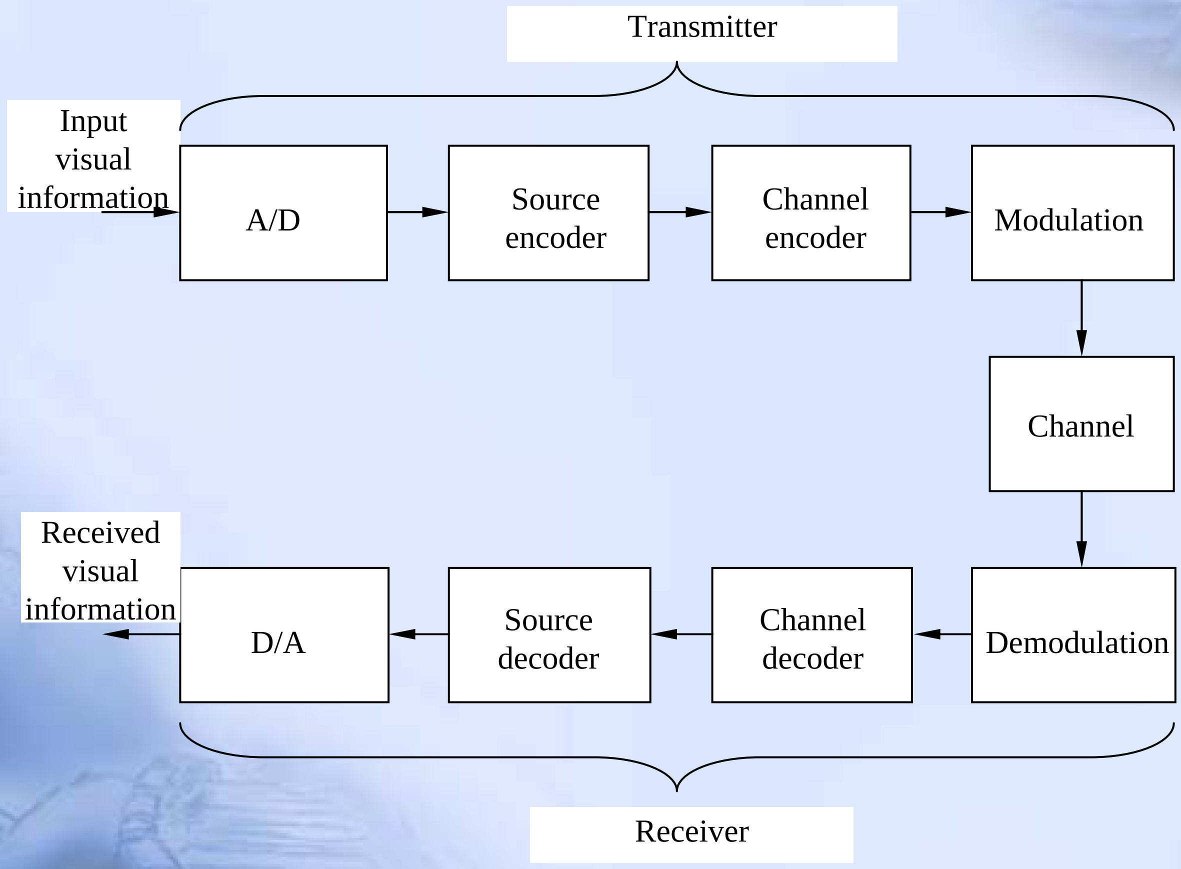
Chapter 2 Quantization

How much can we compress this image losslessly?
How much can we compress this image with a loss?

Outline

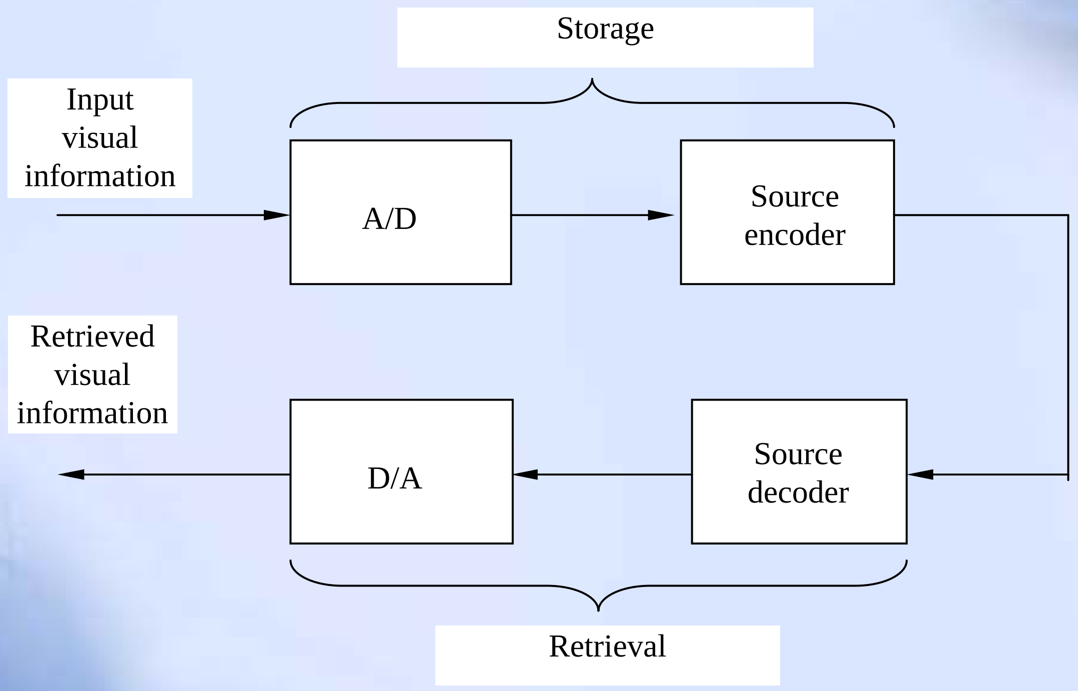
- Quantization and Source encoder
- Uniform quantization
  - Basics
  - Optimum Uniform quantizer

