On the Distribution of Indefinite Quadratic Forms in Gaussian Random Variables

T. Y. Al-Naffouri[1,2], M. Moinuddin[3,4], Nizar Ajeeb[5], Babak Hassibi[6], and Aris L. Moustakas[7]

[1]Electrical Engineering Department, King Fahd University of Petroleum and Minerals, Saudi Arabia
[2]Electrical Engineering Department, King Abdullah University of Science and Technology, Saudi Arabia
[3]Electrical and Computer Engineering Department, King Abdulaziz University, Saudi Arabia
[4] Center of Excellence in Intelligent Engineering Systems (CEIES), King Abdulaziz University,Saudi Arabia
[5]Electrical and Computer Engineering Department, American University of Beirut, Beirut, Lebanon
[6]Electrical Engineering Department, California Institute of Technology, California, U.S.A
[7]Physics Department, National Kapodistrian University of Athens, Greece

***Abstract*—In this work, we propose a unified approach to evaluating the CDF and PDF of indefinite quadratic forms in Gaussian random variables. Such a quantity appears in many applications in communications, signal processing, information theory, and adaptive filtering. For example, this quantity appears in the mean-square-error (MSE) analysis of the normalized least-mean-square (NLMS) adaptive algorithm, and SINR associated with each beam in beam forming applications. The trick of the proposed approach is to replace inequalities that appear in the CDF calculation with unit step functions and to use complex integral representation of the the unit step function. Complex integration allows us then to evaluate the CDF in closed form for the zero mean case and as a single dimensional integral for the non-zero mean case. Utilizing the saddle point technique allows us to closely approximate such integrals in non zero mean case. We demonstrate how our approach can be extended to other scenarios such as the joint distribution of quadratic forms and ratios of such forms, and to characterize quadratic forms in isotropic distributed random variables. We also evaluate the outage probability in multiuser beamforming using our approach to provide an application of indefinite forms in communications.**

Key Words: wireless communications, Correlated Gaussian random vectors, multi-user diversity, weighted norms of Gaussian variables.

I. Introduction

Gaussian random variables (r.v.'s) play a very important role in signal processing, communications, and information theory [1]. One reason why Gaussian r.v.'s are so ubiquitous is the central limit theorem which states that under conditions often reasonable in applications, the probability density function (PDF) of the sum of independent random variables approaches that of a Gaussian random variable.

It is very important to find the distributions of various quantities involving Gaussian random variables, most notably sums of squares of Gaussian random variables (quadratic forms) and ratios of such norms. The quadratic forms in Gaussian random variables [2], [3] appear in many applications in communications, signal processing, and statistics. Some of these applications include non-coherent detection [4], [5], [6], performance analysis of wireless relay networks [7], [8], cooperative diversity in wireless networks [9], diversity combining in communication systems [10], [11], array processing and random beam forming [12], estimation of power spectra [13], $\mathcal{X}^2$ test, analysis of variance [3], probability content of regions under spherical normal distributions [14], and performance analysis of the adaptive filtering algorithms [15], [16], [17]. Table I lists specific applications of indefinite quadratic forms in the field of communications.

A. Characterizing the Behavior of Quadratic Forms: A Literature Review

Several works have been devoted to study quadratic forms and their ratios [13], [14],[18]–[42]. However, the approaches proposed are either restricted to special cases and/or provide approximations or complex solutions in the form of series expansion which limit their usefulness.

