ARTICLE IN PRESS

Signal Processing ∎ (∎∎∎∎) ∎∎∎–∎∎∎

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

# Convergence and tracking analysis of a variable normalised LMF (XE-NLMF) algorithm

Azzedine Zerguine [a,*], Mun K. Chan [b], Tareq Y. Al-Naffouri [a], Muhammad Moinuddin [a], Colin F.N. Cowan [c]

[a] Electrical Engineering Department, King Fahd University of Petroleum & Minerals, Dhahran 31261, Saudi Arabia
[b] Delphi Automotive Systems, 501 Ang Mo Kio Ind, Park 1, 569621, Singapore
[c] School of Electrical and Electronic Engineering, Queen's University of Belfast, Ashby Building, Stranmillis Road, BT9 5AH Belfast, UK

ARTICLE INFO

*Article history:*
Received 13 February 2008
Received in revised form
13 October 2008
Accepted 22 October 2008

*Keywords:*
LMS
LMF
NLMF
NLMS
Adaptive filtering
Equalisation

ABSTRACT

The least-mean-fourth (LMF) algorithm is known for its fast convergence and lower steady state error, especially in sub-Gaussian noise environments. Recent work on normalised versions of the LMF algorithm has further enhanced its stability and performance in both Gaussian and sub-Gaussian noise environments. For example, the recently developed normalised LMF (XE-NLMF) algorithm is normalised by the mixed signal and error powers, and weighted by a fixed mixed-power parameter. Unfortunately, this algorithm depends on the selection of this mixing parameter. In this work, a time-varying mixed-power parameter technique is introduced to overcome this dependency. A convergence analysis, transient analysis, and steady-state behaviour of the proposed algorithm are derived and verified through simulations. An enhancement in performance is obtained through the use of this technique in two different scenarios. Moreover, the tracking analysis of the proposed algorithm is carried out in the presence of two sources of nonstationarities: (1) carrier frequency offset between transmitter and receiver and (2) random variations in the environment. Close agreement between analysis and simulation results is obtained. The results show that, unlike in the stationary case, the steady-state excess mean-square error is not a monotonically increasing function of the step size.



## 1. Introduction

The least-mean-fourth (LMF) algorithm [1] belongs to the class of stochastic gradient descent algorithms, similar to the least-mean-square (LMS) algorithm [2]. The power of LMF lies in its faster initial convergence and lower steady-state error relative to the LMS algorithm. More importantly, its mean-fourth error cost function yields better performance than that of the LMS for noise of sub-Gaussian nature [1], or noise with light-tailed probability density function [3]. However, this higher-order algorithm requires a much smaller step size to ensure stable adaptation [4,5], since the error power 3 in the LMF gradient vector can cause devastating initial instability, resulting in unnecessary performance degradation. The solution proposed here is to normalise the step size in a similar manner as done for the algorithms developed in [6,7]. Although the two normalisation techniques of [6] and [7] are quite similar, the latter (i.e., the XE-NLMF, where XE refers to the use of both input X and error E in the normalisation) offers more flexibility and eventually more improvement in performance than the former.

* Corresponding author. Tel.: +966 3 860 4430; fax: +966 3 860 3535.
*E-mail addresses:* azzedine@kfupm.edu.sa (A. Zerguine), mun.keat.chan@delphi.com (M.K. Chan), naffouri@kfupm.edu.sa (T.Y. Al-Naffouri), moinudin@kfupm.edu.sa (M. Moinuddin), cfn.cowan@ee.qub.ac.uk (C.F.N. Cowan).

0165-1684/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.sigpro.2008.10.022

The recursive equation for the XE-NLMF algorithm is defined as follows [7]:

$$\mathbf{w}_{k+1} = \mathbf{w}_k + \frac{\gamma_{xe} e_k^3 \mathbf{x}_k}{\delta + (1-\alpha)\|\mathbf{x}_k\|^2 + \alpha\|\mathbf{e}_k\|^2}, \tag{1}$$

where $\gamma_{xe}$ represents the step size, the error signal is

$$e_k = d_k - \mathbf{x}_k^{\mathrm{T}} \mathbf{w}_k, \tag{2}$$

the desired response is defined by

$$d_k = \mathbf{x}_k^{\mathrm{T}} \mathbf{w}_{\mathrm{opt}} + \eta_k, \tag{3}$$

$\mathbf{w}_k = [w_0, w_1, \ldots, w_{N-1}]^{\mathrm{T}}$ is the filter coefficient vector of the adaptive filter (with $\mathbf{w}_{\mathrm{opt}}$ its optimal value), $\eta_k$ is the additive noise, $\mathbf{x}_k = [x_k, x_{k-1}, \ldots, x_{k-N+1}]^{\mathrm{T}}$ is the input vector, $\mathbf{e}_k = [e_k, e_{k-1}, \ldots, e_{k-N+1}]^{\mathrm{T}}$ is the error vector, $N$ is the length of the filter, $\alpha$ is the mixing power parameter, and the notation $\|\mathbf{x}_k\|^2$ denotes the squared Euclidean norm of a vector, i.e., $\|\mathbf{x}_k\|^2 = \mathbf{x}_k^{\mathrm{T}}\mathbf{x}_k$.

As shown in (1), the LMF is normalised by both the signal power and error power, which are balanced by $\alpha$. Combining signal power and error power has the advantage that the former normalises the input signal, while the latter can dampen down the outlier estimation errors, thus improving stability while still retaining fast convergence.

This work is an extension of the XE-NLMF algorithm [7]. Instead of a fixed value $\alpha$, a variable $\alpha_k$ is proposed. The value of this mixed-power parameter will compromise between fast convergence and lower steady-state error. Therefore, incorporating a variable value for the mixed-power parameter is prudent and desirable for better adaptive performance. This algorithm finds advantageous applications in environments with highly dynamic channels. The time variation of the mixing parameter allows the algorithm to follow changes in the channel, which is not the case for the same algorithm with a fixed mixing parameter.

The paper is organised as follows. Following the Introduction, Section 2 introduces the variable-mixing XE-NLMF algorithm as well as its convergence analysis. Section 3 treats the transient analysis of the variable-mixing XE-NLMF algorithm, while in Section 4 a derivation of the steady-state excess mean-square error (EMSE) is performed. Section 5 deals with a tracking analysis of the variable-mixing algorithm. Here, it should be noted that the energy conservation principle [8] is used to carry out the different analyses of the proposed algorithm. Section 6 presents and discusses several simulation results on the performance of the variable XE-NLMF algorithm in stationary and non-stationary environments, which substantiates the theoretical predictions. Finally, Conclusions are given in Section 7.

## 2. Variable XE-NLMF algorithm

### 2.1. Algorithm development

The mixing-power parameter, $\alpha_k$, is confined to the interval [0,1] and will be recursively adapted to weight the signal power, $\|\mathbf{x}_k\|^2$, and error power, $\|\mathbf{e}_k\|^2$, for maximum performance. Here, we propose a 1-sample correlation of the error feedback quantity, $\mu_k$, updated according to a variable-step-size-parameter like in [9]

$$\mu_{k+1} = \nu\mu_k + p_k |e_k e_{k-1}|, \tag{4}$$

where the quantity $e_k e_{k-1}$ determines the distance from $\mathbf{w}_k$ to the optimum weight, $\nu$ is a constant, $|\cdot|$ denotes the absolute value, and the quantity $p_k$ is updated according to a weighted sum of the past three samples of $\alpha_k$ in the following way:

$$p_k = a[\alpha_{k-2} + \alpha_{k-1} + \alpha_k], \tag{5}$$

where $a$ is a constant. With this averaging, the recursion curve of $\mu_k$ can be more flexibly controlled. The estimation of the autocorrelation between $e_k$ and $e_{k-1}$ is then used to guide the value of $\alpha_k$ as follows:

$$\alpha_k = \mathrm{erf}(\mu_k), \tag{6}$$

where $\mathrm{erf}(r) = (2/\sqrt{\pi})\int_0^r e^{-y^2}\,dy$ is the error function, with the purpose to confine $\alpha_k$ to the interval [0,1]. Similarly, the parameters $\nu$ and $p$ are restricted to the interval [0,1]. Moreover, to avoid zero in the feedback loop, the initial value of $p_k$ is set at $p_0 = 0.5$. The steady-state mean behaviour of the mixing parameter $\alpha_k$, that is, $E[\alpha_\infty]$ is derived in Appendix A.

This scheme provides an automatic adjustment of $\alpha_k$ according to the estimation of the autocorrelation between $e_k$ and $e_{k-1}$. While the estimation error is large, $\alpha_k$ will approach unity, thus providing fast adaptation. When the error is small, $\alpha_k$ is adjusted to a smaller value for a lower steady-state error. Based on this motivation, the proposed variable XE-NLMF algorithm is expressed as follows:

$$\mathbf{w}_{k+1} = \mathbf{w}_k + \frac{\gamma_{xe} e_k^3 \mathbf{x}_k}{\delta + (1-\alpha_k)\|\mathbf{x}_k\|^2 + \alpha_k\|\mathbf{e}_k\|^2}, \tag{7}$$

where the only difference from (1) is that $\alpha_k$ is now signal dependent.

A convergence analysis of the proposed algorithm will be studied using the weight-error vector defined as follows:

$$\mathbf{v}_k = \mathbf{w}_{\mathrm{opt}} - \mathbf{w}_k. \tag{8}$$

In this regard, the error $e_k$ can be set up in the following way:

$$e_k = \eta_k + \mathbf{x}_k^{\mathrm{T}} \mathbf{v}_k. \tag{9}$$

### 2.2. Mean behaviour of the weight vector

In the ensuing analysis, the following assumptions are used in the convergence analysis of the variable XE-NLMF algorithm. These are quite similar to what is usually assumed in the literature [1,2,5,10] and which can also be justified in several practical instances:

(A1) The noise $\eta_k$ is independent of the input signal $\mathbf{x}_k$, both of which are zero mean, with $x_k$ having variance $\sigma_x^2$ and $\eta_k$ having zero odd moments.
(A2) The step size is small enough for the independence assumption [2] to be valid. As a consequence, the weight-error vector is independent of the input $\mathbf{x}_k$.

Please cite this article as: A. Zerguine, et al., Convergence and tracking analysis of a variable normalised LMF (XE-NLMF)..., Signal Process. (2008), doi:10.1016/j.sigpro.2008.10.022

(A3) The mixing parameter $\alpha_k$ is independent of $e_k$, $\|\mathbf{e}_k\|^2$, and $\|\mathbf{x}_k\|^2$.

The independence assumption [2] is very common in the literature and is justified in several practical instances. The assumption of small step size is not necessarily true in practice but has been commonly used to simplify the analysis [2]. Note that since $\alpha_k$ and $\mathbf{w}_k$ are functions of $\{x_n, \eta_n : n \leqslant k\}$, they will, in general, be dependent. However, when parameters are chosen so that the steady-state variance of $\alpha_k$ and/or $\mathbf{w}_k$ is small, then A3 can be approximately satisfied.

The following observation, which is directly obtained from the various assumptions given above, will also be used in the ensuing derivations.

