Problem 4-6

(a) The Fourier transform integral for this signal becomes

$$X_a(f) = \int_{-\infty}^{\infty} t e^{-\alpha t} u(t) e^{-j2\pi f t}\,dt = \int_0^{\infty} t e^{-\alpha t} e^{-j2\pi f t}\,dt$$

$$= \int_0^{\infty} t e^{-(\alpha + j2\pi f)t}\,dt = \left.\frac{t e^{-(\alpha + j2\pi f)t}}{-(\alpha + j2\pi f)}\right|_0^{\infty} + \frac{1}{(\alpha + j2\pi f)}\int_0^{\infty} e^{-(\alpha + j2\pi f)t}\,dt$$

$$= -\left.\frac{e^{-(\alpha + j2\pi f)t}}{(\alpha + j2\pi f)^2}\right|_0^{\infty} = \frac{1}{(\alpha + j2\pi f)^2}$$

(b) To evaluate this Fourier transform integral, we need the tabulated integral

$$\int t^2 e^{at}\,dt = \frac{e^{at}}{a^3}(a^2 t^2 - 2at + 2)$$

For this signal, the Fourier transform integral is

$$X_b(t) = \int_{-\infty}^{\infty} t^2 u(t) u(1 - t) e^{-j2\pi f t}\,dt = \int_0^1 t^2 e^{-j2\pi f t}\,dt$$

$$= \left\{\frac{e^{-j2\pi f t}}{(-j2\pi f)^3}[(-j2\pi f)^2 t^2 - 2(-j2\pi f)t + 2]\right\}_0^1$$

$$= \left\{\frac{1}{j2\pi f}\left[\frac{1}{j\pi f} - 1\right]e^{-j2\pi f} + \frac{2}{(j2\pi f)^3}\left[1 - e^{-j2\pi f}\right]\right\}$$

(c) The integral for this Fourier transform becomes

$$X_c(f) = \int^1 e^{-(\alpha + j2\pi f)t}\,dt = -\left.\frac{e^{-(\alpha + j2\pi f)t}}{\alpha + j2\pi f}\right|_0^1 = \frac{1 - e^{-(\alpha + j2\pi f)}}{\alpha + j2\pi f}$$

Problem 4-16

(a) The signal may be written as

$$x_a(t) = \Pi(t - 1.5) + \Pi(t + 1.5)$$

Using superposition, time delay, and the Fourier transform pair given, we get

$$X_a(f) = \text{sinc}(f)\, e^{-j2\pi f(1.5)} + \text{sinc}(f)\, e^{j2\pi f(1.5)} = 2\text{sinc}(f)\cos(3\pi f)$$

The second signal may be written as

$$x_b(t) = \Pi(t - 1.5) - \Pi(t + 1.5)$$

Thus, its Fourier transform is

$$X_a(f) = \text{sinc}(f)\, e^{-j2\pi f(1.5)} - \text{sinc}(f)\, e^{j2\pi f(1.5)} = -2j\,\text{sinc}(f)\sin(3\pi f)$$

The third signal may be written as

$$x_c(t) = \Pi(t/4) + \Pi(t/2)$$

Using superposition and the transform pair given, its Fourier transform is

$$X_c(f) = 4\,\text{sinc}(4f) + 2\,\text{sinc}(2f)$$

The fourth signal may be written as

$$x_d(t) = \Pi[(t - 1.5)/3] - \Pi(t - 1.5)$$

Problem 4-25

(a) Use the Fourier transform pairs

$$x_1(t) \leftrightarrow \frac{1}{(\alpha + j2\pi f)^2} \quad \text{(see Prob. 4-6a)}$$

$$\text{and } h_1(t) \leftrightarrow \frac{1}{\beta + j2\pi f}$$

Thus, the Fourier transform of the convolution of the two signals is

$$x_1(t) * h_1(t) = \mathscr{F}^{-1}\left[\frac{1}{(\alpha + j2\pi f)^2}\frac{1}{\beta + j2\pi f}\right]$$

Expand in partial fractions to get

$$x_1(t) * h_1(t) = \mathscr{F}^{-1}\left\{\frac{1}{(\beta - \alpha)^2}\left[\frac{1}{\beta + j2\pi f} - \frac{1}{\alpha + j2\pi f} + \frac{\beta - \alpha}{(\alpha + j2\pi f)^2}\right]\right\}$$

$$= \frac{1}{\beta - \alpha}\left[\frac{1}{\beta - \alpha}e^{-\beta t} - \left(\frac{1}{\beta - \alpha} - t\right)e^{-\alpha t}\right]u(t)$$

(b) For this case, just interchange α and β.

