2.8 Trigonometric Fourier Series

- Importance of freq. repres. periodic $g(t)$
- We can Express a signal g by a trignometric Fourier Series with period $= T_0$

$$\boxed{g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t}$$

① where.

$$a_0 = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} g(t)\, dt$$

$$a_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} g(t) \cos n\omega_0 t \, dt \qquad n=1,2,3,\dots$$

$$b_n = \frac{2}{T_0} \int_{t_1}^{t+T_0} g(t) \sin n\omega_0 t \, dt \qquad n=1,2,3,\dots$$

Compact Trigonometric Fourier Series

Since $a_n \cos n\omega_0 t + b_n \sin n\omega_0 t = C_n \cos(n\omega_0 t + \theta_n)$

where $C_n = \sqrt{a_n^2 + b_n^2}$, $\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$

② $C_0 = a_0$

$$\boxed{g(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)}$$

- note about the periodicity of the trigonometric Fourier Series.
  - $\rightarrow$ $g(t)$ has to be periodic or the F.Series is good for $t_1 \le t \le t_1 + T_0$.
- amplitude spetrum C_n vs. ω , phase spectrum θ_n vs. ω, (Both freq. Spectrum)

2.9 Exponential Fourier Series

Using Euler's formula $C_n \cos(n\omega_0 t + \theta_n) = \frac{C_n}{2}\left[e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)}\right]$

we may write

③ $$\boxed{g(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}}$$

$$D_n = \frac{1}{T_0} \int_{T_0} g(t) e^{-jn\omega_0 t} \, dt$$

* Existance of the Fourier Series : Dirichlet conditions.

$$\int_{T_0} |g(t)|\, dt < \infty$$

- Fourier Spectrum. (Exponential Fourier Spectra)

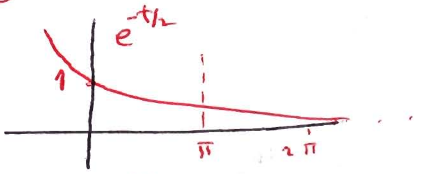
D_n complex we prefer to represent as magnitude & phase.

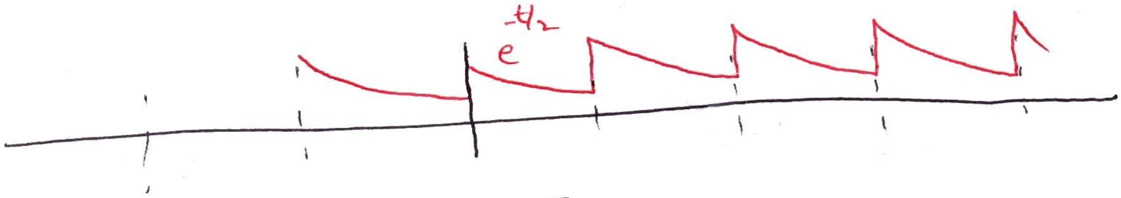
$$\boxed{\begin{array}{lll} |D_n| = |D_{-n}| = \frac{1}{2} C_n & , D_0 = C_0 & \text{even} \\ \angle D_n = \theta_n \text{ and } \angle D_{-n} = -\theta_n & & \text{odd} \end{array}}$$

- we will use Exponential Spectrum.

Example 2.7

Find the compact trigonometric Fourier Series for the exponential $e^{-t/2}$ $g(t)$





$T_0 = \pi$ $\qquad \omega_0 = 2\pi f_0 = \frac{2\pi}{T} = 2$

$$g(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos 2nt + b_n \sin nt)$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} e^{-t/2}\, dt = -\frac{2}{\pi} e^{-t/2} \Big|_0^{\pi} = \frac{-2}{\pi} \left[0.2079 - 1 \right] = 0.504$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} e^{-t/2} \cos 2nt\, dt$$

$$= 0.504 \left(\frac{2}{1 + 16n^2} \right)$$

from p. 774 Appendix D.

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$$

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx)$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} e^{-t/2} \sin 2nt\, dt$$

$$= 0.504 \left(\frac{8n}{1+16n^2} \right)$$

$$\Rightarrow \quad g(t) = 0.504 \left[1 + \sum_{n=1}^{\infty} \frac{2}{1+16n^2} (\cos 2nt + 4n \sin 2nt) \right]$$

The compact form

$C_0 = a_0 = 0.504$

$$C_n = \sqrt{a_n^2 + b_n^2} = 0.504 \sqrt{\frac{4}{(1+16n^2)^2} + \frac{64n^2}{(1+16n^2)^2}} = 0.504 \left(\frac{2}{\sqrt{1+16n^2}} \right)$$