Turin was the first to obtain the characteristic function of Hermitian quadratic forms in complex Gaussian variates [30]. He also derived the value of the CDF at the origin for the special case of zero-mean Gaussian random variables [39]. Tziritas in [40] considered the distribution of positive definite quadratic forms in real and complex Gaussian variables. He provided necessary and sufficient conditions on when the quadratic form can be written as a sum of independent Gamma variables. His approach was to invert the expression for the characteristic function and the expressions he arrived at were almost always in the form of infinite series (for both the central and noncentral Gaussian random variables). The work of Reifler was restricted to the positive definite case only [25]. In [2], Mathai and Provost analyzed the specific scenario of zero-mean Gaussian random variables (central case). Similarly, the works presented in [18], [19], [34], [41] are limited to real Gaussian random variables only. In [5], [28], numerical integration is used to evaluate the distribution of an indefinite quadratic form resulting in approximate solutions. In [38], Raphaeli considered the distribution of special indefinite quadratic forms and computed the resulting CDF as an infinite series of Laguerre polynomials. The series obtained however are difficult to manipulate to find the PDF or moments. Also, it is not clear how Raphaeli's method can be used to treat the real case and how it simplifies in the central case. Shah and Li used the result of [42] to evaluate the distribution of quadratic forms in Gaussian mixtures. Biyari and Lindsey considered a specific indefinite quadratic form and used the characteristic function approach to obtain expressions for the PDF and CDF [26]. Just like the earlier methods, the series expansions obtained are difficult to manipulate. More recently, Simon and Alouini [43] considered the CDF of the difference of two independent chi-square random variables and obtained a closed form expression for the value of the CDF at its zero argument. They used their derivation to evaluate the PDF of a ratio of two such variables. In a related extension, Holm and Alouini evaluated the sum and difference of two correlated Nakagami variate in terms of the McKay distribution and then used that to evaluate the CDF of the ratio of such variables [44]. Some recent works on MIMO systems include the capacity evaluation of spatially correlated MIMO Rayleigh fading channels by Chiani et al. [45] and the derivation of eigenvalue density of correlated random Wishart matrices by Simon and Moustakas [46].

B. Drawbacks of Past Approaches

There are several drawbacks of the approaches highlighted above as we summarize below.

1) The approaches above are not unified in nature[1]. Various techniques are used to treat special cases (complex Gaussian, real Gaussian, central variables, noncentral variables, definite/indefinite forms, ratios of quadratic forms, ... etc).
2) These approaches almost always end up with series expansions whose coefficients are difficult to evaluate. The expansions in turn are difficult to manipulate further to obtain the corresponding moments or CDF's [38].
3) They focus on obtaining the PDF from the characteristic function whereas the CDF is a more useful expression. The reason is that the CDF (just like the PDF) can be used to obtain the moments (through integration by parts). Moreover, the CDF directly gives an expression for the probability (whereas the PDF needs to be integrated to obtain this information).
4) The proposed methods are not generalized to other cases, e.g. to obtain the joint CDF or joint PDF of two or more quadratic forms.

[1]There are other works (including the works of [26], [28], [29]) which also result in a single integral expression and thus provide unification in some sense. But their usefulness is limited as they provide approximate solutions and/or treat special cases.

C. Our Approach

In this work, we aim to study the distribution of various quantities involving weighted norms of a correlated Gaussian vector and show how to find the distribution of these quantities using complex integration. Our aim is to provide a consistent and unified way to approach the distribution of quadratic Gaussian forms by taking advantage of its quadratic structure. More specifically, the approach is unified in the following sense:

1) We transform the inequality that define the CDF into a step function that we represent using its Fourier Transform
2) This makes the CDF expression into an indefinite $M+x$ dimensional integral where M is the number of Gaussian variables and x is the number of step functions employed (x=1 for a single CDF, $x=2$ for a joint CDF, ... etc)
3) This applies for the distribution of any quantity (or joint distribution of a number of quantities). In other words, this quantity need not to be a quadratic form... it could be any other form.
4) What restrict the form we use is our ability to calculate the indefinite (M-dimensional) integral. In the Gaussian and the isotropic cases, the quadratic form are very relevant to many applications and fortunately lend themselves to the evaluation of the M-dimensional integral
5) Once the M-dimensional integral is evaluated, it remains to evaluate the remaining x-dimensional integral(s) (that resulted from the step function(s)).

This with the aid of complex integration allows us to represent the CDF in closed form or at least as a single definite integral. In the latter case, we make use of saddle point approximation technique to evaluate such integrals which is a well known technique for approximating integrals and has been used in numerous works before [47], [48], [49]. As we stress above, the unification is more general than just evaluating this one dimensional integral or not. Specifically, we treat all the following cases using the same methodology described above:

1) General indefinite quadratic form (real, complex, zero mean, non-zero mean, ratios of quadratic forms)
2) Joint distribution of indefinite quadratic forms
3) Other non-Gaussian variables, for example, indefinite quadratic forms in isotropically distributed variables.
4) Alternative proof of Craigs formula for Q function.