Quantization and Source Encoder



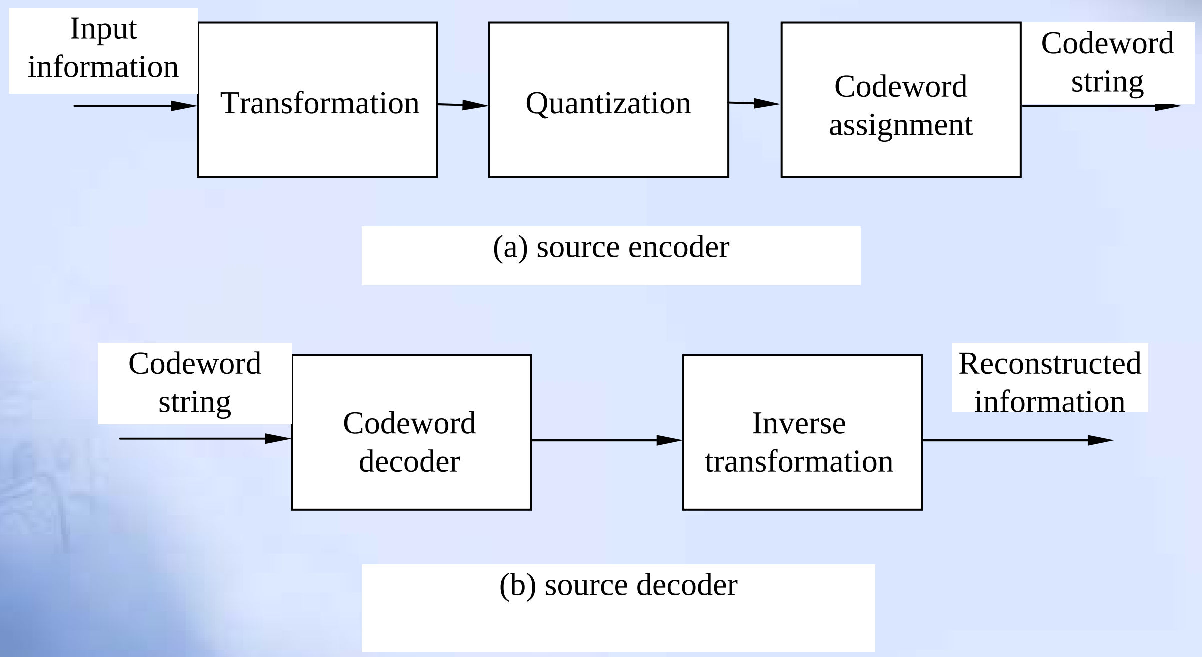
Block diagram of a visual communication system

Quantization and Source Encoder



Block diagram of a visual storage system

Quantization and Source Encoder



Block diagram of a source encoder and decoder

Quantization and Source Encoder

- Quantization:
  - An irreversible process
  - A source of information loss
  - A critical stage in image and video compression
    - It has significant impact on
      - The distortion of reconstructed image and video
      - The bit rate of the compressed bitstream