(c) In this case

$$x_3(t) * h_3(t) = \mathscr{F}^{-1}\left[\frac{1}{\alpha + j2\pi f}\frac{1}{-\beta + j2\pi f}\right]$$

$$= \frac{1}{\alpha - \beta}\left[\frac{1}{\alpha + j2\pi f} + \frac{1}{-\beta + j2\pi f}\right] = \frac{1}{\alpha - \beta}\left[e^{-\alpha t}u(t) + e^{\beta t}u(-t)\right]$$

Problem 4-33

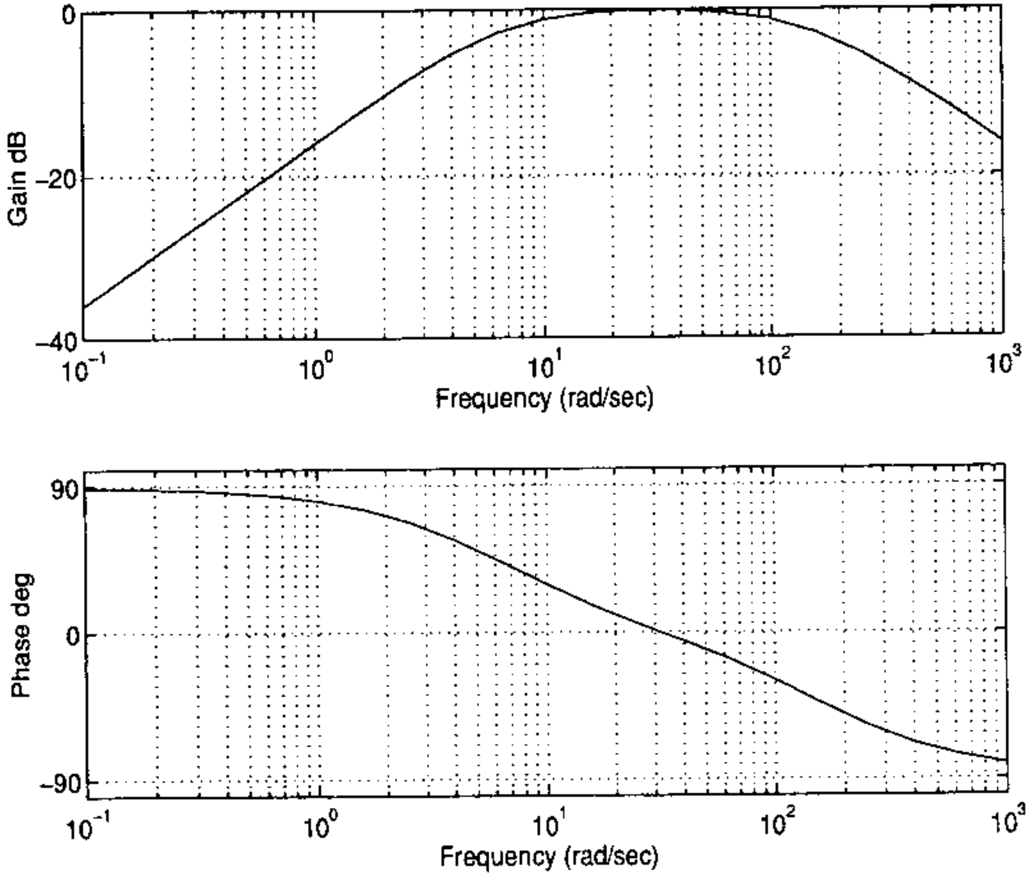
(a) Using the admittance of the parallel combinations, we have

$$H(j\omega) = \frac{1}{Y_{adm}(j\omega)} = \frac{1}{\dfrac{1}{j\omega L} + \dfrac{1}{R} + j\omega C} = R\frac{j\omega\dfrac{L}{R}}{1 - \omega^2 LC + \dfrac{L}{R}j\omega}$$