*Observation 1:* It is straightforward to show that

$$E[e_k^2] = \zeta_{\min} + \mathrm{tr}[\mathbf{R}\mathbf{G}_k], \tag{10}$$

where $\zeta_{\min}$, $\mathbf{G}_k = E[\mathbf{v}_k\mathbf{v}_k^{\mathrm{T}}]$, $\mathbf{R} = E[\mathbf{x}_k\mathbf{x}_k^{\mathrm{T}}]$, and tr[ ] are, respectively, the minimum mean-square error (MSE), the second moment matrix of the misalignment vector, the input correlation matrix, and the matrix trace operator. The EMSE, denoted by $\zeta_k$, is given by

$$\begin{aligned}\zeta_k &= E[e_k^2] - \zeta_{\min} \\ &= \mathrm{tr}[\mathbf{R}\mathbf{G}_k].\end{aligned} \tag{11}$$

Taking the expectation on both sides of (7), under A1–A3, the mean weight-error vector of the variable XE-NLMF algorithm evolves as

$$E[\mathbf{v}_{k+1}] = E[\mathbf{v}_k] - \gamma_{xe} E\left[\frac{e_k^3\mathbf{x}_k}{\delta + (1-\alpha_k)\|\mathbf{x}_k\|^2 + \alpha_k\|\mathbf{e}_k\|^2}\right]. \tag{12}$$

Now, considering the second expectation in the above equation, it can be shown that

$$\begin{aligned}&E\left[\frac{e_k^3\mathbf{x}_k}{\delta + (1-\alpha_k)\|\mathbf{x}_k\|^2 + \alpha_k\|\mathbf{e}_k\|^2}\right] \\ &\quad = E\left[\frac{e_k^3\mathbf{x}_k}{\delta + (1-\alpha_k)\|\mathbf{x}_k\|^2 + \alpha_k\|\tilde{\mathbf{e}}_k\|^2 + \alpha_k e_k^2}\right],\end{aligned} \tag{13}$$

where $\tilde{\mathbf{e}}_k = [e_{k-1}, e_{k-2}, \ldots, e_{k-N+1}]^{\mathrm{T}}$. Since, all the elements of $\tilde{\mathbf{e}}_k$ have around the same magnitude as $e_k$, we have $e_k^2 \ll \|\tilde{\mathbf{e}}_k\|^2$ and therefore, the term $\alpha_k e_k^2$ can be ignored. This will be especially true when the filter is long enough. Consequently, the independence assumption can be invoked to obtain the following:

$$\begin{aligned}&E\left[\frac{e_k^3\mathbf{x}_k}{\delta + (1-\alpha_k)\|\mathbf{x}_k\|^2 + \alpha_k\|\tilde{\mathbf{e}}_k\|^2}\right] \\ &\quad \approx E[e_k^3\mathbf{x}_k]E\left[\frac{1}{\delta + (1-\alpha_k)\,\mathrm{tr}(\mathbf{R}) + \alpha_k\|\tilde{\mathbf{e}}_k\|^2}\right].\end{aligned} \tag{14}$$

To solve the expectation $E[e_k^3\mathbf{x}_k]$ we use the technique of [11], which does not employ any linearisation of $e_k^3$. As a result, $E[e_k^3\mathbf{x}_k]$ is found to be

$$E[e_k^3\mathbf{x}_k] = 3\,(\sigma_\eta^2 + \zeta_k)\mathbf{R}E[\mathbf{v}_k]. \tag{15}$$

The second expectation on the right side of (14) is evaluated in Appendix B and is given by (93). Ultimately, (12) can be set up in the following form:

$$E[\mathbf{v}_{k+1}] \approx \left\{\mathbf{I} - \frac{3\gamma_{xe}(\sigma_\eta^2 + \zeta_k)}{c_k(l_k-2)}\mathbf{R}\right\}E[\mathbf{v}_k], \tag{16}$$

where expressions for $c_k$ and $l_k$ are derived in Appendix B.

If $\mathscr{C} \leqslant \zeta_k$ is the Cramer–Rao bound associated with the problem of estimating the random quantity $\mathbf{x}_k^{\mathrm{T}}\mathbf{w}_{\mathrm{opt}}$ by using $\mathbf{x}_k^{\mathrm{T}}\mathbf{w}_k$, then after taking into account the fact that the eigenvalues of $\mathbf{R}$ are all real and positive, it follows that a sufficient condition for convergence of the proposed algorithm is that the step-size parameter $\gamma_{xe}$ satisfies

$$0 < \gamma_{xe} < \frac{2c_k(l_k-2)}{3(\sigma_\eta^2 + \mathscr{C})\lambda_{\max}}, \tag{17}$$

where $\lambda_{\max}$ is the largest eigenvalue of $\mathbf{R}$.

In general $\lambda_{\max} < \mathrm{tr}(\mathbf{R})$ [12] and with $\mathrm{tr}(\mathbf{R}) = N\sigma_x^2$, one simplifies further the above expression for the step-size parameter. Therefore, a sufficient condition for convergence in the mean of the variable XE-NLMF algorithm is given by

$$0 < \gamma_{xe} < \frac{2c_k\,(l_k-2)}{3N\sigma_x^2\,(\sigma_\eta^2 + \mathscr{C})}. \tag{18}$$

Two extreme scenarios can be considered here for the value of the mixing parameter $\alpha_k$.

1. *Scenario* 1: When $\alpha_k = 0$ (i.e., in the absence of any mixing), the variable XE-NLMF algorithm reduces to the conventional NLMF algorithm [6], and it can be shown that

   $$c_k(l_k-2) \approx \mathrm{tr}(\mathbf{R}). \tag{19}$$

   Consequently, (18) becomes

   $$0 < \gamma_{xe} < \frac{2}{3(\sigma_\eta^2 + \mathscr{C})}. \tag{20}$$

2. *Scenario* 2: When $\alpha_k = 1$, it can be shown that

   $$\begin{aligned}c_k\,(l_k-2) &= \delta\left[1 + \sum_{k=1}^{N-1}(\zeta_{n-k} + \sigma_\eta^2)\right] \\ &\quad - 2\delta\frac{\sum_{k=1}^{N-1}(\zeta_{n-k} + \sigma_\eta^2)^2}{1 + \sum_{k=1}^{N-1}(\zeta_{n-k} + \sigma_\eta^2)}.\end{aligned} \tag{21}$$

   Consequently, (18) becomes

   $$\begin{aligned}0 < \gamma_{xe} &< \frac{2\delta}{3N\sigma_x^2(\sigma_\eta^2 + \mathscr{C})} \\ &\times \left[1 + \sum_{k=1}^{N-1}(\zeta_{n-k} + \sigma_\eta^2) - 2\frac{\sum_{k=1}^{N-1}\,(\zeta_{n-k} + \sigma_\eta^2)^2}{1 + \sum_{k=1}^{N-1}(\zeta_{n-k} + \sigma_\eta^2)}\right].\end{aligned} \tag{22}$$

   This will result in a very small step-size ($\gamma_{xe}$), and with $\delta$ being very small will eventually make the convergence very slow. This is expected, as the update rule for $\alpha_k$ suggests that its value decreases near steady-state, and consequently the effective variable step size in (7) also reduces.

**Remarks.** 1. It can be seen from (7) that the variable XE-NLMF algorithm uses a normalised mixture of signal

Please cite this article as: A. Zerguine, et al., Convergence and tracking analysis of a variable normalised LMF (XE-NLMF)..., Signal Process. (2008), doi:10.1016/j.sigpro.2008.10.022

and error power. The proposed algorithm can be viewed as a variable-step-size LMF algorithm with time varying step size.

2. The error is usually large during the initial adaptation and gradually decreases toward a minimum. Therefore, the signal power, $\|\mathbf{x}_k\|^2$, will act as a threshold to avoid taking large step sizes when the error converges to a minimum. The combination of $(1-\alpha_k)\|\mathbf{x}_k\|^2$ and $\alpha_k\|\mathbf{e}_k\|^2$ has the advantage of normalising the input signal power and improving stability since $\|\mathbf{e}_k\|^2$ will dampen down the outlier distribution of $e_k^3$ in the recursive updating equation.

3. The bound for the step-size ($\gamma_{xe}$) of the proposed algorithm that guarantees convergence of the mean weight vector, given by (18), shows that the mean weight vector stability depends on the Cramer–Rao bound. Therefore, the convergence of the mean weight vector of the proposed algorithm depends on its mean-square stability. A similar fact was observed in [11] for the LMF algorithm.

## 3. Transient analysis of the variable XE-NLMF algorithm

Transient analysis of adaptive algorithms is important in studying their convergence behaviour and to derive steady-state expressions for different error performance measures, e.g., the EMSE. In this work, a unified approach to the transient analysis of adaptive filters with error nonlinearities is used. This approach does not restrict the regression data to be Gaussian and avoids the need for explicit recursions of the covariance matrix of the weight-error vector. In [8], a general framework for the steady-state analysis of adaptive algorithms is developed, which is based on the concept of energy conservation. It holds for all adaptive algorithms whose recursion is of the form

$$\mathbf{w}_{n+1} = \mathbf{w}_k + \gamma_{xe}\mathbf{x}_k f(e_k), \tag{23}$$

where $\gamma_{xe}$ is the step size and $f(e_k)$ denotes a general scalar function of the output estimation error $e_k$, which, in the case of the variable XE-NLMF algorithm, is

$$f(e_k) = \frac{e_k^3}{\delta + (1-\alpha_k)\|\mathbf{x}_k\|^2 + \alpha_k\|\mathbf{e}_k\|^2}. \tag{24}$$

Also, this approach assumes that the adaptive filter is long enough to justify the following assumptions:

(A4) The residual or *a priori* error $e_{ak}$, to be defined later, can be assumed to be Gaussian.
(A5) The norm of the input regressor ($\|\mathbf{x}_k\|^2$) can be assumed to be uncorrelated with $f^2(e_k)$, which is the square of the error nonlinearity, $f(e_k)$.

### 3.1. Error measures

We are interested in studying the time-evolution and steady-state values of $E[|e_k^2|]$ and $E[\|\mathbf{v}_k\|^2]$, where $\mathbf{v}_k$ is the weight-error vector defined in (8). The steady-state values of $E[|e_k^2|]$ and $E[\|\mathbf{v}_k\|^2]$ represent the MSE and the mean-square deviation performance of the filter, respectively, whereas their time evolutions relate to the learning or transient behaviour of the filter.

Using a symmetric positive definite weighting matrix $\mathbf{A}$, to be specified later, weighted *a priori* and *a posteriori* estimation errors are, respectively, defined as

$$e_{ak}^{\mathbf{A}} = \mathbf{x}_k^{\mathrm{T}}\mathbf{A}\mathbf{v}_k \quad \text{and} \quad e_{pk}^{\mathbf{A}} = \mathbf{x}_k^{\mathrm{T}}\mathbf{A}\mathbf{v}_{k+1}. \tag{25}$$

For the special case when $\mathbf{A}=\mathbf{I}$, the weighted *a priori* and *a posteriori* estimation errors defined above are reduced to standard *a priori* and *a posteriori* estimation errors, respectively, that is

$$e_{ak} = e_{ak}^{\mathbf{I}} = \mathbf{x}_k^{\mathrm{T}}\mathbf{v}_k \quad \text{and} \quad e_{pk} = e_{pk}^{\mathbf{I}} = \mathbf{x}_k^{\mathrm{T}}\mathbf{v}_{k+1}. \tag{26}$$

It can be shown that the estimation error, $e_k$, and the *a priori* error, $e_{ak}$, are related via $e_k = e_{ak} + \eta_k$. Also, (23) and (25) result in the following relation:

$$e_{pk}^{\mathbf{A}} = e_{ak}^{\mathbf{A}} - \|\mathbf{x}_k\|_{\mathbf{A}}^2\gamma_{xe}f(e_k), \tag{27}$$

where the notation $\|\mathbf{x}_k\|_{\mathbf{A}}^2$ denotes the weighted squared Euclidean norm of $\mathbf{x}_k$, i.e., $\|\mathbf{x}_k\|_{\mathbf{A}}^2 = \mathbf{x}_k^{\mathrm{T}}\mathbf{A}\mathbf{x}_k$.