Uniform Quantizer

- Simplest
- Most popular
- Conceptually of great importance

Uniform Quantizer -- Basics

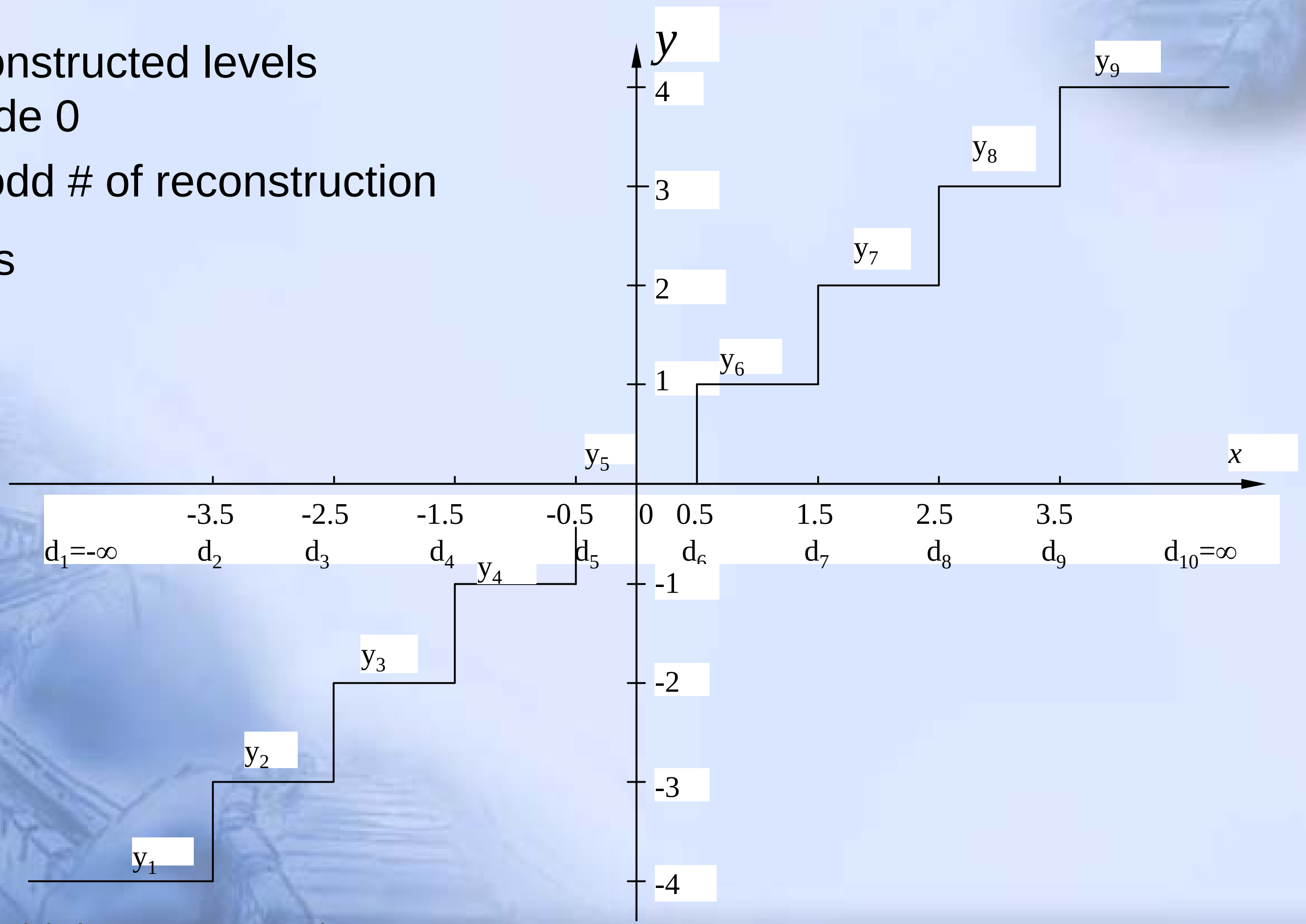
- Definitions
  - The **input-output characteristic** of the quantizer
    - Stair-case like
    - Non-linear

$$y_i = Q(x) \quad if \quad x \in (d_i, d_{i+1}),$$

Where y_i and $Q(x)$ is the output of the quantizer with respect to the input x

Uniform Quantizer – Basics (midtread quantizer)