Further rearrangement after substituting $f = \omega/2\pi$ gives

$$H(f) = R\frac{j(f/f_b)}{1 + j(f/f_b) - (f/f_0)^2} \text{ where } f_0 = \frac{1}{2\pi\sqrt{LC}} \text{ and } f_b = \frac{R}{2\pi L}$$

The amplitude and phase response functions are shown below for $f_0 = 5$ Hz and $f_b = 1$ Hz.



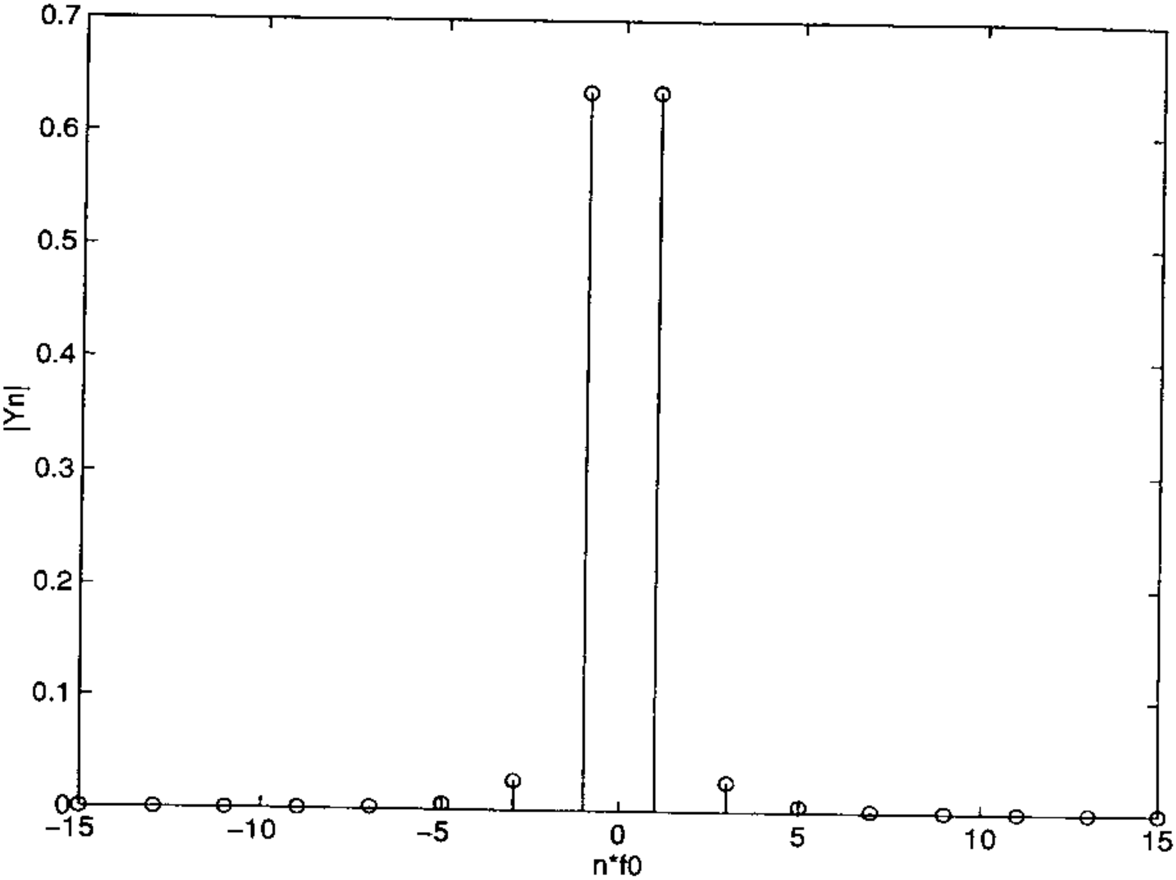
(b) For the square wave input,

$$X_n = \begin{cases} \frac{2A}{|n|\pi}, & n = \pm 1, \pm 5, \cdots \\ -\frac{2A}{|n|\pi}, & n = \pm 3, \pm 7, \cdots \end{cases}$$

