Scaling of the Minimum Euclidean Norm

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***Abstract*— Studying the performance of multiuser MIMO communication systems can be quite challenging. Research in this area has thus focused on the behavior of such systems in the large number of users and antennas regimes which relies on understanding the scaling of a large number of iid random variables. In this paper, we study the scaling behavior of the max-min norm, $\max_{B\geq 0\ \mathrm{Tr}(B)\leq P} \min_{1\leq i\leq n} \|h_i\|_B^2$, where h_i are iid vectors. Such a study is for example important for evaluating the asymptotic performance of multicast systems in which a base station with M antennas broadcasts common information to n users. We study this scaling for various regimes including large M and large n as well as regimes where both n and M grow simultaneously.**

I. Introduction

Communication using multiple antennas has attracted a lot of Research during the last decade. Research initially focused on the use of multiple antennas for point to point links [1], [2], [3],[4]. Recently, researchers have shifted their focus to study the role of multiple antennas on point to multipoint or multiuser broadcast systems [5]. Here a base station with M antennas is to broadcast to n users each equipped with a single antenna. Thus, the received signal by the ith user is given by

$$y_i = s^* h_i + n_i$$

where s is the transmitted signal and is subject to an average power constraint $E[s^* s] \leq P$ and $\nu \sim CN(0, I_M)$ is the additive white complex Gaussian noise. The vector h_i is the channel matrix between the base station and user i and is assumed to be iid over the users.

Quantifying the system performance (sum-rate capacity) of various scheduling schemes can be quite challenging. As such, researchers have resorted to study the asymptotic performance of such systems in the large number of users n and/or large number of antennas M regimes. These asymptotic results are very valuable as they give insights about the system performance when its resources are stressed. Quite often the performance turns out to depend on the Euclidean norm of the users' channels. Two such examples are in order:

A. Opportunistic Scheduling

Here the base station transmits to the user with the most favorable condition, e.g., the user with the largest weighted Euclidean norm, i.e.

$$\max_{B\geq 0\ \mathrm{Tr}(B)\leq P} \max_i \|h_i\|_B^2 \tag{1}$$

which we shall henceforth call the *max-max norm*. Using opportunist communication to broadcast information to independent users has attracted a lot of attention recently (see [6], [7], [8], [9], [10], [11] and the references therein).

B. Multicast

Here the users are all interested in the same information and the base station is to multicast this information at a rate achievable by all users. In this case, the sum-rate capacity is eventually limited by the worst user, i.e.

$$\max_{B\geq 0\ \mathrm{Tr}(B)\leq P} \min_i \|h_i\|_B^2 \tag{2}$$

which we shall refer to as the *max-min norm*. The multicast problem has only recently started to get some attention [12], [13], [14].

The aim of this paper is to study the behavior of the max-min norm for large n and/or M and obtain the scaling of the min norm, $\min_i \|h_i\|^2$, as a special case. Moreover, while most capacities and asymptotic results assume the channel h_i to be Gaussian and the individual elements of h_i to be iid, several of our results apply for more general distributions on $\|h_i\|^2$.

The paper is organized as follows. In the next section, we derive some bounds on the max-min norm that will be very useful as we study the scaling behavior. We subsequently study the scaling of the max-min norm for large n and fixed M in Section III and for large M and fixed n in Section IV. We then turn our attention to the case where M and n grow simultaneously, studying the behavior of the max-min norm for $M = \beta n$ in Section V and for $M = \log n$ in Section VI.

