $\{X \le x, Y \le y\}$ is

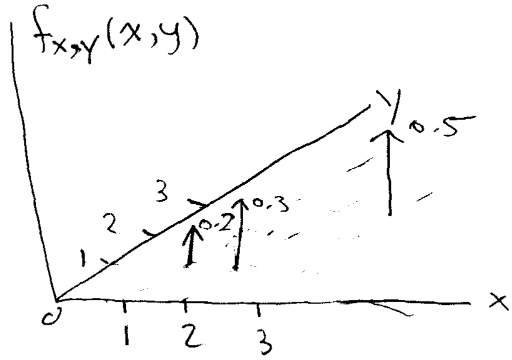
$$F_{X,Y}(x,y) = P\{X \le x, Y \le y\}$$
$$= P\{A \cap B\}$$

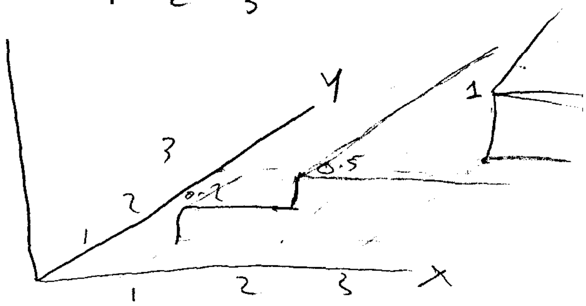
where $A = \{X \le x\}$

Example

~~Let~~ A Joint Sample Space has three elements

$$S_J = \{(1,1), (2,1), (3,3)\}$$
$$P(S_J) = \{0.2, 0.3, 0.5\}$$





In General,

$$F_{X,Y}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} P(x_n, y_m)\, u(x - x_n)\, u(y - y_m)$$

~~in gen~~

For N random variables, the joint pdf is

$$F_{X_1,X_2,\dots,X_N}(x_1,x_2,\dots,x_N) = P\{\underbrace{X_1\le x_1, X_2\le x_2,\dots,X_N\le x_N}_{\text{Joint Event}}\}$$

Proporties of the Joint Distribution

1. $F_{X,Y}(-\infty,-\infty)=0$

   $F_{X,Y}(-\infty,y)=0$

   $F_{X,Y}(x,-\infty)=0$

2. $F_{X,Y}(\infty,\infty)=1$

3. $0\le F_{X,Y}(x,y)\le 1$

4. $F_{X,Y}(x,y)$ is a nondecreasing Function.

5. $P\{x_1<X\le x_2, y_1<Y\le y_2\}$

   $= F_{X,Y}(x_2,y_2)+F_{X,Y}(x_1,y_1) - F_{X,Y}(x_1,y_2)-F_{X,Y}(x_2,y_1)$

6. $\left.\begin{array}{l} F_{X,Y}(x,\infty)=F_X(x) \\ F_{X,Y}(\infty,y)=F_Y(y)\end{array}\right\}$ Marginal Distribution

Marginal Distribution Functions

$F_X(x)=F_{X,Y}(x,\infty)$

$F_Y(y)=F_{X,Y}(\infty,y)$

Why?

Let $A=\{X\le x\}$

$B=\{Y\le y\}$

$F_{X,Y}(x,y)=P(A\cap B)$

Let $y\to\infty \Rightarrow B=\{Y\le\infty\}=S$

Since $A\cap S=A$

$\Rightarrow F_{X,Y}(x,\infty)=P(A\cap S)$

$=P(A)$

$=P\{X\le x\}$

$=F_X(x)$

Example 4.2-2

$$F_{X,Y}(x,y)=P(1,1)u(x-1)u(y-1)+P(2,1)u(x-2)u(y-1)+P(3,3)u(x-3)u(y-3)$$

$$F_X(x)=F_{X,Y}(x,\infty)=P(1,1)u(x-1)+P(2,1)u(x-2)+P(3,3)u(x-3)$$

$$F_Y(y)=F_{X,Y}(\infty,y)=P(1,1)u(y-1)+P(2,1)u(y-1)+P(3,3)u(y-3)$$

4.3 Joint Density and Its Properties

$$f_{X,Y}(x,y)=\frac{\partial^2 F_{X,Y}(x,y)}{\partial x\,\partial y}$$

For the discrete case,

$$f_{X,Y}(x,y)=\sum_{n=1}^{N}\sum_{m=1}^{M}P(x_n,y_m)\delta(x-x_n)\delta(y-y_m)$$