The paper is organized as follows. Following this introduction, the problem is set up in Section II. In Section III, the distribution of indefinite quadratic form in central variables is derived. The non-central quadratic form is investigated in Section V while the real quadratic form is treated in Section VI. Section VII demonstrates how we can extend our technique beyond quadratic forms of Gaussian variables. Section VIII provides the evaluation of outage probability in multiuser beamforming as an application of indefinite forms in communications. Simulation results are presented in section IX to investigate the performance of the derived analytical model. Finally, concluding remarks are given in Section X.

II. Problem Formulation

In this paper, we consider quadratic forms[2] defined by

$$Y_{nc} = \|\mathbf{h} + \mathbf{b}\|_{\mathbf{A}}^2 \tag{1}$$

where $\mathbf{A}$ is a Hermitian matrix of size M, $\mathbf{b}$ is a constant $M \times 1$ vector, and $\mathbf{h}$ is a white circularly symmetric Gaussian vector, i.e., $\mathbf{h} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$, where $\mathbf{0}$ is an $M \times 1$ vector of all zeros and $\mathbf{I}$ is the identity matrix of size M. We are interested in finding the CDF of this variable. The following points are in order.

1) Without loss of generality, we assume that $\mathbf{h}$ is white. To see why this is the case, let $\mathbf{h}_w \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ be the whitened version of $\mathbf{h} \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$, i.e., $\mathbf{h}_w$ and $\mathbf{h}$ are related by[3] $\mathbf{h}_w = \mathbf{R}^{\frac{H}{2}}\mathbf{h}$. Then
$$\|\mathbf{h}+\mathbf{b}\|_{\mathbf{A}}^2 = (\mathbf{h}_w^H\mathbf{R}^{\frac{H}{2}} + \mathbf{b}^H)\mathbf{A}(\mathbf{R}^{\frac{1}{2}}\mathbf{h}_w + \mathbf{b}) = \|\mathbf{h}_w + \tilde{\mathbf{b}}\|_{\tilde{\mathbf{A}}}^2 \tag{2}$$
where $\tilde{\mathbf{b}} = \mathbf{R}^{\frac{1}{2}}\mathbf{b}$ and $\tilde{\mathbf{A}} = \mathbf{R}^{\frac{H}{2}}\mathbf{A}\mathbf{R}^{\frac{1}{2}}$ are the new mean vector and new weight matrix, respectively.
2) In most of our analysis, we will assume $\mathbf{h}$ to have zero mean, that is, we will focus on the central quadratic form
$$Y_c = \|\mathbf{h}\|_{\mathbf{A}}^2 \tag{3}$$
where $\mathbf{h} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$. When the mean is not zero, we can equivalently consider the (noncentral) quadratic form defined in (1).
3) The Hermitian quadratic form (1) is a special case of the real quadratic form
$$Y_r = \|\mathbf{h}_r\|_{\mathbf{A}_r}^2 \triangleq \mathbf{h}_r^T\mathbf{A}_r\mathbf{h}_r \tag{4}$$
where $\mathbf{h}_r$ is a real Gaussian vector $\mathcal{N}(\mathbf{0}, \mathbf{I})$, $\mathbf{A}_r$ is a symmetric matrix, and the notation $()^T$ denotes transposition. We will show in Subsection VI how to deal with the distribution Of such forms.
4) We also apply our technique to scenarios beyond quadratic forms of Gaussian random variables. Specifically, we treat
a) Ratios (divisions) of quadratic forms:
$$Y_d = \frac{\epsilon_b + \|\mathbf{h}\|_B^2}{\epsilon_a + \|\mathbf{h}\|_A^2} \tag{5}$$
b) Joint distributions of quadratic forms
$$\Pr\left\{\|\mathbf{h}\|_{\mathbf{A}}^2 \le x_a, \|\mathbf{h}\|_{\mathbf{B}}^2 \le x_b\right\}, \tag{6}$$
c) Quadratic forms in isotropically distributed random vectors
$$Y_i = \|\boldsymbol{\phi}\|_{\mathbf{A}}^2 \quad \text{and} \tag{7}$$
d) Alternative Proof of Craig's Formula for Q function which is given by [50]
$$Q(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin\theta}}d\theta \tag{8}$$

[2]For any matrix $\mathbf{A}$, the quadratic form $||\mathbf{x}||_{\mathbf{A}}^2$ is defined as $||\mathbf{x}||_{\mathbf{A}}^2 \triangleq \mathbf{x}^H\mathbf{A}\mathbf{x}$ where the notation $()^H$ denotes conjugate transposition.