- Reconstructed levels include 0
- For odd # of reconstruction levels



Uniform Quantizer -- Basics

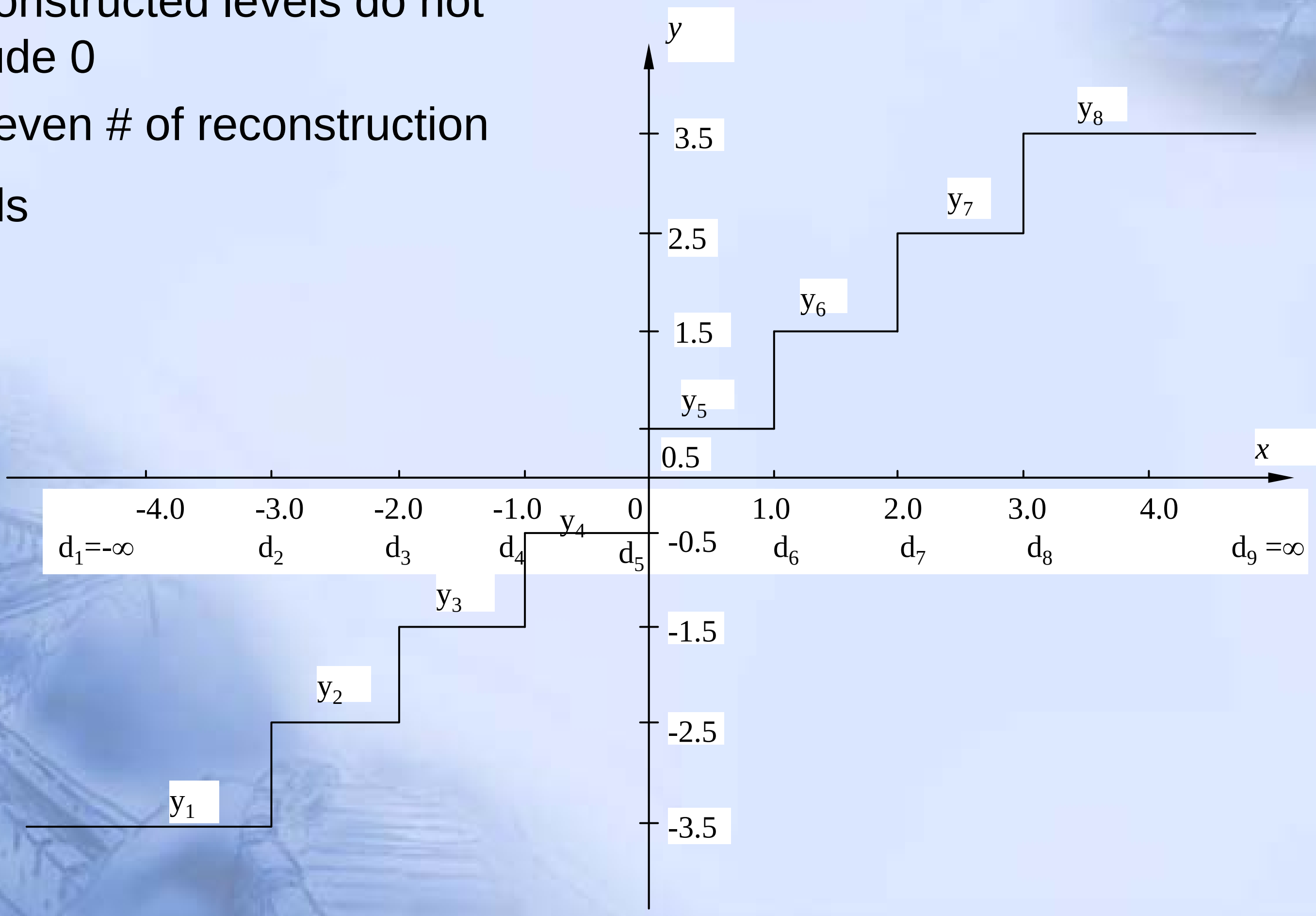
- **Decision levels**
  - **The end points of the intervals, denoted by** d_i **where** ***i*** **: index of intervals**
- **Reconstruction level**
  - **The output of the quantization, denoted by** y_i
- **Step size of the quantizer**
  - **The length of the interval, denoted by** Δ

Uniform Quantizer -- Basics

- **Two features of a uniform quantizer**
  - Except possibly the right-most and left-most intervals, all intervals along the x-axis are uniformly spaced
  - Except possibly the outer intervals, the reconstruction levels of the quantizer are also uniformly spaced
- Furthermore, each inner reconstruction level is the *arithmetic average* of the two decision levels of the corresponding interval

Uniform Quantizer – Basics
(midrise quantizer)

- Reconstructed levels do not include 0
- For even # of reconstruction levels



Uniform Quantizer -- Basics

- **WLOG, assume both input-output characteristics of the midtread and midrise uniform quantizers are odd symmetric with respect to the vertical axis *x=0***
  - **Subtraction of statistical mean of input *x***
  - **Addition of statistical mean back after quantization**
  - ***N*: the total number of reconstruction levels of a quantizer**

