Chapter 4 Transform Coding
Outline

- Principle of block-wise transform coding
- Properties of orthonormal transform
- Discrete cosine transform (DCT)
- Bit allocation for transform coefficients
- Threshold coding
- Typical coding artifacts
Transform Coding (TC)

- TC: another efficient coding scheme based on utilization of inter-pixel correlation.
  - A core technique recommended by JPEG
  - Efficient in terms of coding prediction error in motion compensated predictive coding, adopted by all international video coding standards.

- Transformation decides which format of input source is quantized and encoded
  - In DPCM: the difference signal (smaller variance)
  - In TC: the transformed version of a signal (less correlated)
Hotelling Transform

- Consider an \( N \)-dimensional vector \( \vec{z}_s \)
- The ensemble of such vectors can be modeled by a random vector

\[
\vec{z} = (z_1, z_2, \ldots, z_N)^T
\]

Where each component \( z_i \) is a random variable
- The mean vector of the population

\[
m_{\vec{z}} = E[\vec{z}] = (m_1, m_2, \ldots, m_N)^T, m_i = E[z_i]
\]
- The covariance matrix:

\[
C_{\vec{z}} = E[(\vec{z} - m_{\vec{z}})(\vec{z} - m_{\vec{z}})^T]
\]

with element

\[
C_{i,j} = \text{Cov}(z_i, z_j)
\]

- \( C_{i,i} \) : variance of \( z_i \)
- \( C_{\vec{z}} \) : real and symmetric
Hotelling Transform

- According to theory of linear algebra, always possible to find a set of $N$ orthonormal eigenvectors of the matrix $C_{zz}$ to convert it into a full ranked diagonal matrix
  - $N$ orthonormal eigenvectors $\vec{e}_i$
  - With corresponding eigenvalues: $\lambda_i$
  - Form an orthogonal matrix
    $$\phi = (\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_N)^T$$

- Hotelling transform (or eigenvector transform):
  $$\tilde{y} = \phi(\tilde{z} - m_{\tilde{z}})$$
Hotelling Transform

- Two features
  - $m_{\gamma} = 0$
  - $C_{\gamma} = \phi C_{\zeta} \phi^T = \begin{bmatrix} \lambda_1 & 0 \\ & \lambda_2 & \ddots \\ & & \ddots & \ddots \\ 0 & & & \lambda_N \end{bmatrix}$

- Correlation previously existing between different components has been removed in the transformed domain

- Inverse Hotelling transform
  \[ \tilde{z} = \phi^{-1} \tilde{y} + m_{\tilde{z}} = \phi^T \tilde{y} + m_{\tilde{z}} \]

- Can be viewed as discrete version of the Karhunen-Loeve transform (KLT)
Statistical Interpretation

- The elements in the main diagonal of $C_{\tilde{y}}$
  - Eigenvalues of $C_{\tilde{y}}$.
  - Variances of the components of vector $\tilde{y}$, denoted by $\sigma_{y,1}^2, \sigma_{y,2}^2, \ldots, \sigma_{y,N}^2$
- Arrange eigenvalues (variances) in non-increasing order:
  
  $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$
- Choose an integer $L$ where $L<N$
- Using the corresponding $L$ eigenvectors $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_L$, to form a new matrix $\phi$ of dimension $L \times N$
  - Transform changes to $\tilde{y} = \phi(\tilde{z} - m_\tilde{z})$
  - Inverse transform changes to $\tilde{z}' = \phi^T \tilde{y} + m_\tilde{z}$
Statistical Interpretation

- It can be shown that $MSE$ between $\tilde{z}$ and $\tilde{z}'$ is given by
  \[ MSE_r = \sum_{i=L+1}^{N} \sigma_{y,i}^2 \]
- Quantizing and encoding only $L$ components of vector $\tilde{y}$ lead to higher coding efficiency – basic idea behind transform coding
- The linear unitary transform provides two functions.
  - Decorrelate input data
  - Some transform coefficients more significant than others
    - Some can be discarded
    - Some can be coarsely quantized
    - Some can be finely quantized
Geometrical Interpretation

- A binary image of a car in Fig. 4.1 (a)
  - Each pixel in the shaded object region: a 2-D vector with its two components being coordinates $z_1$ and $z_2$ respectively.
  - The Hotelling transform
  - Note the two features

Fig. 4.1 (b): after transform, object is aligned with principal axes
Procedures of Transform Coding