[3]The representation $\mathbf{R}^{\frac{H}{2}}$ is a short notation for $(\mathbf{R}^{\frac{1}{2}})^H$

III. The Distribution of an Indefinite Hermitian Quadratic Form

Consider the random Hermitian quadratic form defined in (1). The CDF of Y_{nc} is defined by

$$F_{Y_{nc}}(y) = P\{Y_{nc} \le y\} = \int_{\mathcal{A}} p(\mathbf{h})d\mathbf{h} \tag{9}$$

where $p(\mathbf{h})$ is the PDF of $\mathbf{h}$ and $\mathcal{A}$ is area in the M multidimensional complex plane defined by the inequality

$$\|\mathbf{h} + \mathbf{b}\|_{\mathbf{A}}^2 \le y \tag{10}$$

Such an integral would in general be very difficult to evaluate. An alternative way to do so is to express the inequality that appears in (10) as

$$y - \|\mathbf{h} + \mathbf{b}\|_{\mathbf{A}}^2 \ge 0 \tag{11}$$

So, the CDF takes the form

$$F_{Y_{nc}}(y) = \int_{-\infty}^{\infty} p(\mathbf{h})u(y - \|\mathbf{h} + \mathbf{b}\|_{\mathbf{A}}^2)d\mathbf{h}, \tag{12}$$

where $u(\cdot)$ is the unit step function. Since we are dealing with M-dimensional circular white Gaussian random vectors, the PDF of $\mathbf{h}$ is given by

$$p(\mathbf{h}) = \frac{1}{\pi^M}e^{-||\mathbf{h}||^2}, \tag{13}$$

Thus, the CDF can be set up as

$$F_{Y_{nc}}(y) = \frac{1}{\pi^M}\int_{-\infty}^{\infty} e^{-||\mathbf{h}||^2}u(y - \|\mathbf{h} + \mathbf{b}\|_{\mathbf{A}}^2)d\mathbf{h}$$

The above integration is performed over the entire $\mathbf{h}$ plane. Thus, the unit step function allowed us to go around the constraint in limits of integration. However, it is still difficult to deal with the unit step that appears inside the integral. To go around this, we replace the unit step by its Fourier transform representation [51]

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{x(j\omega+\beta)}}{j\omega + \beta}d\omega \tag{14}$$

which is valid for any $\beta > 0$ (and is also independent of the value of β). It is worth pointing out here, the complementary CDF (CCDF) can also be expressed directly using the Fourier representation of the complementary step function 1-u(x) as follows

$$1 - u(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{j\omega-\beta}}{j\omega - \beta}d\omega \tag{15}$$

again for $\beta > 0$. We see that, in contrast to (14) the pole here lies in the lower complex plane, and this will have repercussions in the evaluation of the asymptotic form of the CCDF below.

Thus, the representation (14) yields the the following $M+1$ dimensional integral

$$F_{Y_{nc}}(y) = \frac{1}{2\pi^{M+1}} \iint_{-\infty}^{\infty} e^{-(\|\mathbf{h}\|^2 + \|\mathbf{h}+\mathbf{b}\|^2_{(j\omega+\beta)\mathbf{A}})} d\mathbf{h} \frac{e^{y(j\omega+\beta)}}{j\omega+\beta} d\omega \tag{16}$$

Let $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H$ denote the eigenvalue decomposition of $\mathbf{A}$, then the inner integral in the above can be written equivalently as

$$\int e^{-(\|\mathbf{h}\|^2 + \|\mathbf{h}+\mathbf{b}\|^2_{(j\omega+\beta)\mathbf{A}})} d\mathbf{h} = \int e^{-(\|\tilde{\mathbf{h}}\|^2 + \|\tilde{\mathbf{h}}+\bar{\mathbf{b}}\|^2_{(j\omega+\beta)\mathbf{\Lambda}})} d\tilde{\mathbf{h}} \tag{17}$$

where $\bar{\mathbf{b}} = \mathbf{Q}^H\mathbf{b}$ and $\tilde{\mathbf{h}} = \mathbf{Q}^H\mathbf{h}$ and we have used the fact that $d\tilde{\mathbf{h}} = d\mathbf{h}$. Now, by completing the squares, we can write the sum of (weighted) norms that appear above as a single (noncentral) quadratic form