Quantization Distortion

- In terms of objective evaluation, we define quantization error e_q

$$e_q = x - Q(x),$$

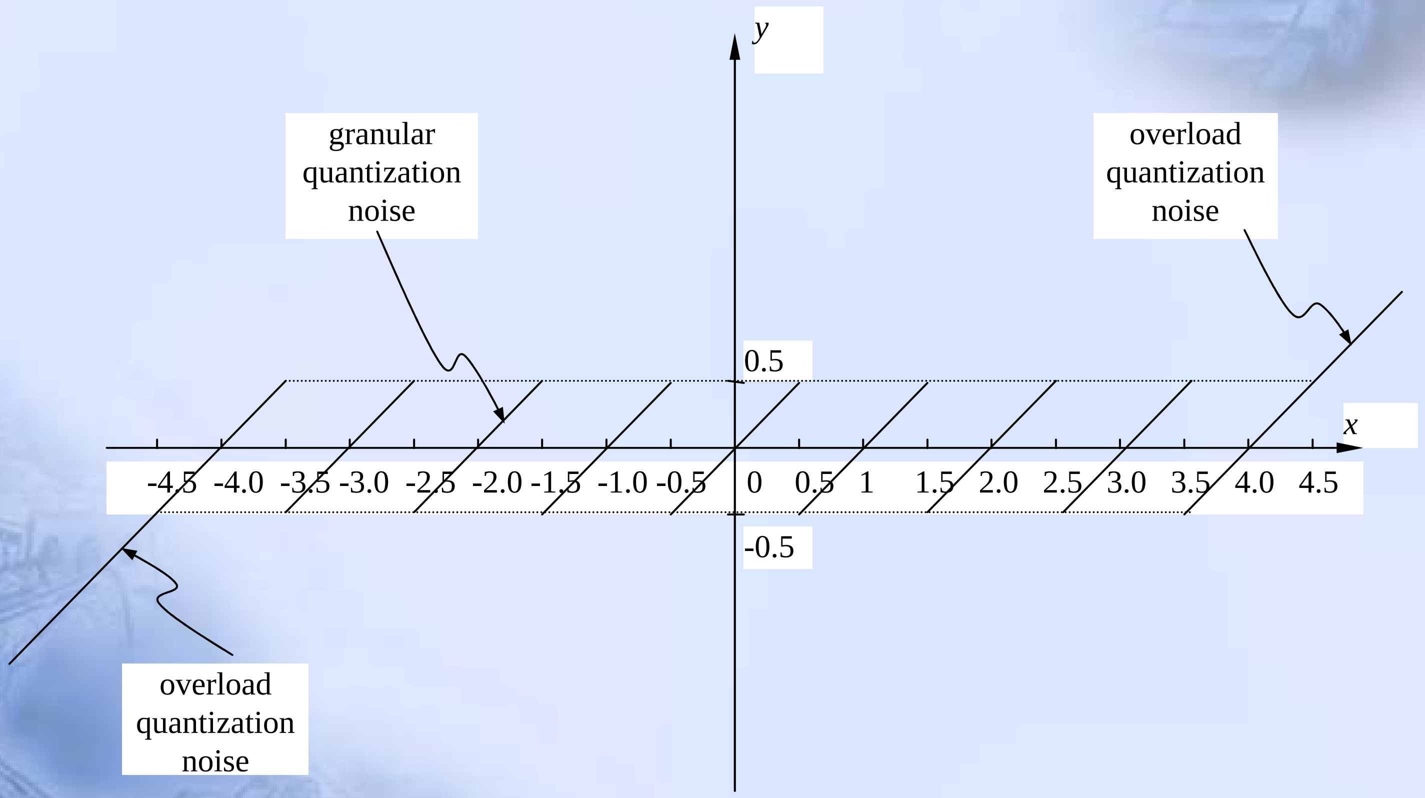
- Quantization error is often referred to as quantization noise
- Mean square quantization error $\boldsymbol{MSE_q}$

$$MSE_q = \sum_{i=1}^{N} \int_{d_i}^{d_{i+1}} (x - Q(x))^2 f_X(x)dx$$

Quantization Distortion

- $f_x(x)$: probability density function (*pdf*)
  - The outer decision levels may be $-\infty$ or ∞
  - When the *pdf* $f_x(x)$ remains unchanged, fewer reconstruction levels (smaller N, coarser quantization) result in more distortion.
  - In general, the mean of e_q is not zero. It is zero when the input x has a uniform distribution. In this case, $\boldsymbol{MSE_q}$ is the variance of the quantization noise e_q.