(a) Transmitter

Input image → Block Division → Linear Transform → Bit allocation

(bit allocation includes Truncation, Quantization, Codeword assignment)

→ Output bitstream

(b) Receiver

Input bitstream → Decoder → Inverse Transform → Block Merge → Reconstructed image
Linear Transforms

- A digital image: a 2-D array \( g(x, y) \)
  - \((x,y)\) coordinates of a pixel in 2-D array
  - \(g()\): gray level value of the pixel
- \(T(u,v): 2-D\) transform of \(g(x,y)\)
  - \((u,v)\): coordinates in transformed domain.
- 2-D forward and inverse transforms:

\[
T(u,v) = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} g(x,y) f(x,y,u,v)
\]

\[
g(x,y) = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} T(u,v) i(x,y,u,v)
\]

- \(f(x,y,u,v)\) and \(i(x,y,u,v)\): forward and inverse transformation Kernels
Separability

A transformation kernel is **separable** if

\[ f(x, y, u, v) = f_1(x, u) f_2(y, v) \]
\[ i(x, y, u, v) = i_1(x, u) i_2(y, v) \]

A 2-D separable transform can be decomposed into two 1-D transforms

\[ T_1(x, v) = \sum_{y=0}^{N-1} g(x, y) f_2(y, v) \]
\[ T(u, v) = \sum_{x=0}^{N-1} T_1(x, v) f_1(x, u) \]

A transformation kernel is **symmetric** if it is separable and

\[ f_1(y, v) = f_2(y, v) \]
Matrix Form

- If a transformation kernel is symmetric (hence separable), then 2-D image transform can be expressed in matrix form.
  - Denote an image matrix by \( G \): \( G = \{g_{i,j}\} = \{g(i-1, j-1)\} \)
  - Denote **forward transform matrix** by \( F \) (NxN)

\[
F = \{f_{i,j}\} = \{f_{i}(i-1, j-1)\}
\]

- Denote inverse transform matrix by \( I \): \( I = \{i_{j,k}\} = \{i_{j}(j-1, k-1)\} \)

\[
I = F^{-1}
\]

- Then

\[
T = F^{T}GF
G = I^{T}TI
\]

- DFT (complex quantities)
  - “\(\ast\)”: complex conjugation
  - Unitary transform

\[
T = F^{*T}GF
G = I^{*T}TI
I = F^{-1} = F^{*T}
\]
Orthogonality

- Definition: a transform is orthogonal if

\[ F^T = F^{-1} \]

- An orthogonal matrix (transform) is a special case of a unitary matrix (transform):
  - only real quantities are involved.
- All 2-D image transforms to be presented:
  - separable, symmetric, and unitary.
Basis Image Interpretation

- Consider 2-D inverse transform

\[ g(x, y) = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} T(u, v)i(x, y, u, v) \]

- Consider an \(N\times N\) basis image for a specific \((u, v)\)

\[ I_{u,v} = \{i(x, y, u, v), 0 \leq x, y \leq N - 1\} \]

  - That is:

\[
I_{u,v} = \begin{bmatrix}
i(0,0,u,v) & i(0,1,u,v) & \cdots & \cdots & i(0,N-1,u,v) \\
i(1,0,u,v) & i(1,1,u,v) & \cdots & \cdots & i(1,N-1,u,v) \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
\vdots & \vdots & \cdots & \ddots & \vdots \\
i(N-1,0,u,v) & i(N-1,1,u,v) & \cdots & \cdots & i(N-1,N-1,u,v)
\end{bmatrix}
\]
Basis Image Interpretation

- The inverse transform in a collective form
  \[ G = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} T(u, v) I_{u,v} \]

- Interpretation:
  - A series expansion of the original image \( G \) into a set of \( N^2 \) basis images
  - Transform coefficients \( T(u, v), 0 \leq u, v \leq N - 1 \), become the coefficients of the expansion
  - Coefficient (weight) \( T(u, v) \): a correlation measure between image \( G \) and basis image [Wintz’72]
Basis Image Interpretation

- Comments on basis images:
  - Have nothing to do with the input image
  - Completely defined by transform itself
    - Different transforms $\Leftrightarrow$ different sets of basis images

- Motivations:
  - With proper transform, transform coefficients are more independent than the gray scales of original input image.
  - Optimum linear transform: uncorrelated coefficients
  - The coefficient variance varies widely
    - Insignificant coefficients ignored
    - Significant coefficients are allocated more bits
  - Coding efficiency is thus enhanced.
Basis Image Interpretation