$$\|\tilde{\mathbf{h}}\|^2 + \|\tilde{\mathbf{h}}+\bar{\mathbf{b}}\|^2_{(j\omega+\beta)\mathbf{\Lambda}} = \|\tilde{\mathbf{h}}+\tilde{\mathbf{b}}\|^2_{\mathbf{B}} + c(\omega)$$

where

$$\begin{aligned} \tilde{\mathbf{b}} &= (\mathbf{I} + \frac{1}{j\omega+\beta}\mathbf{\Lambda}^{-1})^{-1}\bar{\mathbf{b}} && (18)\\ \mathbf{B} &= \mathbf{I} + (j\omega+\beta)\mathbf{\Lambda} && (19)\\ c(\omega) &= \bar{\mathbf{b}}^H(\mathbf{I} + \frac{1}{j\omega+\beta}\mathbf{\Lambda}^{-1})^{-1}\bar{\mathbf{b}} && (20) \end{aligned}$$

which allows us to rewrite (16) as

$$F_{Y_{nc}}(y) = \frac{1}{2\pi^{M+1}} \iint_{-\infty}^{\infty} e^{-\|\tilde{\mathbf{h}}+\tilde{\mathbf{b}}\|^2_{\mathbf{B}}} d\tilde{\mathbf{h}} \frac{e^{y(j\omega+\beta)-c(\omega)}}{(j\omega+\beta)} d\omega \tag{21}$$

The inner integral looks like the area under a Gaussian PDF with mean $\tilde{\mathbf{b}}$ and covariance $\mathbf{B}^{-1}$. In spite of the fact that $\tilde{\mathbf{b}}$ and $\mathbf{B}$ are complex, we show in Appendix A that

$$\frac{1}{\pi^M}\int_{-\infty}^{\infty} e^{-(\tilde{\mathbf{h}}+\tilde{\mathbf{b}})^H\mathbf{B}(\tilde{\mathbf{h}}+\tilde{\mathbf{b}})} d\tilde{\mathbf{h}} = \frac{1}{|\mathbf{B}|} = \frac{1}{|\mathbf{I}+\mathbf{\Lambda}(j\omega+\beta)|} \tag{22}$$

as expected from a Gaussian PDF. This allows us to set up the CDF of Y_{nc} as

$$F_{Y_{nc}}(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{|\mathbf{I}+(j\omega+\beta)\mathbf{\Lambda}|} \frac{e^{y(j\omega+\beta)}}{(j\omega+\beta)} e^{-c(\omega)} d\omega \tag{23}$$

which reduces the $M+1$ dimensional integral into a 1-dimensional problem in the variable $j\omega+\beta$. At this stage of the derivation, we will distinguish between two cases

1. the central (zero mean) case where $\mathbf{b}$ is set to a zero vector (treated in the following section).
2. the non-central (non-zero mean) case (treated in Section V).

IV. Distribution of Central Quadratic Form

In the central quadratic form, the mean vector $\mathbf{b}$ and consequently the vector $\tilde{\mathbf{b}}$ is a zero vector which corresponds to the central quadratic form defined in (3) (as a result Y_{nc} is transformed into Y_c defined in (3)). Thus, the term $c(\omega)$ defined in (20) becomes zero which reduces the CDF of Y_c to

$$F_{Y_c}(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{|\mathbf{I}+(j\omega+\beta)\mathbf{\Lambda}|} \frac{e^{y(j\omega+\beta)}}{(j\omega+\beta)} d\omega \tag{24}$$

To evaluate this integral, we need to first expand the fraction that appears in (24) in a partial fraction expansion. With this in mind, assume that $\mathbf{A}$ has exactly L distinct eigenvalues $\lambda_1, \ldots, \lambda_L$ where λ_l has multiplicity m_l. Then the fraction in (24) can be expanded as

$$\frac{1}{(j\omega+\beta)\prod_{i=1}^{M}(1+\lambda_i(j\omega+\beta))} = \frac{1}{(j\omega+\beta)} + \sum_{l=1}^{L}\sum_{k=1}^{m_l} \frac{\alpha_{k,l}}{(1+\lambda_l(j\omega+\beta))^k} \tag{25}$$

Now, by employing residue theory, we can show that [52]