In General, For N random variables $\vec{X}=(X_1, X_2, \dots, X_N)$

$$f_{X_1,X_2,\dots,X_N}(x_1,x_2,\dots,x_N)=\frac{\partial^N F_{X_1,X_2,\dots,X_N}(x_1,x_2,\dots,x_N)}{\partial x_1\,\partial x_2\cdots\partial x_N}$$

AND

$$F_{X_1,X_2,\dots,X_N}(x_1,x_2,\dots,x_N)=\int_{-\infty}^{x_N}\cdots\int_{-\infty}^{x_2}\int_{-\infty}^{x_1} f_{X_1,X_2,\dots,X_N}(\eta_1,\eta_2,\dots,\eta_N)\,d\eta_1\,d\eta_2\cdots d\eta_N$$

Properties of the Joint Density

① $f_{X,Y}(x,y) \geq 0$

② $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$

③ $F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(\eta_1,\eta_2)\,d\eta_1\,d\eta_2$

④ Marginal CDF

$$F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(\eta_1,\eta_2)\,d\eta_1\,d\eta_2$$

$$F_Y(y) = \int_{-\infty}^{\infty}\int_{-\infty}^{y} f_{X,Y}(\eta_1,\eta_2)\,d\eta_1\,d\eta_2$$

⑤ Marginal pdf

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

K-Dimensional Marginal pdf.

For a random vector with N elements, the k-dimensional marginal pdf is

$$f_{X_1,X_2,\ldots X_k}(x_1,x_2,\ldots,x_k) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{X_1,X_2,\ldots,X_N}(x_1,x_2,\ldots x_N)\,dx_{k+1}\,dx_{k+2}\cdots dx_N$$

4.4 Conditional Distribution and Density.

$$F_X(x|B) = P\{X \le x | B\}$$

Event B

$$= \frac{P\{X \le x \cap B\}}{P(B)}$$

and

$$f_X(x|B) = \frac{dF_X(x|B)}{dx}$$

the event B depends on the other random variable y.

we will consider two cases:

- Point Conditioning.
- Interval conditioning.

— Discrete case

$$F_X(x|Y=y_k) = \sum_{i=1}^{N} \frac{P(x_i, y_k)}{P(y_k)} u(x-x_i)$$

and

$$f_X(x|Y=y_k) = \sum_{i=1}^{N} \frac{P(x_i, y_k)}{P(y_k)} \delta(x-x_i)$$

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case 1: Point Conditioning

— Continuous case:

the event $B=\{y-\Delta y < Y \le y+\Delta y\}$

where Δy very close to zero.

$\Rightarrow$

$$F_X(x|y-\Delta y < Y \le y+\Delta y) = \frac{\int_{y-\Delta y}^{y+\Delta y} \int_{-\infty}^{x} f_{X,Y}(\eta_1, \eta_2)\, d\eta_1\, d\eta_2}{\int_{y-\Delta y}^{y+\Delta y} f_Y(\eta_1)\, d\eta_1}$$

~~as $\Delta y \to 0$~~

$\Rightarrow$ As $\Delta y \to 0$

$\Rightarrow$

$$F_X(x|Y=y) = \frac{\int_{-\infty}^{x} f_{X,Y}(\eta, y)\, d\eta}{f_Y(y)}$$

$$\therefore f_X(x|Y=y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

or

$$\boxed{f_X(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} \quad \text{and} \quad f_Y(y|x) = \frac{f_{XY}(x,y)}{f_X(y)}}$$