- TC coding can be considered a special case of subband coding, although TC was devised much earlier
- An alternative way to define basis images
  - \( I_{u,v} = \overrightarrow{b}_u \overrightarrow{b}_v^T \)
  - the outer product of the basis vectors, where \( \overrightarrow{b}_u \) is the \( u \)th column vector of the inverse transform matrix [Jayant’84]
Subimage Size Selection

- The larger the size, the more decorrelation TC can achieve
- Correlation between image pixels, however, becomes insignificant when the distance becomes large
- Problems with large size:
  - A large block size can NOT adapt to local statistics well
  - A transmission error in TC affects the whole subimage
    - A possibly severe effect of transmission error for large size
  - In video coding, TC is typically used together with motion compensated (MC) coding. Large block size not used in MC.
- Subimage sizes (N) of 4, 8, 16 are used most often
Transforms of Particular Interest

- Discrete Fourier Transform (DFT)
- Discrete Walsh Transform (DWT)
  - DWT transformation kernels [Walsh’23]:

\[
    f(x, y, u, v) = \frac{1}{n} \prod_{i=0}^{n-1} \left[ (-1)^{p_i(x) p_{n-1-i}(u)} (-1)^{p_i(y) p_{n-1-i}(v)} \right]
\]

Where \( n = \log_2 N \)

\( p_i(\text{arg}) \): \( i \)th bit in NBC (natural binary representation) of the arg

The 0th bit: the least significant bit
The \((n-1)\)th bit: the most significant bit

- For instance, consider \( N=16 \) (\( n=4 \))

The NBC of 8 is1000, \( p_0(8) = p_1(8) = p_2(8) = 0, p_3(8) = 1 \)
Discrete Walsh Transforms (DWT)

- DWT transform matrix for $N=4$:

$$
F = \frac{1}{2} \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix}
$$

- The DWT implementation is simple
  - $F$ is simple
  - $i(x, y, u, v) = f(x, y, u, v)$

A set of 16 basis images for $N=4$
Discrete Hadamard Transform

- Closely related to DWT
  - DHT transformation kernels [Hadamard’1893]:
    \[ f(x, y, u, v) = \frac{1}{n} \prod_{i=0}^{n} [(-1)^{P_i(x)}P_i(u) (-1)^{P_i(y)}P_i(v)] \]
    \[ i(x, y, u, v) = f(x, y, u, v) \]
    Where \( n, i, \) and \( P_i(\text{arg}) \) are the same as in DWT

- Walsh-Hadamard transform (DWHT) is frequently used to represent either of the two transforms
Discrete Cosine Transform (DCT)

- **Background**
  - Most commonly used transform for image and video coding
  - Established by Ahmed, Natarajan and Rao [Ahmed’74]

- The basis vectors of 1-D DCT: a good approximation to the eigenvectors of the class of Toeplitz matrices defined as (where $0 < \rho < 1$)

\[
\begin{bmatrix}
1 & \rho & \rho^2 & \ldots & \rho^{N-1} \\
\rho & 1 & \rho & \ldots & \rho^{N-2} \\
\rho^2 & \rho & 1 & \ldots & \rho^{N-3} \\
\vdots & \vdots & \vdots & \ldots & \vdots \\
\rho^{N-1} & \rho^{N-2} & \rho^{N-3} & \ldots & 1
\end{bmatrix}
\]
DCT Transformation Kernel

\[ f(x, y, u, v) = C(u)C(v) \cos\left(\frac{(2x + 1)u\pi}{2N}\right) \cos\left(\frac{(2y + 1)v\pi}{2N}\right) \]

Where

\[ C(u) = \begin{cases} \sqrt{\frac{1}{N}} & \text{for } u = 0 \\ \sqrt{\frac{2}{N}} & \text{for } u = 1, 2, \ldots, N - 1 \end{cases} \]

\[ C(v) \] is defined in the same way

64 basis images for N=8
Relationship with DFT

The DCT of an $N$-point sequence $g_N(n)$ is obtained via the following three steps:

- Form a $2N$-point sequence, $g_{2N}(n)$, then form a periodic sequence $\tilde{g}_{2N}(n)$
- Find Fourier Series (FS) coefficients of $\tilde{g}_{2N}(n)$
- Only keep the $N$ coefficients with indexes $0, 1, \cdots, N - 1$ which are the DCT coefficients of the given $N$-point sequence $g_N(n)$
Relationship with DFT