Quantization Distortion



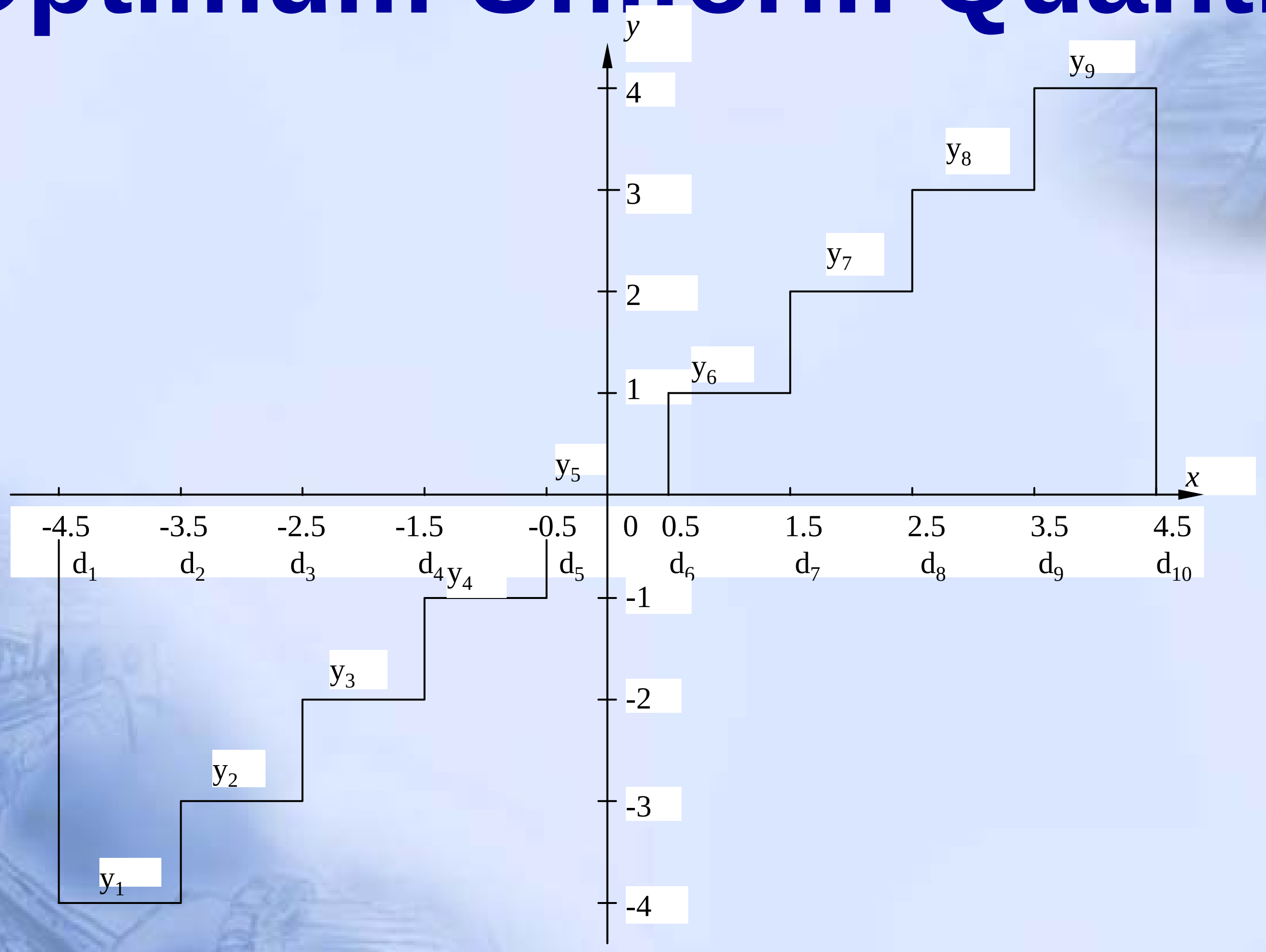
Quantizer Design

- The design of a quantizer (uniform/nonuniform)
  - Choosing the # of reconstruction levels, *N*
  - Selecting the values of decision levels and reconstruction levels
- The design of a quantizer is equivalent to specifying its input-output characteristic
- Optimum quantizer design
  - For a given probability density function of the input random variable, $f_X(x)$, design a quantizer such that the mean square quantization error, MSE_q, is minimized.

Quantizer Design

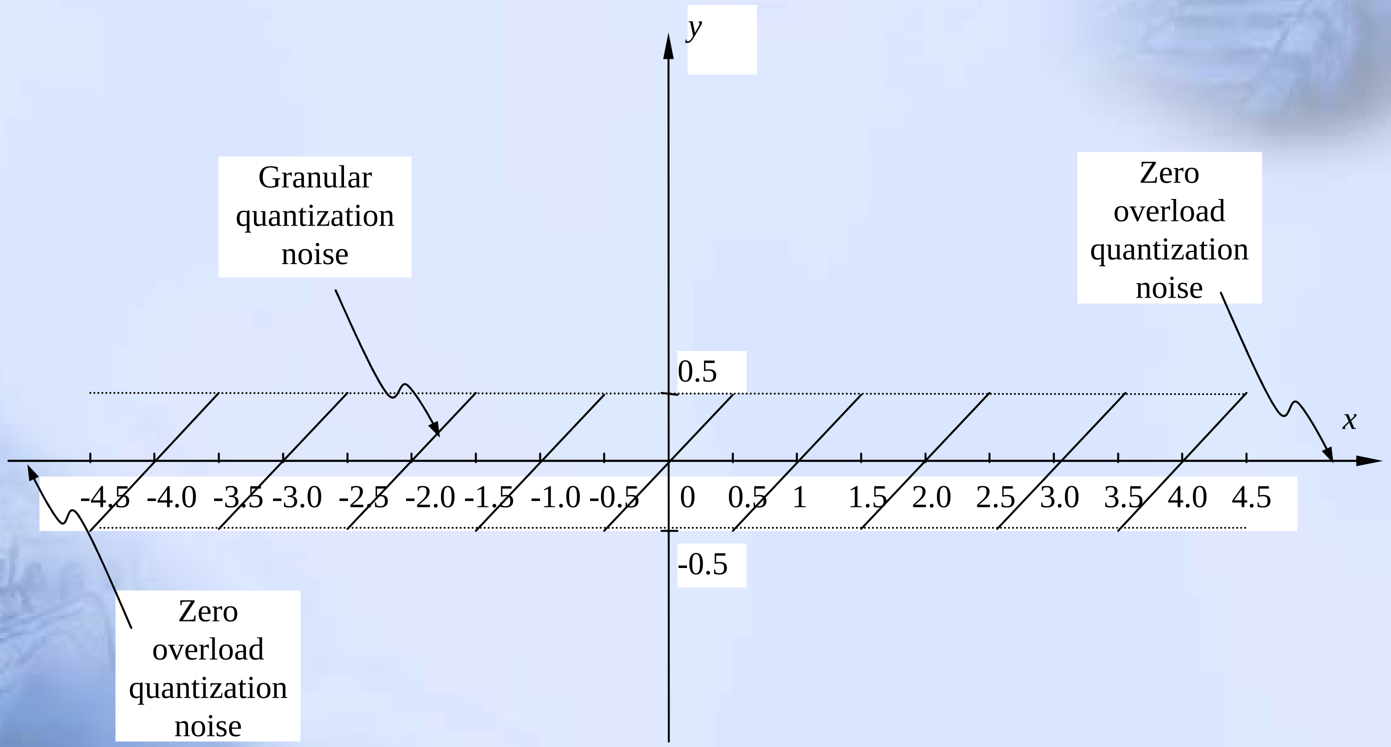
- For uniform quantizer design
  - *N* is usually given.
  - According to the two features of uniform quantizers
    - Only one parameter that needs to be decided: the step size Δ
    - As to the optimum uniform quantizer design, a different *pdf* leads to a different step size