The performance measure in the analysis is the EMSE which is defined in (11). Since $e_{ak} = \mathbf{x}_k^{\mathrm{T}}\mathbf{v}_k$, the EMSE can also be written as follows:

$$\zeta_k = E[|e_{ak}|^2]. \tag{28}$$

### 3.2. Fundamental weighted-energy relation

In this section, the fundamental weighted-energy conservation relation is presented to develop the framework for the transient analysis of the variable XE-NLMF algorithm.

Using (23), the following recursion can be obtained:

$$\mathbf{v}_{k+1} = \mathbf{v}_k - \gamma_{xe}\mathbf{x}_k f(e_k). \tag{29}$$

Thus, by substituting (27) in (29),

$$\mathbf{v}_{k+1} = \mathbf{v}_k - \frac{\mathbf{x}_k}{\|\mathbf{x}_k\|_{\mathbf{A}}^2}[e_{ak}^{\mathbf{A}} - e_{pk}^{\mathbf{A}}]. \tag{30}$$

Now, the fundamental weighted-energy conservation relation can be shown to be

$$\|\mathbf{v}_{k+1}\|_{\mathbf{A}}^2 + \frac{1}{\|\mathbf{x}_k\|_{\mathbf{A}}^2}|e_{ak}^{\mathbf{A}}|^2 = \|\mathbf{v}_k\|_{\mathbf{A}}^2 + \frac{1}{\|\mathbf{x}_k\|_{\mathbf{A}}^2}|e_{pk}^{\mathbf{A}}|^2. \tag{31}$$

This relation shows how the weighted energies of the error quantities evolve in time. It can be shown that different performance measures can be obtained by the proper choice of the weight matrix $\mathbf{A}$, e.g., the EMSE can be recovered when $\mathbf{A}=\mathbf{I}$.

### 3.3. Time evolution of the weighted variance $E[\|\mathbf{v}_k\|_{\mathbf{A}}^2]$

In this section, the time evolution of the weighted variance $E[\|\mathbf{v}_k\|_{\mathbf{A}}^2]$ is derived for the variable XE-NLMF algorithm using the fundamental weighted–energy conservation relation (31). Substituting the expression for *a posteriori* error from (27) in (31) and taking expectation on

Please cite this article as: A. Zerguine, et al., Convergence and tracking analysis of a variable normalised LMF (XE-NLMF)..., Signal Process. (2008), doi:10.1016/j.sigpro.2008.10.022

both sides obtain the following relation:

$$E[\|\mathbf{v}_{k+1}\|^2_{\mathbf{A}}] = E[\|\mathbf{v}_k\|^2_{\mathbf{A}}] - 2\gamma_{xe}E[e^{\mathbf{A}}_{ak}f(e_k)] + \gamma^2_{xe}E[\|\mathbf{x}_k\|^2_{\mathbf{A}}f^2(e_k)]. \tag{32}$$

Now, the two above expectations $E[e^{\mathbf{A}}_{ak}f(e_k)]$ and $E[\|\mathbf{x}_k\|^2_{\mathbf{A}}f^2(e_k)]$ are evaluated. First, the following assumption, which is reasonable for longer filters using central limit arguments, is used:

(A6) For any constant matrix $\mathbf{A}$ and for all $k$, $e_{ak}$ and $e^{\mathbf{A}}_{ak}$ are jointly Gaussian.

This assumption was used in [13] and a similar one was used in [14] to deal directly with the error nonlinearity, avoiding linearisation. Hence, we can simplify the expectation $E[e^{\mathbf{A}}_{ak}e_k]$ using Price's theorem [15, 16] and assumptions A4 and A6 as follows:

$$\begin{aligned} E[e^{\mathbf{A}}_{ak}f(e_k)] &= E[e^{\mathbf{A}}_{ak}f(e_{ak}+\eta_k)] \\ &= E[e^{\mathbf{A}}_{ak}e_{ak}]\frac{E[e_{ak}f(e_k)]}{E[e^2_{ak}]}. \end{aligned} \tag{33}$$

Since $e^{\mathbf{A}}_{ak} = \mathbf{x}^{\mathrm{T}}_k\mathbf{A}\mathbf{v}_k$ and $e_{ak} = \mathbf{x}^{\mathrm{T}}_k\mathbf{I}\mathbf{v}_k$, we can simplify the expectation $E[e^{\mathbf{A}}_{ak}e_{ak}]$ as follows:

$$\begin{aligned} E[e^{\mathbf{A}}_{ak}e_{ak}] &= E[\mathbf{x}^{\mathrm{T}}_k\mathbf{A}\mathbf{v}_k\mathbf{x}^{\mathrm{T}}_k\mathbf{I}\mathbf{v}_k] \\ &= E[\|\mathbf{v}_k\|^2_{\mathbf{A}\mathbf{x}_k\mathbf{x}^{\mathrm{T}}_k\mathbf{I}}] \\ &= E[\|\mathbf{v}_k\|^2_{\mathbf{A}\mathbf{R}}]. \end{aligned} \tag{34}$$

For the case of the variable XE-NLMF algorithm, $E[e_{ak}f(e_k)]$, can be expressed as

$$E[e_{ak}f(e_k)] = E\left[\frac{e_{ak}e^3_k}{\delta + (1-\alpha_k)\|\mathbf{x}_k\|^2 + \alpha_k\|\tilde{\mathbf{e}}_k\|^2 + \alpha_k e^2_k}\right]. \tag{35}$$

As was done in the case of developing (14), similar arguments are used here to reach the following form of (35):

$$E[e_{ak}f(e_k)] \simeq E[e_{ak}e^3_k]E\left[\frac{1}{\delta + (1-\alpha_k)\,\mathrm{tr}(\mathbf{R}) + \alpha_k\|\tilde{\mathbf{e}}_k\|^2}\right]. \tag{36}$$

According to the long-filter assumption A4, $e_{ak}$ is Gaussian. Therefore, the following relation holds:

$$E[e_{ak}e^3_k] = 3E[e^2_{ak}](E[e^2_{ak}] + \sigma^2_\eta). \tag{37}$$

Now, the term $E[e_{ak}f(e_k)]/E[e^2_{ak}]$ simplifies to the following:

$$\frac{E[e_{ak}f(e_k)]}{E[e^2_{ak}]} \simeq 3(\zeta_k + \sigma^2_\eta)E\left[\frac{1}{\delta + (1-\alpha_k)\,\mathrm{tr}(\mathbf{R}) + \alpha_k\|\tilde{\mathbf{e}}_k\|^2}\right]. \tag{38}$$

The last expectation in the above equation is evaluated in Appendix B and is given by (93). Finally, (38) can be shown to be

$$\begin{aligned} \frac{E[e_{ak}f(e_k]}{E[e^2_{ak}]} &\simeq \frac{3(\zeta_k + \sigma^2_\eta)}{c_k\,(l_k - 2)} \\ &\triangleq \mathscr{H}_k, \end{aligned} \tag{39}$$

where $c_k$ and $l_k$ are defined in Appendix B.

Ultimately, (33) can be expressed as

$$E[e^{\mathbf{A}}_{ak}f(e_k)] = E[\|\mathbf{v}_k\|^2_{\mathbf{A}\mathbf{R}}]\mathscr{H}_k. \tag{40}$$

Second, to solve the expectation $E[\|\mathbf{x}_k\|^2_{\mathbf{A}}f^2(e_k)]$, we will resort to assumption A5 which will enable us to split the expectation $E[\|\mathbf{x}_k\|^2_{\mathbf{A}}f^2(e_k)]$ as follows:

$$E[\|\mathbf{x}_k\|^2_{\mathbf{A}}f^2(e_k)] = E[\|\mathbf{x}_k\|^2_{\mathbf{A}}]\cdot E[f^2(e_k)]. \tag{41}$$

To evaluate $E[f^2(e_k)]$, we have used an approximation similar to the one used to calculate the first expectation, that is, $E[e^{\mathbf{A}}_{ak}f(e_k)]$. Consequently, using the same approach to the one we used to evaluate (14) and (36), it can be shown that

$$E[f^2(e_k)] \simeq E[e^6_k]E\left[\frac{1}{(\delta + (1-\alpha_k)\,\mathrm{tr}(\mathbf{R}) + \alpha_k\|\tilde{\mathbf{e}}_k\|^2)^2}\right]. \tag{42}$$

The last expectation in the above equation is solved in Appendix B and is given by (94). Finally, (42) can be shown to be

$$E[f^2(e_k)] \simeq \frac{E[e^6_k]}{c^2_k(l_k-2)(l_k-4)}, \tag{43}$$

where

$$E[e^6_k] = \begin{cases} 15\;(\zeta_k + \sigma^2_\eta)^3 & \text{if noise is Gaussian,} \\ 15\zeta^3_k + 45\sigma^2_\eta\zeta^2_k + 15E[\eta^4_k]\zeta_k + E[\eta^6_k] & \text{otherwise.} \end{cases} \tag{44}$$

Eq. (32) shows the time evolution or transient behaviour of the weighted variance $E[\|\mathbf{v}_k\|^2_{\mathbf{A}}]$ for any constant weight matrix $\mathbf{A}$. As mentioned earlier, different performance measures can be obtained by the proper choice of the weight matrix $\mathbf{A}$.