- Observations:
  - $\tilde{g}_N(n)$ not smooth
    - End-head discontinuities cause high frequency distribution in DFT
  - $\tilde{g}_{2N}(n)$ does not have this discontinuity
- DCT has better energy compaction in low frequency than DFT
  - In terms of energy compaction, DCT is the best among DFT, DWT, DHT and discrete Harr transform
  - KLT: image dependent, not practical

Figure 4.5 An example to illustrate the differences and similarities between DFT and DCT
Relationship with DFT

- DCT can be implemented using the FFT
  - DCT of an $N$-point sequence can be obtained from DFT of a $2N$-point sequence,
  - *Even symmetry* of $\tilde{g}_{2N}(n)$ makes computation required for DCT of an $N$-point sequence equal to that required for DFT of an $N$-point sequence
- As a result, DCT is the most popular image transform used in image and video coding
Performance Comparison

- Energy compaction
  - Mean square approximation error

\[ MSE_r = E\left[(\tilde{z} - \tilde{z}')^2\right] = \sum_{i=L+1}^{N} \sigma_i^2 \]

Where \( \tilde{z}' \) denotes the reconstructed vector

- DWT, DFT, DCT can be implemented using FFT

Figure 4.6 Transform coefficient variances when \( N=16, \rho=0.95 \) [ahmed 1974]
Bit Allocation

- Bit allocation: truncation, quantization, and codeword assignment

- Three types of error in TC
  - Truncation error: majority of coefficients are truncated to zero
  - Quantization error
  - Transmission error

- Two different ways in truncation
  - Zonal coding
  - Threshold coding
Zonal Coding

- No overhead side information
- Coding efficiency, however, may not be high
  - Some coefficients outside the zone might be large
  - Some coefficients inside the zone may be small
- For further improvement, adaptive scheme has to be used.
Threshold Coding

- No predefined zone
- Each transform coefficient compared with a threshold
  - Smaller than threshold, set to zero
  - Larger than threshold, retained for quantization/encoding
  - Threshold determined after evaluation of all coefficients
- Address of retained coefficients has to be sent to receiver as side information
  - Usually a two-pass adaptive technique
Threshold Coding

(a) Transmitter

Input Subimage \( g(x,y) \)

DCT \( C(u,v) \) → Thresholding and shifting \( C_T(u,v) \) → Normalization \( C_{TN}(u,v) \)

| Variable quantization step size |

Roundoff \( C_{TN}(u,v) \) → Zigzag Scan \( C'_{TN}(u,v) \) → Huffman Coding → Rate buffer → Output bitstream

(b) Receiver

Input bitstream → Rate buffer \( C'_{TN}(u,v) \) → Decoding \( C_T(u,v) \) → Arrange Coefficients in blocks → Inverse Normalization

Add shift back \( C'_T(u,v) \) → Inverse DCT \( C'(u,v) \) → Reconstructed Subimage \( g'(x,y) \)
Threshold Coding

- Chen and Pratt: an efficient adaptive coding scheme [Chen’84]
  - A one-pass fast adaptive scheme
  - With several effective techniques, it achieved excellent results in TC
  - Demonstrated satisfactory quality of reconstructed frames at 0.4 bpp for coding color images
    - ⇒ real-time color television transmission over a 1.5 Mbps channel
  - Adopted by JPEG
Thresholding and Shifting

- Formula

\[ C_T(u, v) = \begin{cases} C(u, v) - T & \text{if } C(u, v) > T \\ 0 & \text{otherwise} \end{cases} \]

- \( T \): threshold
- More than 60% of DCT coefficients normally fall below a threshold as low as 5

Figure 4.9 Input-output characteristic of thresholding and shifting

Figure 4.10 Distribution of DCT coefficients

Percentage vs threshold
Normalization and Roundoff

- Normalization is implemented as follows:

\[ C_{TN}(u, v) = \frac{C_T(u, v)}{\Gamma_{u,v}} \]

- \( \Gamma_{u,v} \): normalization factor controlled by the rate buffer

- Then roundoff process converts floating point to nearest integer
  - Uniform midtread quantizer with unit quantization step
  - Normalization is a scaling process
    - Makes resultant uniform midtread quantizer adapt to dynamic range of the coefficients.
  - By adjusting \( \Gamma_{u,v} \), variable bit rate and MSE can be achieved.