Optimum Uniform Quantizer



Uniform quantizer with uniformly distributed input

Optimum Uniform Quantizer



Quantization distortion

Optimum Uniform Quantizer

- The mean square quantization error

$$MSE_q = N \int_{d_1}^{d_2} (x - Q(x))^2 \frac{1}{N\Delta} dx$$

$$MSE_q = \frac{\Delta^2}{12}.$$

$$SNR_{ms} = 10\log_{10} \frac{\sigma_x^2}{\sigma_q^2} = 10\log_{10} N^2.$$

- If we assume $N = 2^n$, we then have

$$SNR_{ms} = 20\log_{10} 2^n = 6.02n \quad dB.$$

Optimum Uniform Quantizer

- The interpretation of the above result
  - If use natural binary code to code the reconstruction levels of a *uniform* quantizer with a *uniformly* distributed input source, then every increased bit in the coding brings out a 6.02 dB increase in the SNR_{ms}
  - Whenever the step size of the uniform quantizer decreases by a half, the MSE_q decreases four times

Quantization Effects

1 bit quantizer

2 bit quantizer

3 bit quantizer

4 bit quantizer

Optimum Uniform Quantizer

- **Conditions of optimum quantization**
  - **Derived by [Lloyd'57, 82; Max'60]**
  - ***Necessary* conditions, for a given *pdf* $f_X(x)$**

$$x_1 = -\infty \qquad x_{N+1} = +\infty$$

$$\int_{d_i}^{d_{i+1}} (x - y_i) f_X(x) dx = 0 \qquad i = 1, 2, \cdots, N$$

Centroid condition

$$d_i = \frac{1}{2}(y_{i-1} + y_i) \qquad i = 2, \cdots, N$$

Nearest neighbor condition

Optimum Uniform Quantizer

- First condition: for an input *x* whose range is $-\infty < x < \infty$
- Second: each reconstruction level is the centroid of the area under the *pdf* and between the two adjacent decision levels
- Third: each decision level (except for the outer intervals) is the arithmetic average of the two neighboring reconstruction levels

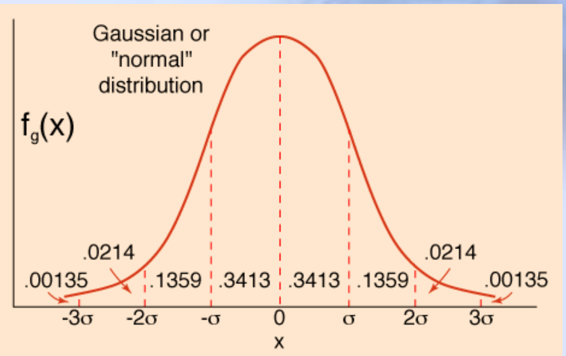
- These conditions are *general* in the sense that there is no restriction imposed on the *pdf*.

Optimum Uniform Quantizer

- **Optimum uniform quantizer with different input distributions**
  - A uniform quantizer is optimum when the input has uniform distribution
  - Normally, if the *pdf* is not uniform, the optimum quantizer is not a uniform quantizer
  - Due to the simplicity of uniform quantization, however, it may sometimes be desirable to design an optimum *uniform* quantizer for an input with a *nonuniform* distribution.