### 3.4. Constructing the learning curves for the EMSE

The learning curves for the EMSE can be obtained using the fact that $\zeta_k = E[e^2_{ak}] = E[\|\mathbf{v}_k\|^2_{\mathbf{R}}]$. If we choose $\mathbf{A} = \mathbf{I}, \mathbf{R}, \ldots, \mathbf{R}^{N-1}$, a set of relations can be obtained from (32) and is given

$$\begin{cases} E[\|\mathbf{v}_{k+1}\|^2_{\mathbf{I}}] = E[\|\mathbf{v}_k\|^2_{\mathbf{I}}] - \mathscr{G}_kE[\|\mathbf{v}_k\|^2_{\mathbf{R}}] \\ \quad +\gamma^2_{xe}E[\|\mathbf{x}_k\|^2_{\mathbf{I}}]E[f^2(e_k)], \\ E[\|\mathbf{v}_{k+1}\|^2_{\mathbf{R}}] = E[\|\mathbf{v}_k\|^2_{\mathbf{R}}] - \mathscr{G}_kE[\|\mathbf{v}_k\|^2_{\mathbf{R}^2}] \\ \quad +\gamma^2_{xe}E[\|\mathbf{x}_k\|^2_{\mathbf{R}}]E[f^2(e_k)], \\ \vdots \\ E[\|\mathbf{v}_{k+1}\|^2_{\mathbf{R}^{N-1}}] = E[\|\mathbf{v}_k\|^2_{\mathbf{R}^{N-1}}] - \mathscr{G}_kE[\|\mathbf{v}_k\|^2_{\mathbf{R}^N}] \\ \quad +\gamma^2_{xe}E[\|\mathbf{x}_k\|^2_{\mathbf{R}^{N-1}}]E[f^2(e_k)], \end{cases} \tag{45}$$

where

$$\mathscr{G}_k = 2\gamma_{xe}\mathscr{H}_k. \tag{46}$$

Now, using the Cayley–Hamilton theorem, we can write

$$\mathbf{R}^N = -p_0\mathbf{I} - p_1\mathbf{R} - \cdots - p_{N-1}\mathbf{R}^{N-1}, \tag{47}$$

where

$$\begin{aligned} p(x) &\triangleq \det(x\mathbf{I} - \mathbf{R}) \\ &= p_0 + p_1x + \cdots + p_{N-1}x^{N-1} + x^N \end{aligned} \tag{48}$$

is the characteristic polynomial of $\mathbf{R}$. Consequently, the following relation is obtained:

$$E[\|\mathbf{v}_{k+1}\|^2_{\mathbf{R}^{N-1}}] = E[\|\mathbf{v}_k\|^2_{\mathbf{R}^{N-1}}] + (p_0 E[\|\mathbf{v}_k\|^2_{\mathbf{I}}] + p_1 E[\|\mathbf{v}_k\|^2_{\mathbf{R}}] + \cdots + p_{N-1} E[\|\mathbf{v}_k\|^2_{\mathbf{R}^{N-1}}])\mathscr{G}_k + \gamma_{xe}^2 E[\|\mathbf{x}_k\|^2_{\mathbf{R}^{N-1}}]E[f^2(e_k)]. \quad (49)$$

Ultimately, using (45) and (49), the transient behaviour of the variable XE-NLMF algorithm can be shown to be governed by the following:

$$\mathscr{W}_{k+1} = \mathscr{A}_k \mathscr{W}_k + \gamma_{xe}^2 E[f^2(e_k)]\mathscr{Y}, \quad (50)$$

where

$$\mathscr{W}_k = [E[\|\mathbf{v}_k\|^2] E[\|\mathbf{v}_k\|^2_{\mathbf{R}}] \cdots E[\|\mathbf{v}_k\|^2_{\mathbf{R}^{N-1}}]]^{\mathrm{T}}, \quad (51)$$

$$\mathscr{Y} = [E[\|\mathbf{x}_k\|^2] E[\|\mathbf{x}_k\|^2_{\mathbf{R}}] \cdots E[\|\mathbf{x}_k\|^2_{\mathbf{R}^{N-1}}]]^{\mathrm{T}}, \quad (52)$$

and

$$\mathscr{A}_k = \begin{bmatrix} 1 & -\mathscr{G}_k & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\mathscr{G}_k & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\mathscr{G}_k \\ p_0\mathscr{G}_k & p_1\mathscr{G}_k & p_2\mathscr{G}_k & \cdots & p_{N-2}\mathscr{G}_k & 1 + p_{N-1}\mathscr{G}_k \end{bmatrix}. \quad (53)$$

Finally, using (50), we can obtain the time evolution (learning curves) of $E[\|\mathbf{v}_k\|^2]$ and $E[\|\mathbf{v}_k\|^2_{\mathbf{R}}] = E[e^2_{ak}]$, that is, the mean-square deviation and the EMSE, respectively.

### 3.5. Mean-square stability

In this section, a Lyapunov approach is adopted for studying the mean-square stability of the variable XE-NLMF algorithm. Consequently, we provide a nontrivial upper bound on $\gamma_{xe}$ for which $E[\|\mathbf{v}_k\|^2]$ remains uniformly bounded for all $k$.

Starting from (32) with $\mathbf{A} = \mathbf{I}$ and using **A4**, it can be shown that the variable XE-NLMF algorithm will be mean-square stable provided that

$$-2\gamma_{xe} E[e^{\mathbf{A}}_{ak} f(e_k)] + \gamma_{xe}^2 E[\|\mathbf{x}_k\|^2_{\mathbf{A}} f^2(e_k)] \leqslant 0. \quad (54)$$

The above inequality, upon substituting the values of the two expectations ($E[e^{\mathbf{A}}_{ak} f(e_k)]$ and $E[\|\mathbf{x}_k\|^2_{\mathbf{A}} f^2(e_k)]$), will lead us to get the following bound on $\gamma_{xe}$:

$$\gamma_{xe} \leqslant \begin{cases} \dfrac{2\mathscr{C}\mathscr{H}_k c_k^2 (l_k - 2)(l_k - 4)}{15(\mathscr{C} + \sigma_\eta^2)^3 \operatorname{tr}(\mathbf{R})} & \text{if noise is Gaussian,} \\[2ex] \dfrac{2\mathscr{C}\mathscr{H}_k c_k^2 (l_k - 2)(l_k - 4)}{15(\mathscr{C}^3 + 45\sigma_\eta^2\mathscr{C}^2 + 15E[\eta_k^4]\mathscr{C} + E[\eta_k^6]) \operatorname{tr}(\mathbf{R})} & \text{otherwise,} \end{cases} \quad (55)$$

where $\mathscr{C} \leqslant E[e^2_{ak}]$ is the Cramer–Rao bound associated with the problem of estimating the random quantity $\mathbf{x}_k^{\mathrm{T}}\mathbf{w}_{\mathrm{opt}}$ by using $\mathbf{x}_k^{\mathrm{T}}\mathbf{w}_k$.

## 4. Steady-state analysis of the variable XE-NLMF algorithm

The purpose of the steady-state analysis of an adaptive filter is to study the behaviour of steady-state EMSE. Now, we analyse (32) for the limiting case when $k \to \infty$. Assuming that the weight-error vector reaches a steady-state MSE value, i.e.,

$$\lim_{k\to\infty} E[\|\mathbf{v}_{k+1}\|^2_{\mathbf{A}}] = \lim_{k\to\infty} E[\|\mathbf{v}_k\|^2_{\mathbf{A}}]. \quad (56)$$

Consequently, for a unity weight matrix ($\mathbf{A} = \mathbf{I}$), (32) reduces to

$$2\lim_{k\to\infty} E[e^2_{ak}] \lim_{k\to\infty} \mathscr{H}_k = \gamma_{xe} \lim_{k\to\infty} E[\|\mathbf{x}_k\|^2] \lim_{k\to\infty} E[f^2(e_k)]. \quad (57)$$

Now, using the definition of the EMSE given by (28), its steady-state value denoted by $\zeta_\infty$ is found to be

$$2\zeta_\infty \lim_{k\to\infty} \mathscr{H}_k = \gamma_{xe} \operatorname{tr}(\mathbf{R}) \lim_{k\to\infty} E[f^2(e_k)]. \quad (58)$$

The terms $\lim_{k\to\infty}\mathscr{H}_k$ and $\lim_{k\to\infty}E[f^2(e_k)]$ can be obtained from (39) and (43), respectively. Since the value of $\alpha_k$ is very small near steady-state, it can be shown that $c_\infty(l_\infty - 4) \approx \operatorname{tr}(\mathbf{R})$. Ultimately, for Gaussian noise, (58) will look like

$$A\zeta_\infty^3 + B\zeta_\infty^2 + C\zeta_\infty + D = 0, \quad (59)$$

where

$$\begin{aligned} A &= -15\gamma_{xe} \operatorname{tr}(\mathbf{R}), \\ B &= 6\operatorname{tr}(\mathbf{R}) - 45\gamma_{xe} \operatorname{tr}(\mathbf{R})\sigma_\eta^2, \\ C &= 6\operatorname{tr}(\mathbf{R})\sigma_\eta^2 - 45\gamma_{xe} \operatorname{tr}(\mathbf{R})\sigma_\eta^4, \\ D &= -15\gamma_{xe} \operatorname{tr}(\mathbf{R})\sigma_\eta^6. \end{aligned} \quad (60)$$

Since the steady-state value of $\zeta_\infty$ is close to zero, the higher powers of $\zeta_\infty$ can be ignored. Hence, the asymptotic expression for the steady-state EMSE of the variable XE-NLMF algorithm can be shown to be

$$\zeta_\infty \approx \frac{5\gamma_{xe}\sigma_\eta^4}{2 - 15\gamma_{xe}\sigma_\eta^2}. \quad (61)$$

## 5. Tracking analysis of the variable XE-NLMF algorithm

Cyclic and random system nonstationarities are a common impairment in communication systems and especially in applications that involve channel estimation, channel equalisation, and inter-symbol-interference cancellation. Random nonstationarity is present due to variations in channel characteristics, which is true in most cases, particularly in the mobile communication environment [17]. Cyclic system nonstationarities arise in communication systems due to mismatches between the transmitter and receiver carrier frequencies. The ability of adaptive filtering algorithms to track such system variations is not yet fully understood. In this regard, Rupp [18] presents a first-order analysis of the performance of the LMS algorithm in the presence of carrier frequency offset. In [19,20], a general framework for the tracking analysis of adaptive algorithms was developed, which handles both cyclic as well as random system nonstationarities simultaneously. This framework, based on an energy conservation principle [8], holds for all adaptive algorithms whose recursions are of the form given by (23). Before presenting the tracking analysis, we first develop a complex version of the variable XE-NLMF algorithm in the next section as a framework.

### 5.1. Complex version

In this case, the update rule for the quantity $\mu_k$ given by (4) is modified as follows:

$$\mu_{k+1} = \nu\mu_k + p_k|e_k e^*_{k-1}|, \tag{62}$$

where $()^*$ represents the complex conjugate operation. The update rule for $p_k$ and $\alpha_k$ will remain the same, as given by (5) and (6), respectively. The weight update equation for the complex variable XE-NLMF algorithm is now modified to the following:

$$\mathbf{w}_{k+1} = \mathbf{w}_k + \frac{\gamma_{xe}|e_k^2|e_k\mathbf{x}_k^*}{\delta + (1-\alpha_k)\|\mathbf{x}_k\|^2 + \alpha_k\|\mathbf{e}_k\|^2}. \tag{63}$$

### 5.2. System model and performance measure

In this section, a general system model is presented that includes both random and cyclic nonstationarities. To start, consider the noisy measurement $d_k$ that arises in a model of the form

$$d_k = \mathbf{x}_k^{\mathrm{T}}\mathbf{w}_k^{\mathrm{o}}\mathrm{e}^{\mathrm{j}\Omega k} + \eta_k, \tag{64}$$

where $\eta_k$ is the measurement noise and $\mathbf{w}_k^{\mathrm{o}}$ is the unknown system to be tracked. The multiplicative term $\mathrm{e}^{\mathrm{j}\Omega n}$ accounts for a possible frequency offset between the transmitter and the receiver carriers in a digital communication scenario. Furthermore, it is assumed that the unknown system vector $\mathbf{w}_k^{\mathrm{o}}$ is randomly changing according to

$$\mathbf{w}_k^{\mathrm{o}} = \mathbf{w}^{\mathrm{o}} + \mathbf{q}_k, \tag{65}$$

where $\mathbf{w}^{\mathrm{o}}$ is a fixed vector, and $\mathbf{q}_k$ is assumed to be a zero-mean stationary random vector process with positive definite autocorrelation matrix $\mathbf{Q}_k = E[\mathbf{q}_k\mathbf{q}_k^{\mathrm{T}}]$. Moreover, it is also assumed that the sequence $\{\mathbf{q}_k\}$ is mutually independent of the sequences $\{\mathbf{x}_k\}$ and $\{\eta_k\}$. Thus, from the generalised system model given by (64) and (65), it can be seen that the effects of both cyclic and random system nonstationarities are included in this system model.