The output spectrum of the filter with this input is

$$Y(f) = \sum_{n=-\infty}^{\infty} X_n H(nf_0)\delta(f - nf_0) = \sum_{n=-\infty}^{\infty} Y_n \delta(f - nf_0)$$

where $f_0 = 1/T$. The amplitude spectrum of the output is shown below for $f_0 = f_b$.



(c) The maginitude of the nth spectral line of the output is

$$|Y_n| = \frac{\left(\frac{2A}{|n|\pi}\right)\left(\frac{nf_0}{f_b}\right)}{\sqrt{\left[1-\left(\frac{nf_0}{f_b}\right)^2\right]^2+\left(\frac{nf_0}{f_b}\right)^2}},\ n \text{ odd}$$

The power in the nth harmonic is proportional to the square of the magnitude of the corresponding line. Thus, the ration of the powers in the fundamental component to that in the nth harmonic is

$$\frac{P_{\text{fund}}}{P_{n\text{th har}}} = \frac{\left[1-\left(\frac{nf_0}{f_b}\right)^2\right]^2+\left(\frac{nf_0}{f_b}\right)^2}{\left[1-\left(\frac{f_0}{f_b}\right)^2\right]^2+\left(\frac{f_0}{f_b}\right)^2}$$

If $f_0 = f_b$, this simplifies to

$$\frac{P_{\text{fund}}}{P_{n\text{th har}}} = (1 - n^2)^2 + n^2$$

(d) We solve the above equation for f_b with $n = 3$ and the left-hand side equal to $10^3 = 1000$:

$$1000 = \frac{\left[1 - \left(\frac{3f_0}{f_b}\right)^2\right]^2 + \left(\frac{3f_0}{f_b}\right)^2}{\left[1 - \left(\frac{f_0}{f_b}\right)^2\right]^2 + \left(\frac{f_0}{f_b}\right)^2}$$

The solution is $f_0/f_b = 1.92$. To get the 3rd harmonic 40 dB down (left-hand side of the above equation equals 10,000), this ratio needs to be 3.36.

Problem 4-39

(a) We may write this waveform as

$$x_a(t) = 2\Lambda(t/3) * \sum_{n=-\infty}^{\infty} \delta(t - 6n)$$

The Fourier transform of this signal, using (4-111), is

$$X_a(f) = \mathscr{F}[2\Lambda(t/3)] \times \frac{1}{6} \sum_{n=-\infty}^{\infty} \delta(f - n/6) = 6\,\mathrm{sinc}^2(3f) \times \frac{1}{6} \sum_{n=-\infty}^{\infty} \delta(f - n/6)$$

$$= \sum_{n=-\infty}^{\infty} \mathrm{sinc}^2(n/2)\,\delta(f - n/6)$$

A plot is provided at the end of the problem.

(b) Write one period of this triangular waveform as

$$x_b(t) = 4\Lambda(t/6) - 2\Pi(t/12), \quad -6 \le t \le 6$$

Therefore, using (4-111), the spectrum is

$$X_b(f) = [4 \times 6\,\mathrm{sinc}^2(6f) - 2 \times 12\,\mathrm{sinc}(12f)] \times \frac{1}{12} \sum_{n=-\infty}^{\infty} \delta(f - n/12)$$

$$= 2 \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \mathrm{sinc}^2(n/2)\,\delta(f - n/12)$$