II. Fixed n and M

In what follows, we derive some inequalities on the max-min norm for fixed values of M and n. This will be useful further ahead when we allow M and/or n to get large. Recall that in the max-min norm problem, we require that $B \geq 0$ and $\mathrm{Tr}(B) \leq P$, from which it is easy to see that $B \leq PI$. Thus,

$$\max_B \|h_i\|_B^2 \leq P\|h_i\|^2 \tag{3}$$

or

$$\max_B \min_i \|h_i\|_B^2 \leq P \min_i \|h_i\|^2 \tag{4}$$

To obtain a lower bound, note that

$$\max_B \|h_i\|_B^2 \geq \|h_i\|_{\frac{P}{M}I}^2 = \frac{P}{M}\|h_i\|^2 \tag{5}$$

which allows us to write

$$\max_B \min_i \|h_i\|_B^2 \geq \frac{P}{M} \min_i \|h_i\|^2 \tag{6}$$

Now, for the case $n \leq M$, consider the set $\mathcal{G} = \{h_1, h_2, \ldots, h_n\}$ and let $\mathcal{A} = \{a_1, a_2, \ldots, a_l\}$ be an orthonormal basis for the space spanned by the set $\mathcal{G}$. Then $l \leq n \leq M$. Define the matrix B' as

$$B' = \frac{P}{n} AA^*$$

where

$$A = \left[\begin{array}{ccccc} a_1 & a_2 & \cdots & a_l & O_{M\times M-l} \end{array} \right]$$

then $B' \geq 0$ and $\mathrm{Tr}(B') = \frac{lP}{n} \leq P$. Moreover, since each h_i is spanned by $\mathcal{A}$, it is easy to see that

$$\|h_i\|_{B'}^2 = \frac{P}{n} \|h_i\|^2$$

so that

$$\max_B \|h_i\|_B^2 \geq \frac{P}{n} \|h_i\|^2 \tag{7}$$

and

$$\max_B \min_i \|h_i\|_B^2 \geq \frac{P}{n} \min_i \|h_i\|^2 \tag{8}$$

Combining (3), (5), and (8) yields

$$\frac{P}{\min\{M,n\}} \|h_i\|^2 \leq \max_B \|h_i\|_B^2 \leq P\|h_i\|^2 \tag{9}$$

Similarly, it is easy to see that

$$\boxed{\frac{P}{\min\{M,n\}} \min_i \|h_i\|^2 \leq \max_B \min_i \|h_i\|_B^2 \leq P \min_i \|h_i\|^2} \tag{10}$$

Note that the bounds in (10) depend on $\min_i \|h_i\|^2$, so let's bound this quantity further. To obtain a lower bound, define the matrix

$$H = \left[\begin{array}{cccc} h_1 & h_2 & \cdots & h_n \end{array} \right] \tag{11}$$

and note that

$$\mathrm{diag}(H^*H) = \left[\begin{array}{cccc} \|h_1\|^2 & \|h_2\|^2 & \cdots & \|h_n\|^2 \end{array} \right]^T$$

It thus follows that

$$\boxed{\lambda_{min}(H^*H) \leq \min_i \|h_i\|^2 \leq \lambda_{max}(H^*H) \leq \max_i \|h_i\|^2} \tag{12}$$

The matrix H helps us obtain another upper bound on $\max_B \min_i \|h_i\|_B^2$ as demonstrated in [13]. The key point is to upper bound the minimization by the sum average

$$\begin{aligned} \max_B \min_i \|h_i\|_B^2 &\leq \frac{1}{n} \max_B \sum_{i=1}^n \|h_i\|_B^2 \\ &= \frac{1}{n} \max_B \sum_{i=1}^n \mathrm{Tr}(Bh_ih_i^*) \\ &= \frac{1}{n} \max_B \mathrm{Tr}\left(B \sum_{i=1}^n h_ih_i^*\right) \\ &= \frac{1}{n} \max_B \mathrm{Tr}\left(BHH^*\right) \end{aligned}$$

which yields the sum-average inequality

$$\boxed{\max_B \min_i \|h_i\|_B^2 \leq \frac{P}{n} \lambda_{max}(H^*H)} \tag{13}$$

III. Scaling with n, Fixed M

In this section, we study the behavior of $\max_B \min_i \|h_i\|_B^2$ for large n and fixed M. To do so, we rely on the following theorem which characterizes the behavior of the minimum x_{min} of a large number of nonnegative iid random variables $\{x_1, x_2, \ldots, x_n\}$ having CDF $F(x)$ and characteristic function $\phi(s)$. The proof of this lemma is omitted and can be found in [14].