In the steady-state analysis of adaptive algorithms, an important measure of performance is their steady-state EMSE defined in (11). In the case of tracking, the weight-error vector $\tilde{\mathbf{v}}_k$ is defined this time as

$$\tilde{\mathbf{v}}_k = \mathbf{w}_k^{\mathrm{o}}\mathrm{e}^{\mathrm{j}\Omega k} - \mathbf{w}_k. \tag{66}$$

### 5.3. Fundamental energy conservation relation

Using (23), (65), and (66) the following recursion is obtained:

$$\tilde{\mathbf{v}}_{k+1} = \tilde{\mathbf{v}}_k - \gamma_{xe}\mathbf{x}_k^* f(e_k) + \mathbf{c}_k\mathrm{e}^{\mathrm{j}\Omega k}, \tag{67}$$

where $\mathbf{c}_k$ is defined as

$$\mathbf{c}_k = \mathbf{w}^{\mathrm{o}}(\mathrm{e}^{\mathrm{j}\Omega} - 1) + \mathbf{q}_{k+1}\mathrm{e}^{\mathrm{j}\Omega} - \mathbf{q}_k. \tag{68}$$

Now, let us define the following *a priori* estimation error, $e_{\mathrm{a}k} = \mathbf{x}_k^{\mathrm{T}}\tilde{\mathbf{v}}_k$ and *a posteriori* estimation error, $e_{\mathrm{p}k} = \mathbf{x}_k^{\mathrm{T}}(\tilde{\mathbf{v}}_{k+1} - \mathbf{c}_k\mathrm{e}^{\mathrm{j}\Omega k})$. Then, it is very easy to show that the estimation error and the *a priori* error are related via $e_k = e_{\mathrm{a}k} + \eta_k$. Also, the *a posteriori* error is defined in terms of the *a priori* error as

$$e_{\mathrm{p}k} = e_{\mathrm{a}k} - \frac{\gamma_{xe}}{\hat{\mu}_k}f(e_k), \tag{69}$$

where $\hat{\mu}_k = 1/\|\mathbf{x}_k\|^2$. Substituting (69) into (67) results in the following update relation:

$$\tilde{\mathbf{v}}_{k+1} = \tilde{\mathbf{v}}_k - \hat{\mu}_k\mathbf{x}_k^*[e_{\mathrm{a}k} - e_{\mathrm{p}k}] + \mathbf{c}_k\mathrm{e}^{\mathrm{j}\Omega k}. \tag{70}$$

By evaluating the energies of both sides of the above equation (taking into account that $\hat{\mu}_k\|\mathbf{x}_k\|^2 = 1$), the following relation is obtained:

$$\|\tilde{\mathbf{v}}_{k+1} - \mathbf{c}_k\mathrm{e}^{\mathrm{j}\Omega k}\|^2 + \hat{\mu}_k|e_{\mathrm{a}k}|^2 = \|\tilde{\mathbf{v}}_k\|^2 + \hat{\mu}_k|e_{\mathrm{p}k}|^2. \tag{71}$$

It can be seen that if $\Omega = 0$ (i.e., no frequency offset), the above equation reduces to the basic fundamental energy conservation relation.

### 5.4. Tracking analysis

The energy relation (71) will be used to evaluate the EMSE at steady state. But before starting the analysis, first the following assumption is stated:

(A7) In steady state, the weight-error vector $\tilde{\mathbf{v}}_k$ takes the generic form $\mathbf{z}_k\mathrm{e}^{\mathrm{j}\Omega k}$, with the stationary random process $\mathbf{z}_k$ independent of the frequency offset $\Omega$.

Using (69), assumption A7, and taking expectation of both sides of (71) with the fact that at steady state $E[\tilde{\mathbf{v}}_{k+1}] = E[\tilde{\mathbf{v}}_k]$, the following relation can be obtained:

$$\begin{aligned} E[\hat{\mu}_k\|e_{\mathrm{a}k}\|^2] = {} & 2\mathrm{tr}\{\mathbf{Q}_k\} + \|\mathbf{w}^{\mathrm{o}}\|^2|1-\mathrm{e}^{\mathrm{j}\Omega}|^2 \\ & - 2\mathrm{Re}\{E[\mathbf{q}_k^*(\mathbf{z}_k - \gamma_{xe}\mathbf{x}_k^* f(e_k)\mathrm{e}^{-\mathrm{j}\Omega k})]\} \\ & - 2\mathrm{Re}\{(1-\mathrm{e}^{\mathrm{j}\Omega})^*\mathbf{w}^{\mathrm{o}*} \times E[\mathbf{z}_k - \gamma_{xe}\mathbf{x}_k^* f(e_k)\mathrm{e}^{-\mathrm{j}\Omega k}]\} \\ & + E\left[\hat{\mu}_k|e_{\mathrm{a}k} - \frac{\gamma_{xe}}{\hat{\mu}_k}f(e_k)|^2\right], \end{aligned} \tag{72}$$

which can be used to solve for the steady-state EMSE.

To find the value of $\mathbf{z} = E[\mathbf{z}_k]$, (67) is used, where it is multiplied by the term $\mathrm{e}^{-\mathrm{j}\Omega n}$ and then expectation is taken on both sides to get

$$(1-\mathrm{e}^{\mathrm{j}\Omega})\mathbf{z} = \gamma_{xe}E\left[\mathbf{x}_k^* f(e_k)\mathrm{e}^{-\mathrm{j}\Omega k}\right] + \mathbf{w}^{\mathrm{o}}\,(1-\mathrm{e}^{\mathrm{j}\Omega}). \tag{73}$$

For the variable XE-NLMF algorithm, the function $f(e_k)$ can be approximated by

$$f(e_k) \approx \frac{3e_{\mathrm{a}k}\eta_k^2 + \eta_k^3}{\delta + (1-\alpha_k)\|\mathbf{x}_k\|^2 + \alpha_k\|\mathbf{e}_k\|^2} \tag{74}$$

Please cite this article as: A. Zerguine, et al., Convergence and tracking analysis of a variable normalised LMF (XE-NLMF)..., Signal Process. (2008), doi:10.1016/j.sigpro.2008.10.022

which yields the value of $\mathbf{z}$ at steady state:

$$\mathbf{z} = \left[\mathbf{I} - \frac{3\gamma_{xe}\sigma_\eta^2\mathbf{R}}{(1-e^{j\Omega})c_k(l_k-2)}\right]^{-1}\mathbf{w}^o, \quad (75)$$

where $l_k$ and $c_k$ are defined by (90) and (95), respectively.

It can be observed that $c_\infty(l_\infty - 2) \approx N$. Ultimately, the steady-state EMSE, $\zeta_{\text{tracking}}$, for the variable XE-NLMF algorithm is obtained from (72)

$$\zeta_{\text{tracking}} = \frac{\sigma_\eta^2}{\sigma_\eta^2 - 3\gamma_{xe}\phi_\eta^4}\left[\text{tr}\{\mathbf{Q}_k\mathbf{R}\} + \frac{\gamma_{xe}\phi_\eta^6}{12\sigma_\eta^2} + \frac{\Gamma N}{12\gamma_{xe}\sigma_\eta^2}\right], \quad (76)$$

where $\Gamma$, $\mathbf{X}_1$, and $\mathbf{X}_2$ are defined, respectively, as

$$\Gamma = |1-e^{j\Omega}|^2\text{Re}\{\text{tr}[\|\mathbf{w}^o\|^2(\mathbf{I}-2\mathbf{X}_1\mathbf{X}_2)]\},$$

$$\mathbf{X}_1 = \mathbf{I} - \frac{3\gamma_{xe}\sigma_\eta^2}{N}\mathbf{R},$$

and

$$\mathbf{X}_2 = \left[(1-e^{j\Omega})\mathbf{I} - \frac{3\gamma_{xe}\sigma_\eta^2}{N}\mathbf{R}\right]^{-1}.$$

For a white Gaussian input signal, an approximate expression for the steady-state EMSE of the variable XE-NLMF algorithm is found to be

$$\zeta_{\text{tracking}} = \frac{\sigma_\eta^2}{\sigma_\eta^2 - 3\gamma_{xe}\phi_\eta^4}\left\{\sigma_x^2\text{tr}\{\mathbf{Q}_k\} + \frac{\gamma_{xe}\phi_\eta^6}{12\sigma_\eta^2} + \frac{N\Omega^2\|\mathbf{w}^o\|^2}{12\gamma_{xe}\sigma_\eta^2}\left[1 + \frac{2(N-3\gamma_{xe}\sigma_\eta^2\sigma_x^2)}{3\gamma_{xe}\sigma_\eta^2\sigma_x^2}\right]\right\}. \quad (77)$$

## 6. Simulation results

This section presents the results of some simulation experiments that were carried out to investigate the performance and behavior of the proposed variable XE-NLMF algorithm. The results demonstrate several good properties of the variable XE-NLMF algorithm, and its performance superiority over the XE-NLMF with fixed mixing parameter and NLMS algorithms. The results also confirm the theoretical findings. Finally, the system performance will be evaluated both in a stationary and a non-stationary environment, as discussed below.

Since the performance of the fixed XE-NLMF algorithm is governed by the choice of the right $\alpha$, the following study considers the effect of $\alpha$ on the performance of the fixed XE-NLMF algorithm. The convergence characteristics of different values of $\alpha$ ($0 \leqslant \alpha \leqslant 0.9$) are studied in this part of the simulations and these are shown in Fig. 1. Typically for $\alpha = 0.1$, the convergence curves are close to the NLMF algorithm, which slows down after the initial fast convergence. Lower weight errors are observed for $\alpha = 0.1$ and 0.5, but their convergence speeds are slower than that observed when $\alpha = 0.9$. The performance for $\alpha = 0.5$ is in between that of $\alpha = 0.1$ and 0.9. Therefore, the value of $\alpha = 0.5$ is considered for the rest of this study. The experiment for this scenario is carried out in the same system used in the next section under a signal-to-noise ratio (SNR) of 30 dB, and the noise is additive white Gaussian.