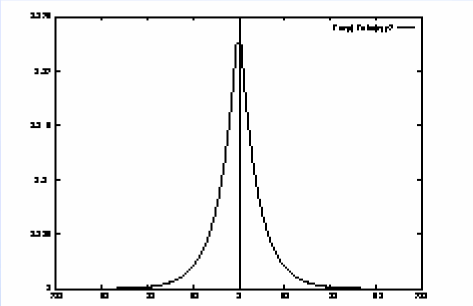
Some Typical Distributions

Distribution	Functional Form	Mean	Standard Deviation
Gaussian	$f_g(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-a)^2}{2\sigma^2}}$	a	σ



Laplacian

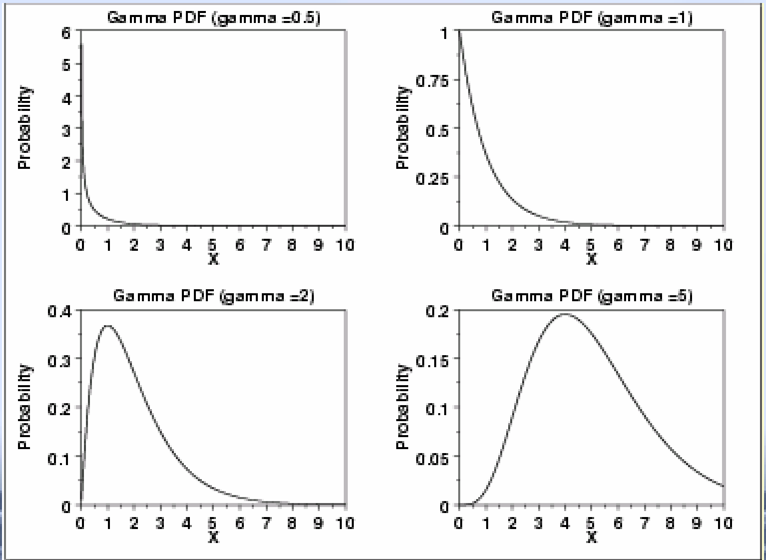
$$p(x) = \frac{\lambda}{2} e^{-\lambda|x|}$$



Gamma

$$f(x) = \frac{\left(\frac{x-\mu}{\beta}\right)^{\gamma-1} \exp\left(-\frac{x-\mu}{\beta}\right)}{\beta\Gamma(\gamma)} \qquad x \geq \mu; \gamma, \beta > 0$$

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$$



Optimum Uniform Quantizer

- Optimal symmetric *uniform* quantizer for Gaussian, Laplacian and Gamma distribution (with zero mean and unit variance). [Max’60][Paez’72]. The numbers enclosed in rectangular are the step sizes.

	Uniform			Gaussian			Laplacian			Gamma		
N	d_i	y_i	MSE	d_i	y_i	MSE	d_i	y_i	MSE	d_i	y_i	MSE
2	-1.000 0.000 1.000	-0.500 0.500	8.33 $\times 10^{-2}$	-1.596 0.000 1.596	-0.798 0.798	0.363	-1.414 0.000 1.414	-0.707 0.707	0.500	-1.154 0.000 1.154	-0.577 0.577	0.668
4	-1.000 -0.500 0.000 0.500 1.000	-0.750 -0.250 0.250 0.750	2.08 $\times 10^{-2}$	-1.991 -0.996 0.000 0.996 1.991	-1.494 -0.498 0.498 1.494	0.119	-2.174 -1.087 0.000 1.087 2.174	-1.631 -0.544 0.544 1.631	1.963 $\times 10^{-1}$	-2.120 -1.060 0.000 1.060 2.120	-1.590 -0.530 0.500 1.590	0.320
8	-1.000 -0.750 -0.500 -0.250 0.000 0.250 0.500 0.750 1.000	-0.875 -0.625 -0.375 -0.125 0.125 0.375 0.625 0.875	5.21 $\times 10^{-3}$	-2.344 -1.758 -1.172 -0.586 0.000 0.586 1.172 1.758 2.344	-2.051 -1.465 -0.879 -0.293 0.293 0.879 1.465 2.051	3.74 $\times 10^{-2}$	-2.924 -2.193 -1.462 -0.731 0.000 0.731 1.462 2.193 2.924	-2.559 -1.828 -1.097 -0.366 0.366 1.097 1.828 2.559	7.17 $\times 10^{-2}$	-3.184 -2.388 -1.592 -0.796 0.000 0.796 1.592 2.388 3.184	-2.786 -1.990 -1.194 -0.398 0.398 1.194 1.990 2.786	0.132

Optimum Uniform Quantizer

- Under these circumstances, however, three equations are not a set of simultaneous equations one can hope to solve with any ease
- Numerical procedures were suggested for design of optimum uniform quantizers
  - E.g., Newton method
- Max derived uniform quantization step size for an input with a Gaussian distribution [Max '60]

Optimum Uniform Quantizer

- Paez and Glisson found step size for Laplacian and Gamma distributed input signals [Paez’ 72]
- Zero mean: if the mean is not zero, only a shift in input is needed when applying these results
- Unit variance: if the standard deviation is not unit, the tabulated step size needs to be multiplied by the standard deviation.

Nonuniform Quantizer

- In general, an optimal quantizer is a nonuniform quantizer.
  - Depends on statistic (pdf) of input source
- Companding quantization
  - Using uniform quantizer to realize non-uniform quantization
  - Reading: Section 2.3.2
- Adaptive quantization
  - Adapt to changing statistic of input source
  - Reading: Section 2.4
- HW #1: Ex. 2-2