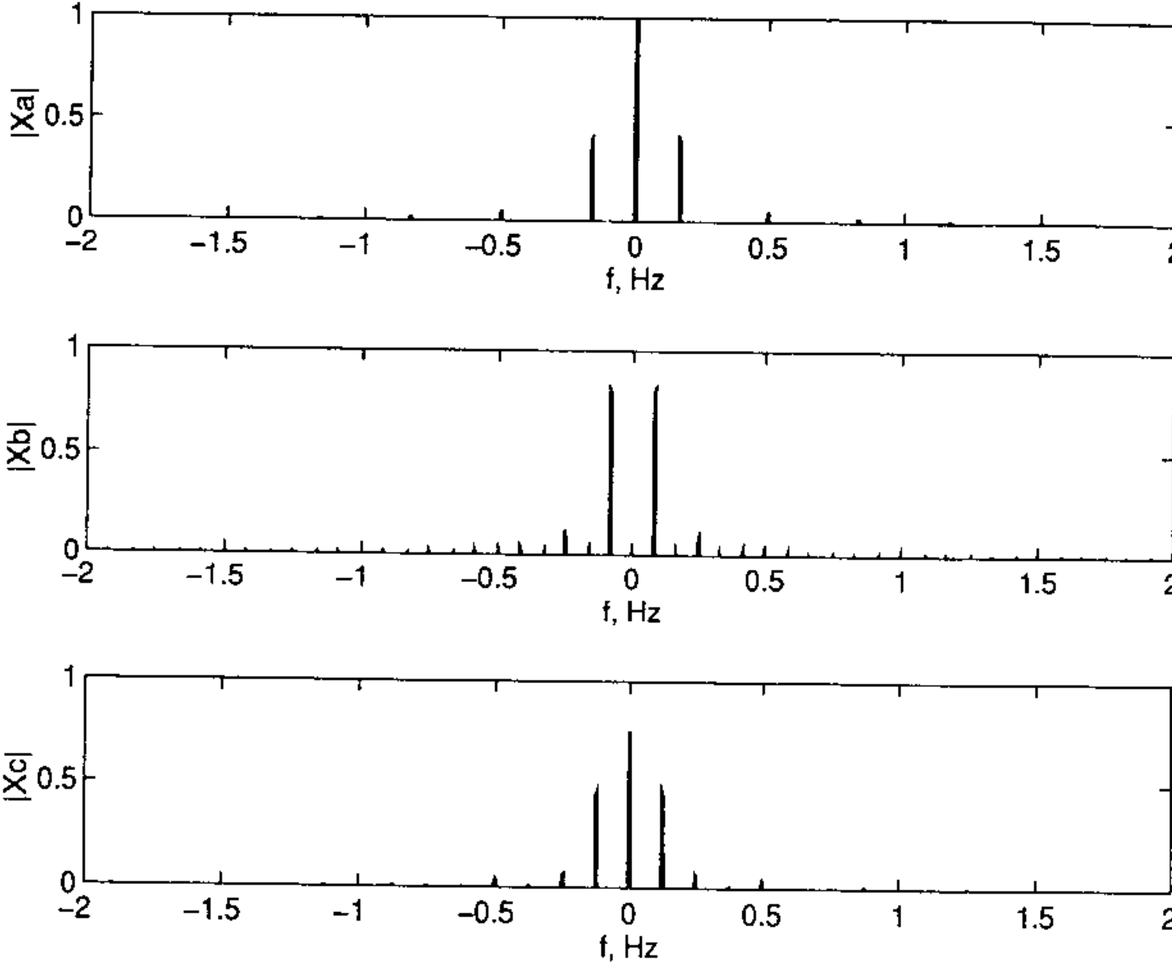
(c) Write one period of this waveform as

$$x_c(t) = 2\Lambda(t/3)$$

Again using (4-111), the spectrum is

$$X_c(f) = \frac{3}{4} \sum_{n=-\infty}^{\infty} \mathrm{sinc}^2(3n/8)\,\delta(f - n/8)$$

Plots are given below for all three cases. The delta functions are approximated as square pulses and their heights are equal to their weights.


|Xa|
1
0.5
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
f, Hz
|Xb|
1
0.5
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
f, Hz
|Xc|
1
0.5
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
f, Hz