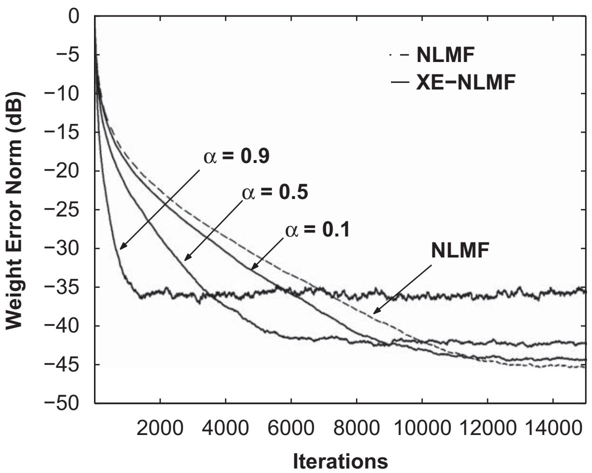


**Fig. 1.** Effect of $\alpha$ on the convergence performance of the fixed XE-NLMF algorithm.

### 6.1. System performance in a stationary environment

In the first experiment, a system identification setup is used to evaluate the algorithm's performance. The unknown system is modelled by a time-invariant finite impulse response filter with $\mathbf{w}_{\text{opt}} = [0.035 \ \ -0.068 \ \ 0.12 \ \ -0.258 \ \ 0.9 \ \ -0.25 \ \ 0.10 \ \ -0.07 \ \ 0.067 \ \ -0.067]^T$. The coloured input signal, $x_k$, is obtained by passing a white Gaussian noise, $u_k$, through a channel defined by $x_k = 0.5x_{k-1} + 0.6u_k$. The SNR is set at 20 dB. Two types of additive noises, white Gaussian noise and binary additive noise (sub-Gaussian), are tested. The performance measure considered here is the normalised weight-error norm, $10\log_{10}(\|\mathbf{w}_{\text{opt}} - \mathbf{w}_k\|^2/\|\mathbf{w}_{\text{opt}}\|^2)$. The results are averaged over 300 independent runs. The step size for the proposed variable XE-NLMF algorithm is $\gamma_{xe} = 0.1$, whereas that of the NLMS is set to 0.2. The parameters $v$ and $a$ are set to 0.98 and 0.9, respectively. The unknown system length is $N = 6$.

Fig. 2 depicts the convergence behaviour of the proposed variable XE-NLMF, the XE-NLMF (with $\alpha = 0.5$), and the NLMS with the same convergence rate in an additive white Gaussian noise (AWGN) environment. As can be seen from this figure, the variable XE-NLMF algorithm adapts faster than the XE-NLMF and NLMS algorithms, and at the same time, produces a lower steady-state weight-error norm of more than 15 dB. This demonstrates the advantages of incorporating a variable mixed-power parameter in the XE-NLMF algorithm.

In Fig. 3, the variable XE-NLMF algorithm converges faster than both the XE-NLMF (with $\alpha = 0.5$) and the NLMS algorithms with a lower-steady state error than the NLMS algorithm in a binary (sub-Gaussian) additive noise environment. Here, the difference of 23 dB in weight-error norm is more apparent than in the case of the AWGN environment. Thus the LMF-based algorithm performs better in sub-Gaussian noise.

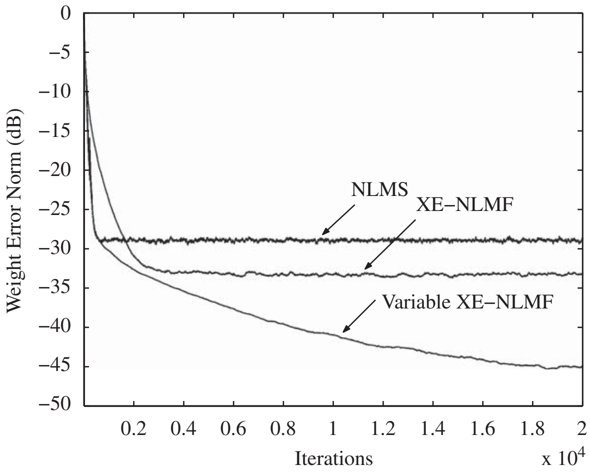


**Fig. 2.** Convergence performance of the proposed variable XE-NLMF algorithm, the fixed XE-NLMF algorithm ($\alpha = 0.5$), and the NLMS algorithm in a AWGN environment.

In the second experiment, a channel equaliser, similar to that of [7], is used to study the performance of the proposed variable XE-LMF algorithm in terms of bit-error rate (BER). The channel is $h(z) = 1 + 0.4z^{-1}$ and the co-channel is $c(z) = 1 + 0.2z^{-1}$. The equaliser length is $N = 8$. The BER in an AWGN and co-channel interference scenario are depicted in Figs. 4 and 5, respectively. As expected, in AWGN, the BER performance of the three algorithms is almost the same. However, the most interesting result is that of Fig. 5 where the variable XE-NLMF algorithm outperforms the NLMS algorithm. A 2 dB improvement over the NLMS algorithm is achieved at a BER of $10^{-6}$. Again the difference in performance between the two algorithms is more apparent in the case of a sub-Gaussian noise environment. Also, as can be seen from this figure, the performance of the XE-NLMF and the NLMF algorithms are almost the same, but both are outperformed by the variable XE-NLMF algorithm.

Finally, Fig. 6 depicts the time evolution of the MSE, namely, $E[\|\mathbf{v}_k\|^2_{\mathbf{R}}] + \zeta_{\min}$, obtained for both the theoretical

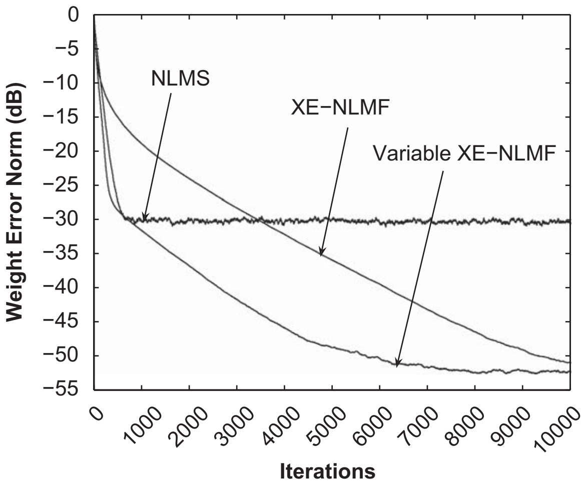


**Fig. 3.** Convergence performance in binary additive noise environment.

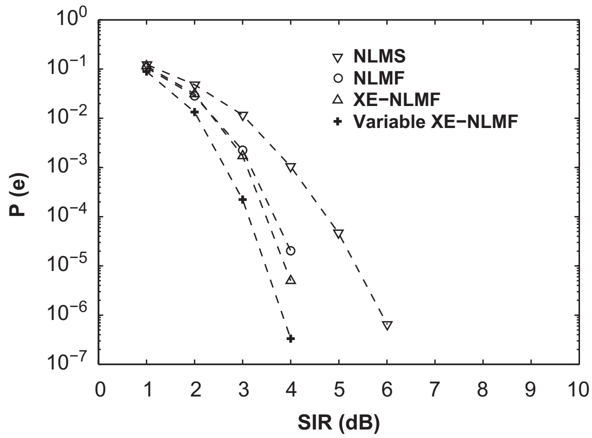


**Fig. 5.** BER performance in a co-channel interference environment.

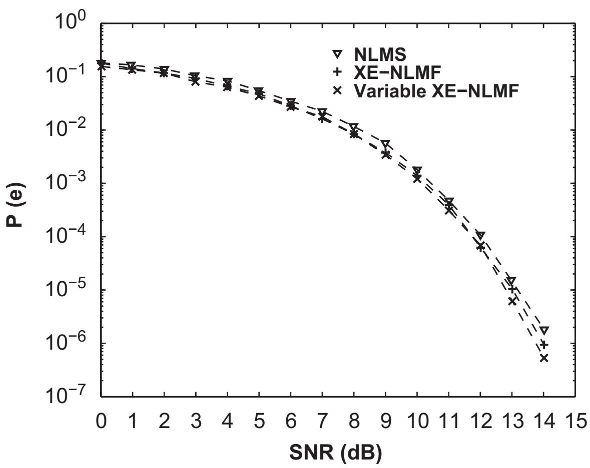


**Fig. 4.** BER performance in AWGN environment.

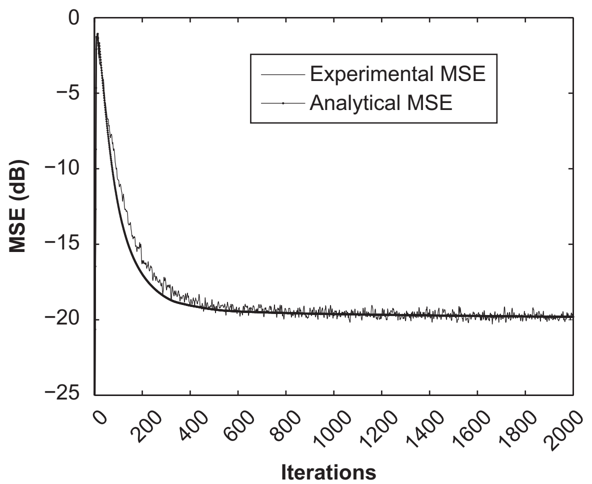


**Fig. 6.** MSE comparison of analytical and simulation results for the variable XE-NLMF algorithm.

Please cite this article as: A. Zerguine, et al., Convergence and tracking analysis of a variable normalised LMF (XE-NLMF)..., Signal Process. (2008), doi:10.1016/j.sigpro.2008.10.022

analysis, i.e., the second entry of (51), and the simulations. Good agreement between the experimental and analytical results is observed. The good matching between these results not only confirms the correctness of the derivations, but also justifies the usefulness of the various simplifying assumptions and approximations used.

### 6.2. System performance in a non-stationary environment

The results presented in this part of the simulation are presented to validate the theoretical findings embodied in (77) for different values of $\Omega$. While the system characteristics are time-varying, the unknown system is given by $[1.0119 - j0.7589,\ -0.3796 + j0.5059]^T$. Experiments are carried out at 10 dB SNR and two values are considered for $\mathrm{tr}\{\mathbf{Q}_k\}$: a very small value of $\mathrm{tr}\{\mathbf{Q}_k\} = 10^{-7}$, and a very large one of $\mathrm{tr}\{\mathbf{Q}_k\} = 10^{-2}$.

Fig. 7 depicts the comparison between both the theoretical and simulation results for three different values of $\Omega$, i.e., 0.01, 0.02, and 0.03. As can be seen from

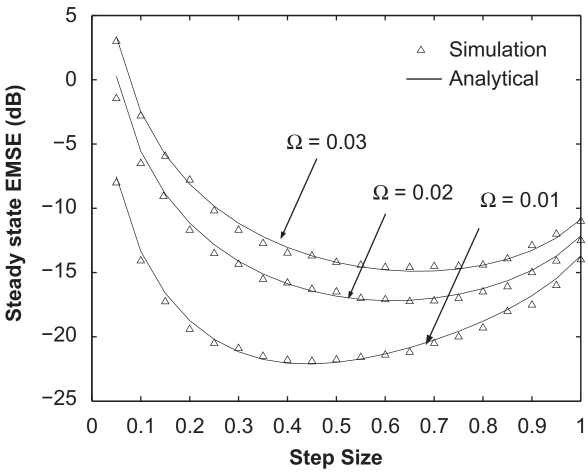


**Fig. 7.** Analytical (–) and experimental (△) steady-state EMSE for $\Omega = 0.01$, 0.02 and 0.03, and $\mathrm{tr}\{\mathbf{Q}_k\} = 10^{-7}$.