Case II: Interval conditioning

$$F_X(x \mid y_a < Y \le y_b) = \frac{\int_{y_a}^{y_b}\int_{-\infty}^{x} f_{XY}(q,y)\,dq\,dy}{\int_{y_a}^{y_b}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy}$$

and

$$f_X(x \mid y_a < Y \le y_b) = \frac{\int_{y_a}^{y_b} f_{XY}(x,y)\,dy}{\int_{y_a}^{y_b}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy}$$

4.5

Statistical Independence

Two events A and B are statistically independent if and only if

$$P(A \cap B) = P(A)P(B)$$

Also, Let $A = \{X \le x\}$ and $B = \{Y \le y\}$

$$P\{X \le x, Y \le y\} = P\{X \le x\}P\{Y \le y\}$$

$$\Rightarrow F_{XY}(x,y) = F_X(x)F_Y(y)$$

and $$f_{XY}(x,y) = f_X(x)f_Y(y)$$

Also, ~~the~~ if A and B are independent

$$\Longrightarrow F_X(x|Y \le y) = F_X(x)$$

and $$F_Y(y|X \le x) = F_Y(y)$$

Also, ~~the~~ same applies for the pdf.

Example 4.5-1

Let the joint pdf of x and y be

$$f_{X,Y}(x,y) = u(x)u(y)\,x\,e^{-x(y+1)}$$

- Find $f_X(x)$, $f_Y(y)$
- Are X and Y independent?

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_0^{\infty} u(x)\,x\,e^{-x(y+1)}\,dy$$

$$= u(x)\,x e^{-x}\int_0^{\infty} e^{-xy}\,dy$$

$$= u(x)\,x e^{-x}\left(\frac{1}{x}\right) = u(x)e^{-x}$$

$$f_Y(y) = \int_0^{\infty} u(y)\,x e^{-x(y+1)}\,dx$$

$$= \frac{u(y)}{(y+1)^2} \rightarrow \text{See Appendix C}$$

Since

$$f_X(x)f_Y(y) = u(x)u(y)\frac{e^{-x}}{(y+1)^2}$$

$$\ne f_{XY}(x,y)$$

$\Longrightarrow$ X and Y are not independent

Example 4.5-2

$$f_{XY}(x,y) = \frac{1}{12} u(x) u(y) e^{-\left(\frac{x}{4}\right)-\left(\frac{y}{3}\right)}$$

$$f_X(x) = \int_0^\infty \frac{1}{12} u(x) e^{-\frac{x}{4}} e^{-\frac{y}{3}} dy$$

$$= \frac{1}{4} u(x) e^{-\frac{x}{4}}$$

$$f_Y(y) = \int_0^\infty \frac{1}{12} u(y) e^{-y/3} e^{-\frac{x}{4}} dx$$

$$= \frac{1}{3} u(y) e^{-y/3}$$

Since $f_X(x) f_Y(y) = f_{XY}(x,y)$

$\Rightarrow$ X and Y are indepondent

4.6

Distribution and Density of a sum of Random Variables

Let X and Y be two independent Random Variables. ~~such that~~ the pdf of X and Y are $f_Y(y)$ and $f_X(x)$ then the pdf of the sum

$$W = X + Y$$

is the convolution of their individual pdfs.