Lemma 1: Let i_0 be the first non-zero derivative of $F(x)$ at zero, i.e., $F^{(j)}(0) = 0$ for all $j < i_0$ and $F^{(i_0)}(0) \neq 0$. Then $n^{\frac{1}{i_0}} x_{min}(n)$ converges in distribution to a random variable y with CDF

$$F_y(y) = 1 - \exp\left(-\frac{F^{(i_0)}(0)}{i_0!} y^{i_0}\right) \tag{14}$$

Furthermore, we can find $F^{(i_0)}(0)$ using the initial value theorem applied to the pair

$$F^{(i_0)}(x) \leftrightarrow s^{i_0-1}\phi(s) \tag{15}$$

i.e., from

$$\lim_{x\to 0} F^{(i_0)}(x) = \lim_{s\to\infty} s^{i_0}\phi(s)$$

We thus say that

$$\boxed{x_{\min} \text{ converges to } \frac{E}{n^{\frac{1}{i_0}}}}$$

where E is the expectation that arises from the distribution (14)

$$\begin{aligned} E &= \int_0^\infty \exp\left(-\frac{F^{(i_0)}(0)}{i_0!} x^{i_0}\right) && (16) \\ &= \frac{C_{i_0}}{F^{(i_0)}(0)^{\frac{1}{i_0}}} && (17) \end{aligned}$$

and where

$$C_{i_0} = \frac{\Gamma(\frac{1}{i_0})(i_0!)^{\frac{1}{i_0}}}{i_0}$$

Let's employ this theorem to find the scaling of $\min_i \|h_i\|^2$ when h_i are iid $CN(0,R)$. While the CDF and pdf of $\|h_i\|^2$ will both have different forms depending on whether some of the eigenvalues λ_l of R are the same or different, the characteristic function takes one form and is given by

$$\phi(s) = \prod_{l=1}^M \frac{1}{1+\lambda_l s}$$

From this, it is easy to see that

$$\lim_{s\to\infty} s^i \phi(s) = F^{(i)}(0) = 0 \text{ for } i < M$$

and that

$$\lim_{s\to\infty} s^M \phi(s) = F^{(M)}(0) = \frac{1}{\prod_{l=1}^M \lambda_l} = \frac{1}{\det(R)}$$

We thus conclude that

$$\boxed{\min_i \|h_i\|^2 \text{ scales as } C_M \det(R)^{\frac{1}{M}} \frac{1}{n^{\frac{1}{M}}}}$$

Now, by employing the inequality (10), we conclude that for $h_i \sim CN(0,R)$ and for large n

$$\boxed{\det(R)^{\frac{1}{M}} \frac{PC_M}{Mn^{\frac{1}{M}}} \le \max_B \min_i \|h_i\|_B^2 \le \det(R)^{\frac{1}{M}} \frac{PC_M}{n^{\frac{1}{M}}}} \quad (18)$$

Using Lemma 1 and (10) we can similarly study the scaling of the min max norm when h_i is non-Gaussian.

IV. Scaling with M, n fixed

To find the scaling of the max min norm for large M and fixed n, we start from inequality (10)

$$\frac{P}{n} \min_i \|h_i\|^2 \le \max_B \min_i \|h_i\|_B^2 \le P \min_i \|h_i\|^2$$

Note now that by the law of large numbers,

$$\lim_{M\to\infty} \frac{\|h_i\|^2}{M} = 1$$

This result applies even upon minimization over a finite number of norms, i.e.

$$\lim_{M\to\infty} \min_i \frac{\|h_i\|^2}{M} = 1$$

We thus obtain

$$\boxed{\tfrac{1}{n} PM \le \lim_{M\to\infty} \max_B \min_i \|h_i\|_B^2 \le PM}$$

