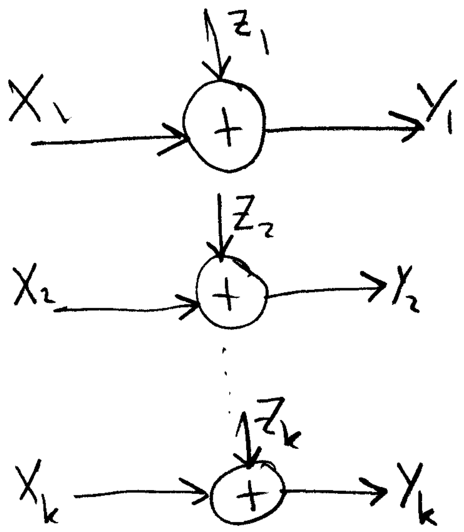
10.4 Parallel Gaussian Channels.

- Consider k independent Gaussian channels in parallel with a common power constraint.
- The objective is to distribute the total power among the channels so as to maximize the capacity.



$$Y_j = X_j + Z_j \;, \quad j = 1, 2, \dots, k$$

with $Z_j \sim \mathcal{N}(0, N_j)$

and $E\left[\sum_{j=1}^{k} X_j^2\right] \leq P$ ← Power constraint

→ We wish to distribute the power among the various channels so as to maximize the total capacity:

$$C = \max_{f(x_1, x_2, \dots, x_k): \sum E x_i^2 \leq P} I(X_1, X_2, \dots, X_k ; Y_1, Y_2, \dots, Y_k)$$

Since $Z_1, Z_2, \dots, Z_k$ are independent

$$\Rightarrow I(X_1, X_2, \dots, X_k ; Y_1, Y_2, \dots, Y_k)$$

$$= h(Y_1, Y_2, \dots, Y_k) - h(Y_1, Y_2, \dots, Y_k | X_1, X_2, \dots, X_k)$$

$$= h(Y_1, Y_2, \dots, Y_k) - h(Z_1, Z_2, \dots, Z_k | X_1, X_2, \dots, X_k)$$

$$= h(Y_1, Y_2, \dots, Y_k) - h(Z_1, Z_2, \dots, Z_k)$$

$$= h(Y_1, Y_2, \dots, Y_k) - \sum_i h(Z_i)$$

$$\leq \sum_i h(Y_i) - h(Z_i)$$

$$\leq \sum_i \frac{1}{2} \log\left(1 + \frac{P_i}{N_i}\right)$$

where $P_i = E X_i^2$, and $\sum P_i = P$

Equality is achieved by

$$(X_1, X_2, \dots, X_k) \sim \mathcal{N}\left(0, \begin{bmatrix} P_1 & 0 & \cdots & 0 \\ 0 & P_2 & \cdots & 0 \\ 0 & 0 & P_3 \cdots & P_j \\ \vdots & \vdots & \cdots & P_k \end{bmatrix}\right)$$

- So the problem is reduced to finding the power allotment that maximizes the capacity subject to the constraint that $\sum P_i = P$.

- Using Lagrange multipliers

$$J(P_1, \ldots, P_k) = \sum \frac{1}{2}\log\left(1+\frac{P_i}{N_i}\right) + \lambda\left(\sum P_i\right)$$

- Differentiating with respect to P_i, we get

$$\frac{1}{2}\frac{1}{P_i+N_i} + \lambda = 0$$

$$\Rightarrow \quad \frac{1}{P_i+N_i} = -2\lambda$$

$$P_i + N_i = \frac{-1}{2\lambda}$$

$$P_i = \frac{-1}{2\lambda} - N_i$$

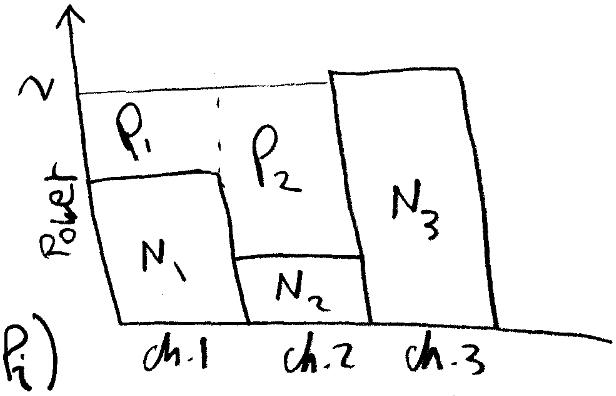
$$\boxed{P_i = \nu - N_i}$$

Since the P_i's must be non-negative, we restrict P_i to be:

$$P_i = (\nu - N_i)^+$$ where $(x)^+$ denotes the positive part.

