2.1 The Random variable Concept

- a real random variable is a real function of the elements of a sample space S.
- Thus, this function maps every element in the sample space to a real number.
- We denote the random variable function by a capital Letter (such as W, X or Y)
- Any Particular ~~number~~ value of the random variable is denoted by a small letter (w, x or y)
- Example 2.1-1
- Example 2.1-2

* Conditions for a function to Be a Random Variable.

1- Every Point in S must correspond to only one value of the random variable.

2- The set $\{X \le x\}$ should be an event for any real number x

$\Rightarrow P\{X \le x\}$ exist

3- $P\{X = -\infty\} = 0$

- The Random Variables can be Discrete or Continuous

* A Discrete random variable has discrete values only

- See example 2.1-3

* A continuous random variable has a continuous range of values.

- see example 2.1-4

Fact: the probability of occurrence of any discrete value of a continuous random variable is zero.

(why? see example 2.1-4)

* A mixed R.V. has discrete and Continuous values.

2.2 and 2.3 Distribution and Density Functions

— Cumulative Prob. Distribution Function
" Distribution Function"
" CDF "

$F_X(x) = P\{X \leq x\}$; where x is a real number

Random variable

— Properties of $F_X(x)$:

① $F_X(-\infty) = 0$

② $F_X(\infty) = 1$

③ $0 \leq F_X(x) \leq 1$

④ $F_X(x_1) \leq F_X(x_2)$ if $x_1 < x_2$ "non decreasing"

⑤ $P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1)$

⑥ $F_X(x^+) = F_X(x)$ "Continuous from the right"

— Probability Density Function (Pdf)

$$f_X(x) = \frac{dF_X(x)}{dx}$$

— Properties of $f_X(x)$

"nonnegative" ① $0 \leq f_X(x)$ all x

"Unit Area" ② $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

③ $F_X(x) = \int_{-\infty}^{x} f_X(\eta)\,d\eta$

④ $P\{x_1 < X \leq x_2\} = \int_{x_1}^{x_2} f_X(x)\,dx$

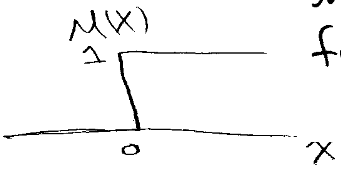
— Discrete random Variable

Let $X = \{x_1, x_2, \cdots, x_N\}$ be a discrete random variable, ~~then~~ and Let $P\{X = x_i\} = P(x_i)$ then, the discrete CDF will be:

$$F_X(x) = \sum_{i=1}^{N} P\{X = x_i\} u(x - x_i)$$

$$= \sum_{i=1}^{N} P(x_i) u(x - x_i)$$

where $u(x)$ is the unit-step function:



$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

then, the pdf will be

$$f_X(x) = \frac{dF_X(x)}{dx}$$

$$= \sum_{i=1}^{N} P(x_i) \delta(x - x_i)$$

where $\delta(x)$ is unit-impulse function "Delta function"

$$\delta(x) = \frac{du(x)}{dx}$$

and $$u(x) = \int_{-\infty}^{x} \delta(\eta) \, d\eta$$

See Example 2-2-1 Page 46

— Continuous random Variable

Let $f_X(x)$ be the pdf ~~of~~ a continuous random variable, then the CDF will be

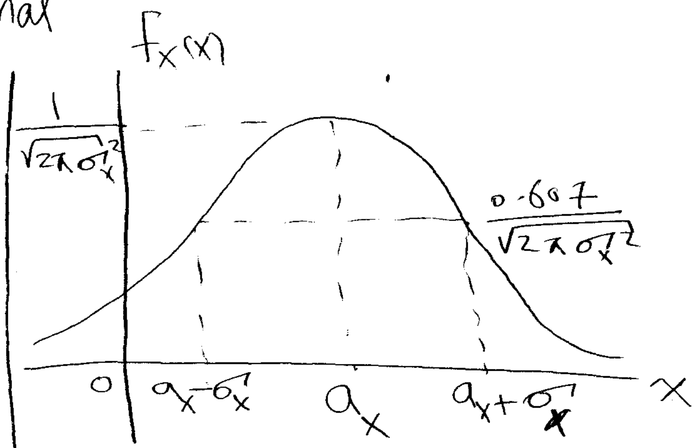
$$F_X(x) = \int_{-\infty}^{x} f_X(\eta) \, d\eta$$