V. Scaling with M and n, $M = \beta n$

Now let's study the behavior of the max min norm when M and n are linearly related ($M = \beta n$) and growing to ∞. Without loss of generality, we assume that $\beta \le 1$ so that $M \le n$. We start with (12)

$$\lambda_{min}(\frac{H^*H}{M}) \le \min_i \frac{\|h_i\|^2}{M} \le \lambda_{max}(\frac{H^*H}{M})$$

and note that as $n, M \to \infty$ with $\frac{M}{n} = \beta$, the eigenvalues of $\frac{H_i^* H_i}{M}$ become uniformly distributed in the range $[(1-\sqrt{\beta})^2, (1+\sqrt{\beta})^2]$. We can thus write

$$\boxed{(1-\sqrt{\beta})^2 \le \lim_{\frac{M}{n}=\beta,\, n\to\infty} \min_i \frac{\|h_i\|^2}{M} \le (1+\sqrt{\beta})^2}$$

This together with (10) yields the following lower bound on the max-min norm

$$P(1-\sqrt{\beta})^2 \le \max_B \min_i \|h_i\|^2$$

To obtain an upper bound, we use the sum-average inequality (13)

$$\max_B \min_i \|h_i\|^2 \le P\lambda_{max}(\frac{H^*H}{n})$$

and note that the as $n, M \to \infty$ with $\frac{M}{n} = \beta$, the eigenvalues of $\frac{HH^*}{n}$ will be confined to the range $[(1-\frac{1}{\sqrt{\beta}})^2, (1+\frac{1}{\sqrt{\beta}})^2]$. This yields the upper bound

$$\max_B \min_i \|h_i\|^2 \le P(1+\frac{1}{\sqrt{\beta}})^2$$

and we thus conclude that

$$\boxed{P(1-\sqrt{\beta})^2 \le \lim_{\frac{M}{n}=\beta,\, n\to\infty} \max_B \min_i \|h_i\|^2 \le P(1+\frac{1}{\sqrt{\beta}})^2} \quad (19)$$

VI. Scaling with M and n, $M = \log n$

The bounds in the previous section tell us that by allowing the size of h_i to increase linearly with the total number of vectors, we guarantee that $\max_B \min_i \|h_i\|_B^2$ will have a constant norm. The question that begs itself now is whether we can guarantee a constant norm for a sublinear growth in M.

We can get an intuitive answer for this from Section III where we proved that the max-min norm behaves like $\alpha C_M \frac{1}{n^{\frac{1}{M}}}$. Now it is easy to see that for large M, $C_M \simeq 1$. Thus, we need to guarantee that $\frac{1}{n^{\frac{1}{M}}}$ behaves like a constant which would be the case for $M = \log n$. As such, we study in this section the scaling of the max-min norm for $M = \log n$. We confine our attention here to Gaussian distributed vectors ($h_i \sim \mathcal{N}(0, I)$) but our approach applies for more general distributions.

To study the scaling, we rely on on the Chernof bound. To this end, let $Y = \frac{\|h_i\|^2}{M}$, and define $g(Y)$ by

$$g(Y) = \begin{cases} 1 & \text{if } Y \le 1-\epsilon \\ 0 & \text{if } Y > 1-\epsilon \end{cases}$$

Then for $\nu \ge 0$

$$g(Y) \le e^{-\nu(Y-(1-\epsilon))}$$

and hence

$$E[g(Y)] = P(Y \le 1-\epsilon) \le e^{\nu(1-\epsilon)} E[e^{-\nu Y}]$$

or

$$P(\frac{\|h_i\|^2}{M} \le 1-\epsilon) \quad = \quad e^{\nu(1-\epsilon)} \frac{1}{\left(1+\frac{\nu}{M}\right)^M} \quad (20)$$

Now we can tighten the upper bound by choosing the optimum ν, which, upon setting the first derivative to zero, turns out to be

$$\nu = M\frac{\epsilon}{1-\epsilon} > 0$$

and the bound reads

$$\begin{aligned} P(\frac{\|h_i\|^2}{M} \le 1-\epsilon) &\le e^{M\epsilon}(1-\epsilon)^M & (21) \\ &= e^{M(\epsilon+\log(1-\epsilon))} & (22) \end{aligned}$$