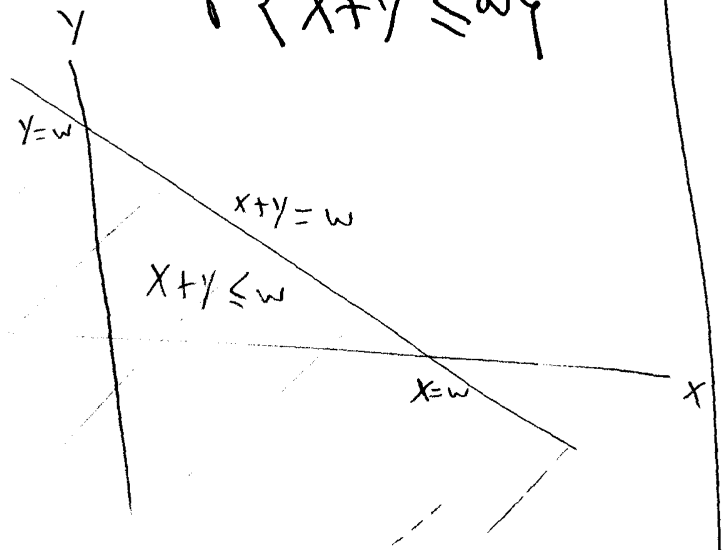
Proof:

Let $W = X + Y$

the CDF of W is

$$F_W(w) = P\{W \le w\}$$

$$= P\{X + Y \le w\}$$



$$F_W(w) = \int_{-\infty}^{\infty} \int_{x=-\infty}^{w-y} f_{XY}(x,y)\,dx\,dy$$

Since X and Y are independent

$$\Rightarrow \quad f_{XY}(x,y) = f_X(x) f_Y(y)$$

$$\therefore \quad F_W(w) = \int_{-\infty}^{\infty} f_Y(y) \int_{x=-\infty}^{w-y} f_X(x)\,dx\,dy$$

$\therefore$ Using Leibniz's Rule,

$$f_W(w) = \int_{-\infty}^{\infty} f_Y(y) f_X(w-y)\,dy$$

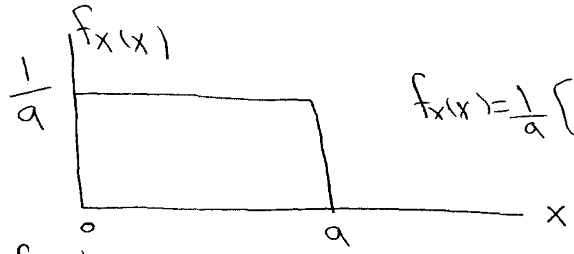
$$\therefore \quad f_W(w) = f_Y(y) \underset{\text{convolution.}}{*} f_X(x)$$

In General, for N independent random variables, Let

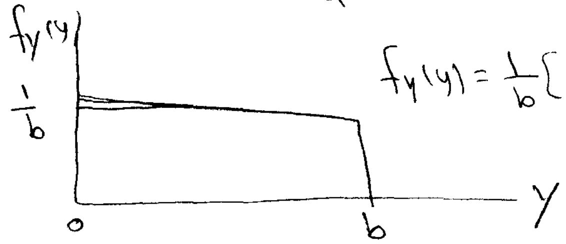
$$Y = X_1 + X_2 + \cdots + X_N$$

$$\therefore \quad f_Y(y) = f_{X_N}(x_N) * f_{X_{N-1}}(x_{N-1}) * \cdots * f_{X_1}(x_1)$$

Example 4.6-1



$$f_X(x) = \frac{1}{a}\left[u(x) - u(x-a)\right]$$



$$f_Y(y) = \frac{1}{b}\left[u(y) - u(y-b)\right]$$

Find the pdf of $W = X + Y$

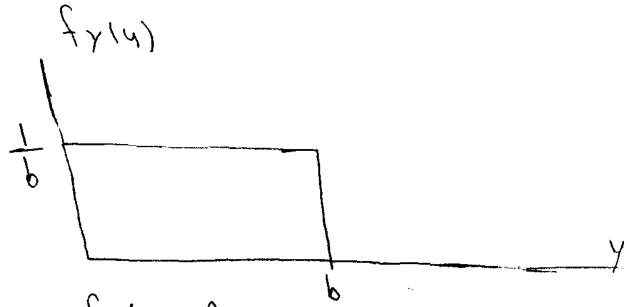
$$f_W(w) = \int_{-\infty}^{\infty} f_Y(y)\, f_X(w-y)\, dy$$