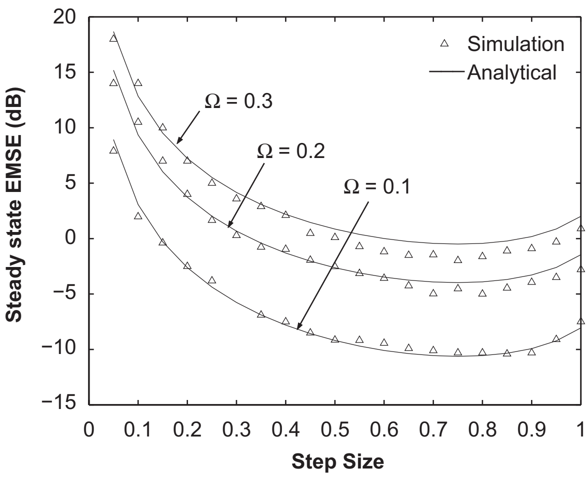


**Fig. 8.** Analytical (–) and experimental (△) steady-state EMSE for $\Omega = 0.1$, 0.2 and 0.3, and $\mathrm{tr}\{\mathbf{Q}_k\} = 10^{-7}$.

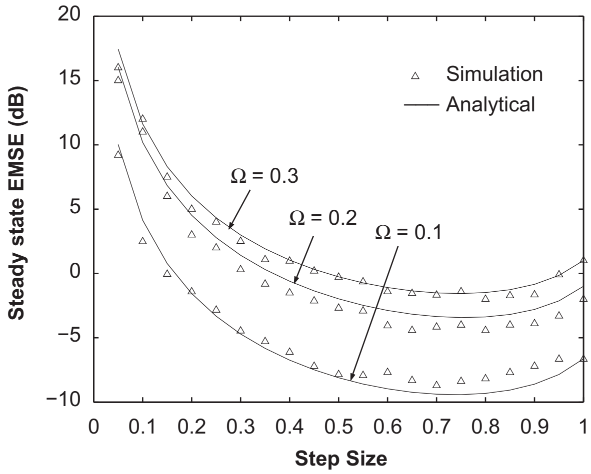


**Fig. 9.** Analytical (–) and experimental (△) steady-state EMSE for $\Omega = 0.1$, 0.2 and 0.3, and $\mathrm{tr}\{\mathbf{Q}_k\} = 10^{-2}$.

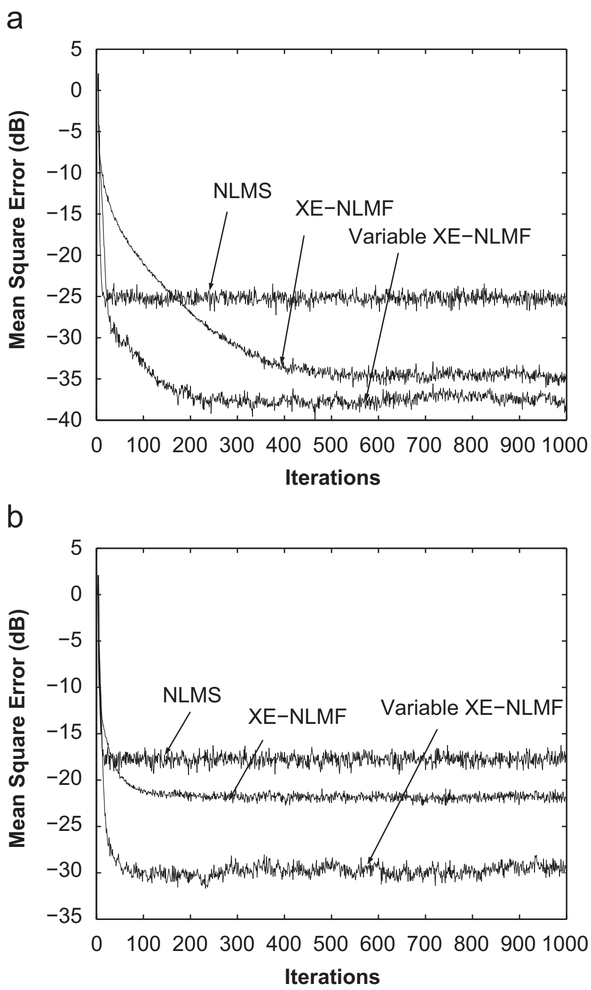


**Fig. 10.** Tracking performance of the proposed variable XE-NLMF algorithm, the fixed XE-NLMF algorithm ($\alpha = 0.5$), and the NLMS algorithm in a AWGN environment for $\mathrm{tr}\{\mathbf{Q}_k\} = 10^{-7}$ (a) $\Omega = 0.001$ and (b) $\Omega = 0.01$.

Please cite this article as: A. Zerguine, et al., Convergence and tracking analysis of a variable normalised LMF (XE-NLMF)..., Signal Process. (2008), doi:10.1016/j.sigpro.2008.10.022

this figure, close agreement between theory and simulation is obtained. This figure shows also that the steady-state EMSE has a minimum value for a certain value of the step-size $\gamma_{xe}$, e.g., for $\Omega = 0.01$, $\gamma_{xe}$ is around 0.43. Moreover, unlike in the stationary case, the steady-state EMSE is not a monotonically increasing function of the step-size $\gamma_{xe}$. Furthermore, it is observed from this figure that degradation in performance is obtained by increasing the frequency offset $\Omega$.

Similar behaviour is observed in Fig. 8 for the case of $\Omega = 0.1$, $\Omega = 0.2$, and $\Omega = 0.3$. As expected in this case too, the steady-state EMSE of the variable XE-NLMF algorithm gets larger for larger values of $\Omega$. More importantly, both theory and simulation are in close agreement in this part of the simulations.

Figs. 7 and 8 are obtained for the case when $\mathrm{tr}\{\mathbf{Q}_k\} = 10^{-7}$, which represents a small value. Increasing this value to $10^{-2}$, the results depicted in Fig. 9 for three different values of $\Omega$, i.e., 0.1, 0.2, and 0.3, still show that the previously stated observations are similar to those obtained for a larger value of $\mathrm{tr}\{\mathbf{Q}_k\}$.

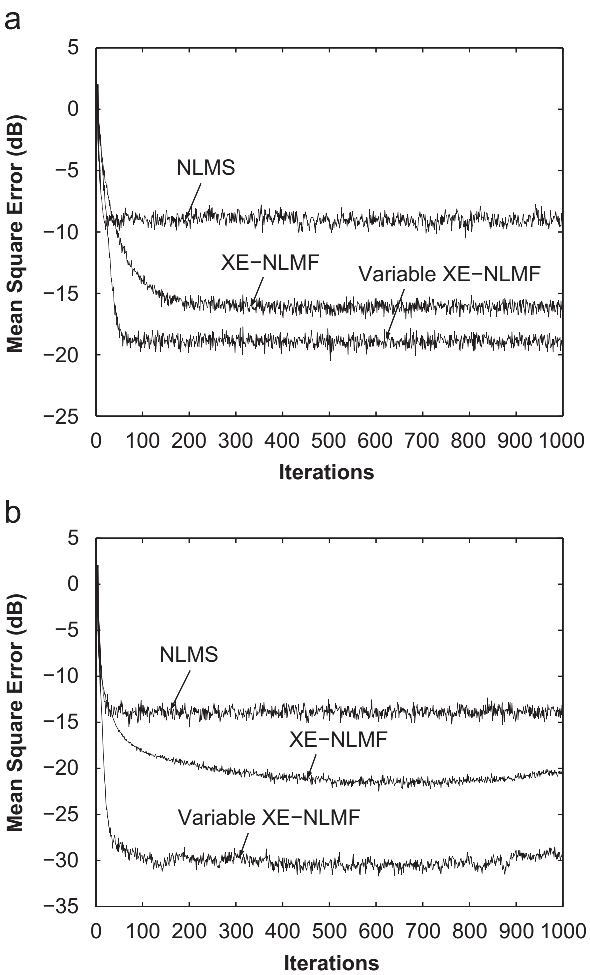


**Fig. 11.** Tracking performance of the proposed variable XE-NLMF algorithm, the fixed XE-NLMF algorithm ($\alpha = 0.5$), and the NLMS algorithm in a binary additive noise environment for $\Omega = 0.001$ (a) $\mathrm{tr}\{\mathbf{Q}_k\} = 10^{-2}$ and (b) $\mathrm{tr}\{\mathbf{Q}_k\} = 10^{-7}$.

Figs. 10(a) and (b) depict the tracking behaviour of the proposed variable XE-NLMF algorithm, the fixed XE-NLMF algorithm ($\alpha = 0.5$), and the NLMS algorithm in an AWGN environment for $\mathrm{tr}\{\mathbf{Q}_k\} = 10^{-7}$, $\Omega = 0.01$ and 0.001, respectively. Finally, Figs. 11(a) and (b) depict the tracking behaviour of the proposed variable XE-NLMF algorithm, the fixed XE-NLMF algorithm ($\alpha = 0.5$), and the NLMS algorithm in binary additive noise environment for $\Omega = 0.001$, $\mathrm{tr}\{\mathbf{Q}_k\} = 10^{-2}$ and $10^{-7}$, respectively. Two important observations can be seen from these figures. First, the consistency of performance of the proposed algorithm in both scenarios over the other algorithms. Second, the MSE increases with $\Omega$.

## 7. Conclusions

This work has proposed a variable XE-NLMF algorithm with a variable mixed-power parameter ($\alpha_k$). The variable mixed-power parameter follows the scheme of a variable-step-size LMS and is effective in controlling the mixed-power parameter. This variable normalisation strategy provides an optimised mixed normalisation of signal and error powers for the LMF algorithm. Hence, replacing the fixed user-selected mixed-power parameter by an automatic adaptation provides an extra performance gain for the XE-NLMF algorithm.

The results of our study can be summarised briefly as follows:

1. A thorough theoretical performance analysis of the algorithm was carried out and showed good agreement with simulation results.
2. The analytical results of the steady-state EMSE were derived for the variable XE-NLMF algorithm in the presence of both random and cyclic nonstationarities. Close agreement between theory and simulation in both cases for two different values of $\mathrm{tr}\{\mathbf{Q}_k\}$ and different values of $\Omega$ was obtained. The results, show that unlike in the stationary case, the steady-state EMSE is not a monotonically increasing function of the step-size $\gamma_{xe}$, while the ability of the variable XE-NLMF algorithm to track variations in the environment degrades with frequency offset $\Omega$.
3. The variable XE-NLMF requires three extra multiplications and three extra additions in comparison with the fixed XE-NLMF algorithm with fixed mixing parameter, and $(N+5)$ extra multiplications and $(N+3)$ extra additions in comparison with the NLMS algorithm.
4. Finally, these properties make the proposed variable XE-NLMF algorithm more suitable for modern day applications, e.g., wireless communications where a non-Gaussian environment is more likely to be present, rather than the fixed XE-NLMF algorithm with fixed mixing parameter or the NLMS algorithm.

## Acknowledgment

The first author acknowledges KFUPM for the support received under fast track grant FT-2004/2.