We can use this to bound the probability $P(\min_i \frac{\|h_i\|^2}{M} \le 1-\epsilon)$

$$\begin{aligned} P(\min_i \frac{\|h_i\|^2}{M} \le 1-\epsilon) &= 1-(1-P(\frac{\|h_i\|^2}{M} \le 1-\epsilon))^n & (23) \\ &\le 1-(1-e^{M(\epsilon+\log(1-\epsilon))})^n & (24) \\ &= 1-(1-n^{\epsilon+\log(1-\epsilon)})^n & (25) \end{aligned}$$

where the last line follows from the fact that $M = \log n$. For the above probability to vanish as n grows, we require that

$$\epsilon + \log(1-\epsilon) < -1$$

Let ϵ_l be the infimum of the set $\{\epsilon : \epsilon + \log(1-\epsilon) < -1\}$, (i.e. ϵ_l satisfies $\epsilon_l + \log(1-\epsilon_l) = -1$ or $\epsilon_l \simeq .8414$). Then,

$$\boxed{\lim_{M=\log n,\, n\to\infty} P(\min_i \frac{\|h_i\|^2}{M} \ge 1-\epsilon_l) = 1} \quad (26)$$

Now let's obtain an upper bound for $\min_i \frac{\|h_i\|^2}{M}$. Employing Chernof bound again, it is easy to show that for $\nu \geq 0$

$$P(\frac{\|h_i\|^2}{M} \geq 1+\epsilon) \leq e^{-\nu(1+\epsilon)} E[e^{\nu \frac{\|h_i\|^2}{M}}] \quad (27)$$

$$= e^{-\nu(1+\epsilon)} \frac{1}{(1-\frac{\nu}{M})^M} \quad (28)$$

Moreover, the upper bound is tightest for

$$\nu = M \frac{\epsilon}{1+\epsilon}$$

We thus have

$$P(\frac{\|h_i\|^2}{M} \geq 1+\epsilon) \leq e^{-M\epsilon}(1+\epsilon)^M = e^{M(-\epsilon+\log(1+\epsilon))}$$

or

$$P(\min_{h_i} \frac{\|h_i\|^2}{M} \geq 1+\epsilon) \leq (n^{(-\epsilon+\log(1+\epsilon))})^n$$

where we used the fact that $n = \log M$. This probability vanishes provided that $-\epsilon + \log(1+\epsilon) < 0$ and the infimum for which this is true is $\epsilon_u = 0$. We can thus write

$$\boxed{\lim_{M=\log n,\ n\to\infty} P\left(\min_i \frac{\|h_i\|^2}{M} \leq 1\right) = 1} \quad (29)$$

From (26) and (29), we see that

$$\boxed{\lim_{M=\log n,\ n\to\infty} \min_i \frac{\|h_i\|^2}{M} = \mathcal{H} \in [1-\epsilon_l, 1] \quad \text{w.p.1}} \quad (30)$$

We now are ready to bound the max-min norm. Specifically, from the lower bound in (10), we have that

$$\lim_{M=\log n,\ n\to\infty} \min_i \frac{\|h_i\|^2}{M} \geq \mathcal{H}P$$

Moreover, and since the max-min norm is upper bounded by a constant when M and n are related linearly (see (19), so will the norm be when $M = \log n$.

VII. Conclusion

Random vectors and matrices play an important role in the study of communication systems. The asymptotic behavior these variables is of particular importance in multiuser communication. In this paper, we studied the scaling behavior of the max-min weighted Euclidean norm (where the minimization is taken over a large number of n iid random vectors of size M). Specifically, we obtained bounds on the max-min norm for fixed M and n. We then showed that the norm scales as $\frac{1}{n^{\frac{1}{M}}}$ for fixed M and scales as M for fixed n. We then studied the scaling of the max-min norm when M and n grow simultaneously and showed that by setting $M = \log n$, the max-min norm converges to a nonzero constant. We obtained, as a by-product of our study, the scaling behavior of the minimum Euclidean norm. Moreover, several of our results apply for generically distributed (i.e. non-Gaussian) random vectors.

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