$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n} \right) = \tan^{-1}(-4n) = -\tan^{-1} 4n$$

$$g(t) = 0.504 + 0.504 \sum_{n=1}^{\infty} \frac{2}{\sqrt{1+16n^2}} \cos(2nt - \tan^{-1} 4n)$$

$$= 0.504 + 0.244 \cos(2t - 75.96^\circ) + \cdots$$

n	0	1	2	3	4
C_n	0.504	0.244	0.125	0.084	0.063
θ_n	0	−75.96	−85.24	−86.42	−87.14

look at Exs 2.8 & 2.9

Ex. 2.10

Find the complex Fourier series

$T_0 = \pi$, $\omega_0 = \frac{2\pi}{T_0} = 2$

$$\varphi(t) = \sum_{n=-\infty}^{\infty} D_n e^{j2nt}$$

$$D_n = \frac{1}{T_0} \int_{T_0} \varphi(t) e^{-j2nt} dt = \frac{1}{\pi} \int_0^{\pi} e^{-t/2} e^{-j2nt} dt$$

$$= \frac{1}{\pi} \int_0^{\pi} e^{-(\frac{1}{2} + j2n)t} dt = \frac{-1}{\pi(\frac{1}{2} + j2n)} e^{-(\frac{1}{2} + j2n)t} \Big|_0^{\pi}$$

$$\varphi(t) = 0.504 \sum_{n=-\infty}^{\infty} \frac{1}{1+j4n} e^{j2nt}$$

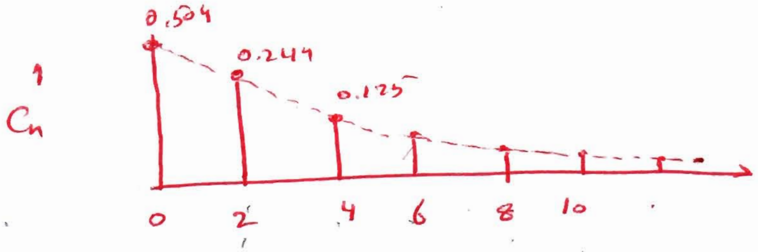
$$= 0.504 \left[1 + \frac{1}{1+j4} e^{j2t} + \cdots \right.$$

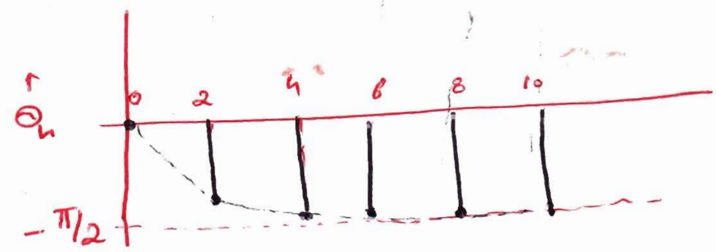
$$\left. + \frac{1}{1-j4} e^{-j2t} + \cdots \right]$$

D_n complex D_n & D_{-n} are complex conjugates

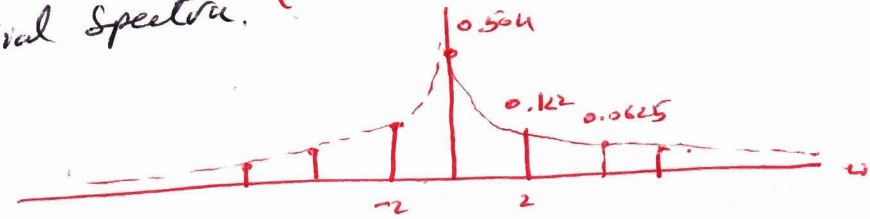
Spectrum.

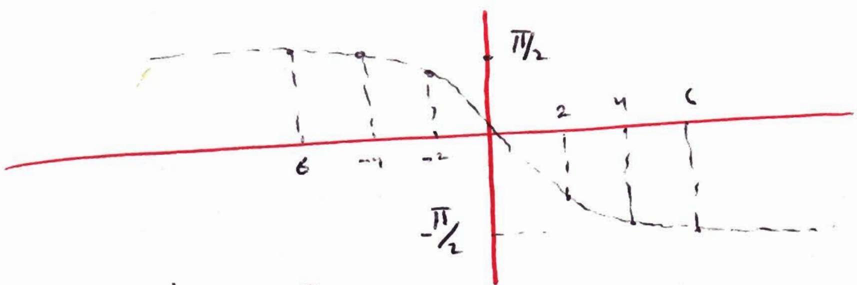
Trigonometric.





Exponential Spectra. (real $g(t)$)





What is a negative frequency?

just a way of representation indicating that there is a component at $(-n)$

Parsavals theorem.

$$g(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

(orthogonal add up).

$$P_g = C_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} C_n^2$$

$$g(t) = D_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} D_n e^{jn\omega_0 t}$$

$$P_g = \sum_{n=-\infty}^{\infty} |D_n|^2 = D_0^2 + 2\sum_{n=1}^{\infty} |D_n|^2$$

for real $g(t)$ $|D_{-n}| = |D_n|$