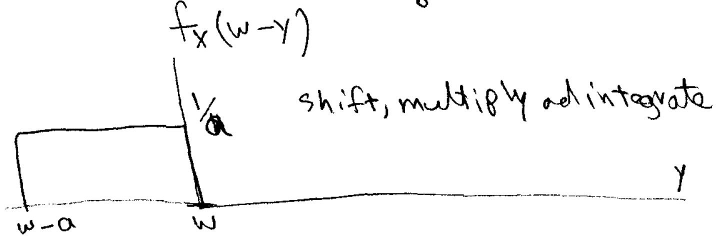
$$= \int_{-\infty}^{\infty} \frac{1}{ab}\left[u(y) - u(y-b)\right]\left[u(w-y) - u(w-y-a)\right] dy$$

$$= \frac{1}{ab}\int_{0}^{\infty}\left[1 - u(y-b)\right]\left[u(w-y) - u(w-y-a)\right] dy$$

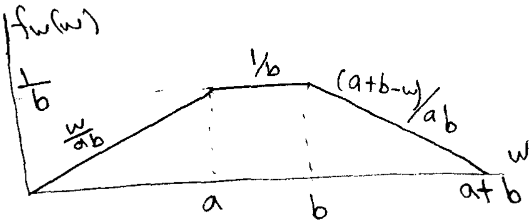
$$= \frac{1}{ab}\Big[\int_0^{\infty} u(w-y)\,dy - \int_0^{\infty} u(w-y-a)\,dy$$
$$- \int_0^{\infty} u(y-b)\,u(w-y)\,dy$$
$$+ \int_0^{\infty} u(y-b)\,u(w-y-a)\,dy\Big]$$

this convolution can be easily evaluated by using graphical techniques.





shift, multiply and integrate.



4.7 Central Limit Theorem

Central limit Theorem says that the prob. distribution function of the sum of a large number of random variables approaches a gaussian distribution.

Case 1 Unequal Distributions

Let $Y_N = X_1 + X_2 + \cdots + X_N$

and Let $\overline{X_i}$ be the mean of X_i

and $\sigma_{X_i}^2$ be the ~~mean~~ variance of X_i

thus, we can approximate the pdf of Y_N to be gaussian with mean $\overline{Y_N} = \overline{X_1} + \overline{X_2} + \cdots + \overline{X_N}$

and Variance $\sigma_{Y_N}^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \cdots + \sigma_{X_N}^2$

Case 2 Equal Distributions

$$Y_N = X_1 + X_2 + \cdots X_N$$

$\Rightarrow$ mean $\overline{Y_N} = \sum_{i=1}^{N} \overline{X_i} = N\overline{X_1}$

Variance $\sigma_{Y_N}^2 = \sum_{i=1}^{N} \sigma_{X_i}^2 = N\sigma_{X_1}^2$

Example 4.7-1

Let X_1 and X_2 be two independent uniformly distributed random variables X_1 and X_2 with pdf

$$f_X(x) = \frac{1}{a}\{u(x) - u(x-a)\}$$

where $a > 0$

- the mean of X_1 and X_2 is $\overline{X} = \frac{a}{2}$

  the variance of X_1 and X_2 is $\sigma_X^2 = \frac{a^2}{12}$

  the pdf of the sum $W = X_1 + X_2$ is

  $$f_X(w) = \frac{1}{a}\,\mathrm{tri}\left(\frac{w}{a}\right)$$

- The gaussian approximation has mean $\overline{W} = 2\overline{X} = a$

  and Variance $\sigma_W^2 = 2\sigma_X^2 = \frac{a^2}{6}$

  $$f_W(w) \simeq \frac{e^{-(w-a)^2/(a^2/3)}}{\sqrt{\pi\left(\frac{a^2}{3}\right)}}$$