or we can say that

$$f_X(x) = \frac{dF_X(x)}{dx}$$

See Example 2-2-2 Page 47

2.4 The Gaussian Random Variable. "Normal"

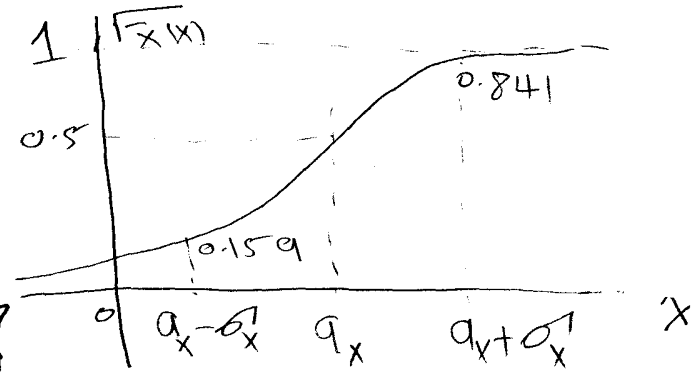


pdf $f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-(x-a_x)^2}{2\sigma_x^2}}$

Def: $a_x \equiv$ the mean

$\sigma_x \equiv$ standard deviation

$\sigma_x^2 \equiv$ variance



CDF $F_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{x} e^{\frac{-(\eta-a_x)^2}{2\sigma_x^2}} d\eta$

- Can we evaluate the integral and find a closed-form solution?
  - The answer is NO. there are no closed-form solutions for this integral.
  - Thus, we use numerical or approximation methods to evaluate the CDF.
- In Appendix B, there is a numerical table for the normalized CDF which is

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\eta^2/2} d\eta$$

where $a_x = 0$

Thus, to find $F_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{x} e^{-\frac{(\eta-a_x)^2}{2\sigma_x^2}} d\eta$

Let $u = \frac{\eta - a_x}{\sigma_x} \Rightarrow du = \frac{d\eta}{\sigma_x}$

$$\Rightarrow F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-a_x}{\sigma_x}} e^{-\frac{u^2}{2}} du$$

$$= F\left(\frac{x-a_x}{\sigma_x}\right)$$

See Examples 2.4-1 and Example 2.4-2

Note: $F(-x) = 1 - F(x)$

what is the releation between the Gaussian CDF and the Q-Funtion. ?

A Famous function in probability theory and Mathmatics is the Q-Funtion.

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\eta^2/2}\, d\eta$$

Note that

$$F(x) = 1 - Q(x)$$

the Q-Function has the approximation

$$Q(x) \simeq \left[\frac{1}{(1-a)x + a\sqrt{x^2+b}} \right] \frac{e^{-x^2/2}}{\sqrt{2\pi}} \quad x \geq 0$$

where $a = 0.339$
and $b = 5.510$

See Example 2.4-3

2.5 Other Distribution and Density Functions

(1) Binomial

Let $0<p<1$ and $N=1,2,\dots$ then the Binomial pdf is

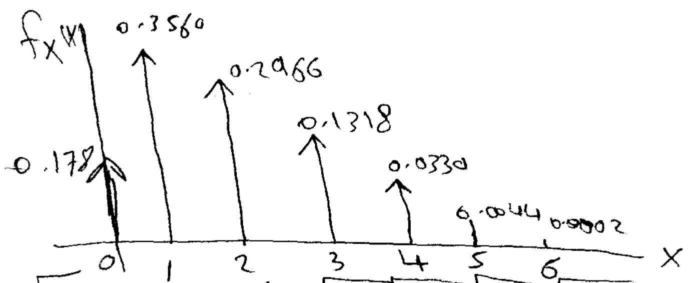
$$f_X(x)=\sum_{k=0}^{N}\binom{N}{k}p^k(1-p)^{N-k}\,\delta(x-k)$$

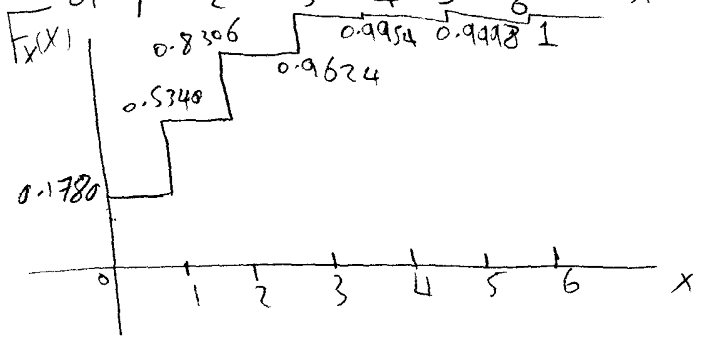
Binomial coefficient $\binom{N}{k}=\frac{N!}{k!(N-k)!}$

the CDF is

$$F_X(x)=\sum_{k=0}^{N}\binom{N}{k}p^k(1-p)^{N-k}\,u(x-k)$$

Example at $N=6$ and $p=0.25$





(2) Poisson

pdf $f_X(x)=e^{-b}\sum_{k=0}^{\infty}\frac{b^k}{k!}\delta(x-k)$

CDF $F_X(x)=e^{-b}\sum_{k=0}^{\infty}\frac{b^k}{k!}u(x-k)$

where $b>0$ is a real constant.