So, the total Power will be

$$P = \sum (\nu - N_i)^+$$



"Water-Filling"

The capacity in this case will be

$$C = \sum_{i=1}^{k} \frac{1}{2}\log\left(1+\frac{(\nu-N_i)^+}{N_i}\right)$$

Ch.10 [Cover]

- ~~Consider~~

10.5 correlated Gaussian channels.

- Consider a Gaussian channel with Memory.
- A Block of n uses of the channel
- Let K_Z be the covariance matrix of the noise.
- Let K_X be the input covariance matrix
- The power constraint on the input is $\frac{1}{n}\sum_i E[X_i^2] \le P$

or $\frac{1}{n} tr(K_X) \le P$

- For channels with memory, consider a block of n consecutive uses of the channel as n channels in parallel with dependent noise.
- As the case of independent channels,

$$I(X_1, X_2, \dots, X_n; Y_1, Y_2, \dots, Y_n)$$
$$= h(Y_1, Y_2, \dots, Y_n) - h(Z_1, Z_2, \dots, Z_n)$$

The maximum Mutual information is obtained when Y is normal with covariance

$$K_Y = K_X + K_Z$$

$$\Rightarrow h(Y_1, Y_2, \dots, Y_n) = \frac{1}{2}\log\left[(2\pi e)^n |K_X + K_Z|\right]$$

- Now, we want to choose K_X to maximize $|K_X + K_Z|$

- Decompose K_Z to be:

$$K_Z = Q \Lambda Q^t, \text{ where } QQ^t = I$$

$$\Rightarrow |K_X + K_Z| = |K_X + Q\Lambda Q^t|$$
$$= |Q|\,|Q^t K_X Q + \Lambda|\,|Q^t|$$
$$= |Q^t K_X Q + \Lambda|$$
$$= |A + \Lambda|$$

where $A = Q^t K_X Q$

since For any matrices B and C

$$tr(BC) = tr(CB)$$

$$\Rightarrow tr(A) = tr(Q^t K_X Q)$$
$$= tr(QQ^t K_X)$$
$$= tr(K_X)$$

- ~~Now we apply Hadamard's inequality~~
- The problem is reduced to maximize $|A + \Lambda|$ subject to a trace constraint $tr(A) \le nP$
- Using Hadamard's inequality, the determinant of any positive definite matrix K is less than the product of its diagonal elements,

$$|K| < \prod K..$$

$\Rightarrow |A+\Lambda| \leq \prod_i (A_{ii}+\lambda_i)$

with equality iff A is diagonal.

Since $tr(A) = tr(K_X) \leq nP$

$\Rightarrow \frac{1}{n}\sum_i A_{ii} \leq P$

and $A_{ii} \geq 0$. (why? ~~cause~~ covariance matrices are positive definite)

The maximum value of $\prod_i (A_{ii}+\lambda_i)$ is attained when

$$\boxed{A_{ii} + \lambda_i = \nu}$$

Since A_{ii} must be ≥ 0

we put the condition

$$A_{ii} = (\nu - \lambda_i)^+$$

where ν is chosen so that $\sum A_{ii} = nP$

This value of A maximizes the entropy of Y and hence the mutual information.

Summary

Additive correlated Gaussian Noise channels:

$$Y_i = X_i + Z_i, \quad Z^n \sim N(0, K_Z)$$

the capacity is

$$C = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{2}\log\left(1 + \frac{(\nu-\lambda_i)^+}{\lambda_i}\right)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of K_Z and ν is chosen so that

$$\sum_i (\nu - \lambda_i)^+ = nP.$$