## Appendix A. Derivation of $E[\alpha_\infty]$

In this appendix, the steady-state mean value of the power mixing parameter ($\alpha_k$) is derived. The update rule for the quantity $p_k$, given by (5), can be set up to the following:

$$p_k = a[\mathrm{erf}(\mu_{k-2}) + \mathrm{erf}(\mu_{k-1}) + \mathrm{erf}(\mu_k)], \tag{78}$$

where we have used the relation given by (6). Consequently, the recursion for $\mu_k$, given by (4), can be modified as follows:

$$\mu_{k+1} = v\mu_k + a|e_k e_{k-1}|[\mathrm{erf}(\mu_{k-2}) + \mathrm{erf}(\mu_{k-1}) + \mathrm{erf}(\mu_k)]. \tag{79}$$

We assume that $\mu_k$ is independent of $e_k$ for all values of $k$. This is not true in general, but it is applicable in the case when $v$ is close to 1. Taking the expectation on both sides of (79) and using this independent assumption then gives

$$E[\mu_{k+1}] = vE[\mu_k] + aE[|e_k e_{k-1}|] \times [E[\mathrm{erf}(\mu_{k-2})] + E[\mathrm{erf}(\mu_{k-1})] + E[\mathrm{erf}(\mu_k)]]. \tag{80}$$

Near steady-state, $\mu_k$ has a small value, which makes it possible to use the following approximation:

$$E[\mathrm{erf}(\mu_\infty)] \approx \mathrm{erf}(E[\mu_\infty]), \tag{81}$$

Consequently, at steady-state, the recursion (80) takes the following form:

$$E[\mu_\infty] = vE[\mu_\infty] + aE[|e_\infty^2|][\mathrm{erf}(E[\mu_\infty]) + \mathrm{erf}(E[\mu_\infty]) + \mathrm{erf}(E[\mu_\infty])\}]. \tag{82}$$

Ultimately, it can be shown that

$$E[\mu_\infty] = \frac{3a}{(1-v)} E[|e_\infty|^2]\mathrm{erf}(E[\mu_\infty]). \tag{83}$$

Knowing that $E[|e_\infty|^2] = \zeta_\infty + \sigma_\eta^2$ and using expression (61), we can rewrite (83) as follows:

$$E[\mu_\infty] = \frac{3a}{(1-v)} \left[ \frac{5\gamma_{xe}\,\mathrm{tr}(\mathbf{R})\sigma_\eta^4}{2c_\infty\,(l_\infty - 4) - 10\gamma_{xe}\,\mathrm{tr}(\mathbf{R})\sigma_\eta^2} + \sigma_\eta^2 \right] \mathrm{erf}(E[\mu_\infty]). \tag{84}$$

This is a fixed-point equation that can be solved for $E[\mu_\infty]$.

Finally, $E[\alpha_\infty]$ can be obtained using $E[\alpha_\infty] = \mathrm{erf}(E[\mu_\infty])$.

## Appendix B. Derivation of (14) and (42)

In this appendix, the expectation terms

$$E\left[\frac{1}{\delta + (1-\alpha_k)\,\mathrm{tr}(\mathbf{R}) + \alpha_k \|\tilde{\mathbf{e}}_k\|^2}\right]$$

and

$$E\left[\frac{1}{(\delta + (1-\alpha_k)\,\mathrm{tr}(\mathbf{R}) + \alpha_k \|\tilde{\mathbf{e}}_k\|^2)^2}\right]$$

are evaluated. Let us define a new random variable $Y$, such that

$$Y = \delta + (1-\alpha_k)\,\mathrm{tr}(\mathbf{R}) + \alpha_k \|\tilde{\mathbf{e}}_k\|^2, \tag{85}$$

where we are using the instantaneous value of the mixing parameter $\alpha_k$. Next, $Y$ can be set up to the following:

$$\begin{aligned} Y &= [\delta + (1-\alpha_k)\,\mathrm{tr}(\mathbf{R})] \cdot \left[1 + \frac{\alpha_k}{\delta + (1-\alpha_k)\,\mathrm{tr}(\mathbf{R})} \sum_{k=1}^{N-1} e_{n-k}^2 \right] \\ &= [\delta + (1-\alpha_k)\,\mathrm{tr}(\mathbf{R})] \\ &\quad \times \left[1 + \frac{\alpha_k}{\delta + (1-\alpha_k)\,\mathrm{tr}(\mathbf{R})} \sum_{k=1}^{N-1} \sigma_{e_{n-k}}^2 (e'_{n-k})^2 \right], \end{aligned} \tag{86}$$

where $\sigma_{e_{n-k}}^2$ is the variance of the error term $e_{n-k}$ and

$$e'_{n-k} = \frac{e_{n-k}}{\sigma_{e_{n-k}}} \tag{87}$$

is the normalised error term with unit variance.

Now, by the long-filter assumption A4, $e_{ak}$ is a Gaussian random variable with zero mean. Consequently, the error term $e_k$ also becomes Gaussian in the presence of Gaussian noise. Ultimately, the term $\sum_{k=1}^{N-1} \sigma_{e_{n-k}}^2 (e'_{n-k})^2$ can be considered as a weighted sum of central chi-square random variables with one degree of freedom. There is no closed form expression for this sum for general weights. In this case, it is common practice in statistics to approximate a weighted sum of chi-square variables by a single one with different degrees of freedom and an appropriate scaling factor. Therefore, using the procedure outlined in [21], we can write

$$1 + \frac{\alpha_k}{\delta + (1-\alpha_k)\,\mathrm{tr}(\mathbf{R})} \sum_{k=1}^{N-1} \sigma_{e_{n-k}}^2 (e'_{n-k})^2 \approx \beta \mathscr{X}^2(l), \tag{88}$$

where $\beta$ is a scaling factor and $\mathscr{X}^2(l)$ is a central chi-square random variable with $l$ degrees of freedom. The parameters $\beta$ and $l$ should be chosen such that both sides of the above equation have the same first two moments. Comparing the above equation with [21, Eq. (10)], the parameters $\beta$ and $l$ can be found directly to be

$$\beta_k = \frac{b_k}{a_k} \tag{89}$$

and

$$l_k = \frac{a_k^2}{b_k}, \tag{90}$$

where

$$a_k = 1 + \frac{\alpha_k}{\delta + (1-\alpha_k)\,\mathrm{tr}(\mathbf{R})} \sum_{k=1}^{N-1} (\zeta_{n-k} + \sigma_\eta^2) \tag{91}$$

and

$$b_k = \frac{\alpha_k^2}{[\delta + (1-\alpha_k)\,\mathrm{tr}(\mathbf{R})]^2} \sum_{k=1}^{N-1} (\zeta_{n-k} + \sigma_\eta^2)^2. \tag{92}$$

Finally, using the fact that $E[g(y)] = \int_{-\infty}^{\infty} g(y) f_Y(y)\,dy$ where $g(y)$ is any function of the random variable $Y$, it can be shown that

$$E\left[\frac{1}{\delta + (1-\alpha_k)\,\mathrm{tr}(\mathbf{R}) + \alpha_k \|\tilde{\mathbf{e}}_k\|^2}\right] = \frac{1}{c_k(l_k - 2)} \tag{93}$$

and

$$E\left[\frac{1}{(\delta + (1-\alpha_k)\,\mathrm{tr}(\mathbf{R}) + \alpha_k \|\tilde{\mathbf{e}}_k\|^2)^2}\right] = \frac{1}{c_k^2(l_k - 2)(l_k - 4)}, \tag{94}$$

Please cite this article as: A. Zerguine, et al., Convergence and tracking analysis of a variable normalised LMF (XE-NLMF)..., Signal Process. (2008), doi:10.1016/j.sigpro.2008.10.022

where $c_k$ is given by

$$c_k = [\delta + (1 - \alpha_k)\,\mathrm{tr}(\mathbf{R})]\beta_k. \tag{95}$$

## References

[1] E. Walach, B. Widrow, The least mean fourth (LMF) adaptive algorithm and its family, IEEE Trans. Inf. Theory IT-30 (February 1984) 275–283.
[2] S. Haykin, Adaptive Filter Theory, fourth ed., Prentice-Hall, Upper-Saddle River, NJ, 2002.
[3] J.A. Chambers, O. Tanrikulu, A.G. Constantinides, Least mean mixed-norm adaptive filtering, IEE Electron. Lett. 30 (19) (September 1994) 1574–1575.
[4] S. Koike, Stability conditions for adaptive algorithms with non-quadratic error criteria, in: EUSIPCO-2000, Tempere, Finland, September 5–8, 2000, pp. 131–134.
[5] A. Zerguine, A novel approach for the convergence analysis of the LMF algorithm, in: EUSIPCO-2004, Vienna, Austria, September 6–10, 2004, pp. 1801–1804.
[6] A. Zerguine, Convergence and steady-state analysis of the normalized least mean fourth algorithm, Digital Signal Process. 17 (1) (January 2007) 17–31.
[7] M.K. Chan, C.F.N. Cowan, Using a normalised LMF algorithm for channel equalisation with co-channel interference, in: EUSIPCO-2002, Toulouse, France, September 3–6, 2002, pp. 48–51.
[8] A.H. Sayed, Fundamentals of Adaptive Filtering, Wiley, New York, 2003.
[9] R.H. Kwong, E.W. Johnston, A variable step size LMS algorithm, IEEE Trans. Signal Process. 40 (7) (July 1992) 1633–1642.
[10] J.E. Mazo, On the independence theory of equalizer convergence, Bell Syst. Tech. J. 58 (May–June 1979) 963–993.
[11] P.I. Hubscher, J.C.M. Bermudez, An improved statistical analysis of the least mean fourth (LMF) adaptive algorithm, IEEE Trans. Signal Process. 51 (3) (March 2003) 664–671.
[12] J.G. Proakis, Digital Communications, McGraw-Hill, Singapore, 2001.
[13] T.Y. Al-Naffouri, A.H. Sayed, Transient analysis of adaptive filters with error nonlinearities, IEEE Trans. Signal Process. 51 (3) (March 2003) 653–663.
[14] N.J. Bershad, M. Bonnet, Saturation effects in LMS adaptive echo cancellation for binary data, IEEE Trans. Acoust. Speech Signal Process. 38 (10) (October 1990) 1687–1696.
[15] A. Papoulis, Probability, Random Variables, and Stochastic Processes, McGraw-Hill, New York, 1991.
[16] S.C. Douglas, T.H.-Y. Meng, Stochastic gradient adaptation under general error criteria, IEEE Trans. Signal Process. 42 (6) (June 1994) 1352–1365.
[17] T.S. Rappaport, Wireless Communications, Prentice-Hall, New Jersey, USA, 1996.
[18] M. Rupp, LMS tracking behavior under periodically changing systems, in: EUSIPCO-1998, Island of Rhodes, Greece, September 8–11, 1998, pp. 1253–1256.
[19] N.R. Yousef, A.H. Sayed, A unified approach to the steady-state and tracking analysis of adaptive filters, IEEE Trans. Signal Process. 49 (2) (February 2001) 314–324.
[20] M. Moinuddin, A. Zerguine, Tracking analysis of the NLMS algorithm in the presence of both random and cyclic non-stationarities, IEEE Signal Process. Lett. 10 (9) (September 2003) 256–258.
[21] Q.T. Zhang, D.P. Liu, A simple capacity formula for correlated diversity Rician fading channels, IEEE Commun. Lett. 6 (11) (November 2002) 481–483.

Please cite this article as: A. Zerguine, et al., Convergence and tracking analysis of a variable normalised LMF (XE-NLMF)..., Signal Process. (2008), doi:10.1016/j.sigpro.2008.10.022