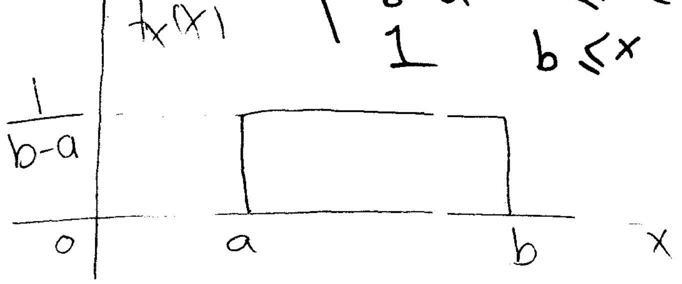
- Binomial Approximation using Poisson when N is large and p is very small, we can approximate the Binomial R.V. with a Poisson R.V. with $\boxed{b=Np}$

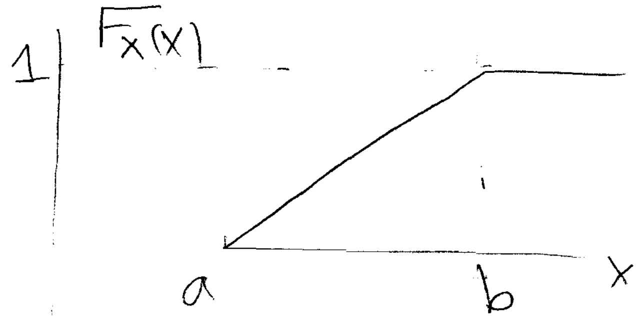
- Time of Arrival Problems. See Example 2.5-1.

③ Uniform

pdf $f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$

CDF $F_X(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x < b \\ 1 & b \le x \end{cases}$



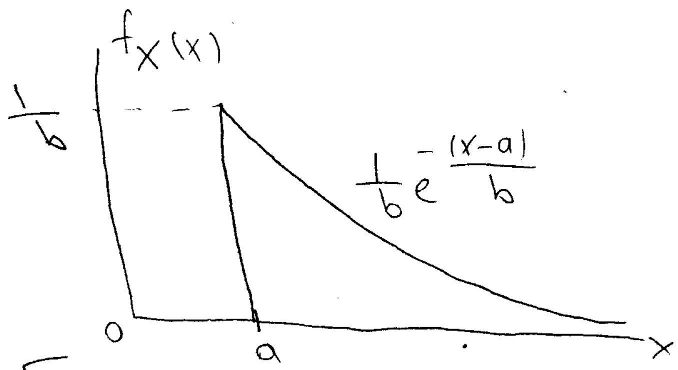


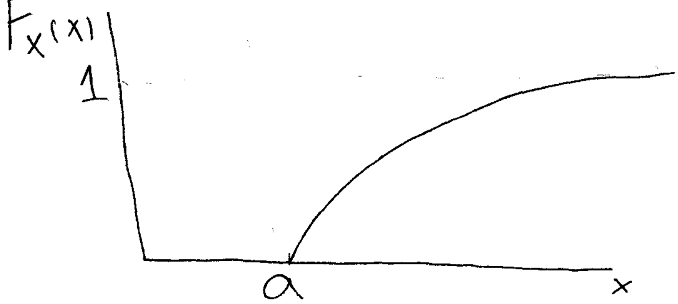
④ Exponential

Pdf $f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{(x-a)}{b}} & x > 0 \\ 0 & x < a \end{cases}$

CDF $F_X(x) = \begin{cases} 1 - e^{-\frac{(x-a)}{b}} & x > a \\ 0 & x < a \end{cases}$

where $-\infty < a < \infty$ and $b > 0$





Solve Example 2.5-2

⑤ Rayleigh

pdf $f_X(x) = \begin{cases} \frac{2}{b}(x-a)e^{-\frac{(x-a)^2}{b}} & x \geq a \\ 0 & x < a \end{cases}$

CDF $F_X(x) = \begin{cases} 1 - e^{\frac{-(x-a)^2}{b}} & x \geq a \\ 0 & x < a \end{cases}$

where $-\infty < a < \infty$ and $b > 0$