- Take into account image statistics and HVS characteristics.
Normalization and Roundoff

- HVS more sensitive to luminance component than to chrominance components
- HVS more sensitive to low frequency components
- Quantization table in JPEG
  - a matrix consisting of all normalization factors

<table>
<thead>
<tr>
<th>Luminance quantization table</th>
</tr>
</thead>
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<tr>
<td>16 11 10 16 24 40 51 61</td>
</tr>
<tr>
<td>12 12 14 19 26 58 60 55</td>
</tr>
<tr>
<td>14 13 16 24 40 57 69 56</td>
</tr>
<tr>
<td>14 17 22 29 51 87 80 62</td>
</tr>
<tr>
<td>18 22 37 56 68 109 103 77</td>
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<td>24 35 55 64 81 104 113 92</td>
</tr>
<tr>
<td>49 64 78 87 103 121 120 101</td>
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<td>72 92 95 98 112 100 103 99</td>
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</table>

<table>
<thead>
<tr>
<th>Chrominance quantization table</th>
</tr>
</thead>
<tbody>
<tr>
<td>17 18 24 47 99 99 99 99</td>
</tr>
<tr>
<td>18 21 26 66 99 99 99 99</td>
</tr>
<tr>
<td>24 26 56 99 99 99 99 99</td>
</tr>
<tr>
<td>47 66 99 99 99 99 99 99</td>
</tr>
</tbody>
</table>
Zigzag Scan

- Most quantized coefficients are zero
- Run-length code (RLC)
  - Very efficient to encode address information of nonzero coefficients
  - Run-length of zero coefficients: # of consecutive zeros in zigzag scan
  - Zigzag scanning minimizes use of RLC in the block
Huffman Coding

- Statistical studies of magnitude of nonzero DCT coefficients and run-length of zeros were conducted in [Chen’84]
  - Domination of coefficients with small amplitude and short run-lengths was found
- This justifies the applications of Huffman coding to magnitude of nonzero coefficients and run-lengths of zeros

Special codewords
- End of block (EOB)
  - Indicate the termination of coding a block
  - save bits
- Run-length prefix:
  - differentiate run-length codewords from amplitude codewords.
Rate Buffer Feedback and Equalization

- A rate buffer accepts a *variable*-rate data input from the encoding process and provides a *fixed*-rate data output to the channel.
- The status of the rate buffer is monitored and fed back to control the threshold and the normalization factor.
  - A one-pass adaptation is achieved.
Some Issues

- **Effect of transmission error**
  - In TC, each pixel in reconstructed image relies on all transform coefficients of the subimage where the pixel is located
  - A bit reversal transmission error will be spread
    - Subimage size should be limited
  - The effect on the reconstructed image varies
    - An error in DC or low frequency AC coefficients may be objectionable
    - An error in high frequency AC coefficients less noticeable
Reconstruction Error Sources

- Three sources for reconstruction error
  - Truncation, quantization, transmission
- Block artifacts
  - TC are usually conducted blockwise
  - Block boundary artifacts can be annoying to HVS, especially at low bit rate
- To alleviate blocking artifacts
  - Block overlapping
  - Post-filtering reconstructed image along block boundary
  - Advanced transform
Reconstruction Error Sources

- Block overlapping
  - Each pixel in overlapped regions takes an average of all its reconstructed pixel values from multiple blocks
  - Extra bits are used for those pixels in overlapped regions
    - Overlapped region is usually only one pixel wide

- Post-filtering
  - *Low pass* filtering block boundaries
  - No bit overhead, better result
  - Adopted by international coding standards.
Comparison: DPCM vs TC

- Both utilize inter-pixel correlation, and are efficient
- In terms of computational complexity, DPCM simpler
- In terms of memory requirement and processing delay, DPCM superior
- Design of DPCM system, and its performance, is sensitive to image-to-image variation.
  - Optimum DPCM design is matched to statistics of certain image
  - TC less sensitive to variation in image statistics
- In general, optimum DPCM with a third or higher order predictor performs better than TC for bit rate about 2-3 bits per pixel
  - When bit rate is lower, TC is preferred
Hybrid Coding

- Hybrid transform/waveform coding
  - In waveform coding, the waveform of a signal is coded, e.g., DPCM
  - Hybrid coding combines TC and DPCM coding, i.e., TC applied first rowwise followed by DPCM coding columnwise, or vice versa.
  - Both used in JPEG

- Two techniques complement each other: has TC’s small sensitivity to variable statistics and DPCM’s simplicity

- A successful hybrid coding scheme in interframe coding
  - Use motion compensated predictive coding
  - Prediction error is transform coded
  - Efficient and adopted by international video coding standards

- HW #2: Ex. 4-8