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{+j\omega p}}{(a+j\omega)^\nu} d\omega &= \begin{cases} \frac{p^{\nu-1}}{\Gamma(\nu)} e^{-ap} u(p) & \text{for } a>0 \\ -\frac{(-p)^{\nu-1}}{\Gamma(\nu)} e^{-ap} u(-p) & \text{for } a<0 \end{cases} && (26)\\ &= \frac{\operatorname{sign}^\nu(a)}{\Gamma(\nu)} (p)^{\nu-1} e^{-ap} u(ap) && (27) \end{aligned}$$

We can use the above results to evaluate the integrals in (24). Specifically, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{y(j\omega+\beta)}}{j\omega+\beta} d\omega = e^{\beta y} e^{-\beta y} u(y) = u(y) \tag{28}$$

and

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{y(j\omega+\beta)}}{(1+\lambda_l(j\omega+\beta))^k} d\omega \\ &= \frac{e^{y\beta}}{\lambda_l^k} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{yj\omega}}{(\beta+\frac{1}{\lambda_l}+j\omega)^k} = \frac{e^{-\frac{y}{\lambda_l}} u\left(\frac{y}{\lambda_l}\right)}{\Gamma(k)|\lambda_l|^k} y^{k-1} \end{aligned} \tag{29}$$

where in arriving at (29), we used the fact that β is chosen such that $\beta + \frac{1}{\lambda_l} > 0$ and hence $\operatorname{sign}\left(\beta+\frac{1}{\lambda_l}\right) = \operatorname{sign}(\lambda_l)$. Note that both integrals in (28) and (29) are independent of β as they should. This allows us to write the CDF $F_{Y_c}(y)$ in the following closed form

$$F_{Y_c}(y;\boldsymbol{\lambda},\mathbf{m}) = u(y) + \sum_{l=1}^{L}\sum_{k=1}^{m_l} \frac{\alpha_{k,l}}{\Gamma(k)|\lambda_l|^k} y^{k-1} e^{-\frac{y}{\lambda_l}} u\left(\frac{y}{\lambda_l}\right) \tag{30}$$

where $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_L]$ is the vector of eigenvalues and $\mathbf{m} = [m_1, m_2, \cdots, m_L]$ is the corresponding multiplicity vector (the dependence on $\boldsymbol{\lambda}$ and/or $\mathbf{m}$ can be dropped if this dependence is understood).

To get more insight, let's consider the special case when no eigenvalue of $\mathbf{A}$ is repeated, i.e., $m_l = 1,\ \forall\ l$ and $L = M$ (and we can thus write $\mathbf{m} = \mathbf{1}$, where $\mathbf{1}$ is the vector with all entries equal to 1). In this case, the partial fraction expansion takes the form

$$\frac{1}{(j\omega+\beta)\prod_{l=1}^{M}(1+\lambda_l(j\omega+\beta))} = \frac{1}{j\omega+\beta} + \sum_{l=1}^{M} \frac{\alpha_l}{1+\lambda_l(j\omega+\beta)} \tag{31}$$

where $\alpha_l = \frac{-\lambda_l}{\prod_{i=1,i\neq l}^{M}(1-\frac{\lambda_i}{\lambda_l})}$. Upon carrying out the integration using (26), we finally arrive at

$$F_{Y_c}(y) = u(y) - \sum_{l=1}^{M} \frac{\lambda_l^M}{\prod_{i=1,i\neq l}^{M}(\lambda_l - \lambda_i)} \frac{1}{|\lambda_l|} e^{-\frac{y}{\lambda_l}} u\left(\frac{y}{\lambda_l}\right) \quad (32)$$

Remark

By differentiating (24), we get an integral representation for the PDF of y, denoted by $f_{Y_c}(y)$, and it is given by

$$f_{Y_c}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\prod_{l=1}^{M}(1+\lambda_l(j\omega+\beta))} e^{y(j\omega+\beta)} d\omega \quad (33)$$

which can be solved using the same approach described for the CDF. For the special case when no eigenvalue is repeated, it can be shown that

$$f_{Y_c}(y) = \sum_{l=1}^{M} \frac{\lambda_l^{M-1}}{\prod_{i=1,i\neq l}^{M}(\lambda_l - \lambda_i)} \frac{1}{|\lambda_l|} e^{-\frac{y}{\lambda_l}} u\left(\frac{y}{\lambda_l}\right) \quad (34)$$

V. Non-central Quadratic Forms

In this section, we deal with the non-central case for which $\mathbf{b}$ in (1) is non-zero. In this case, the term c in (20) is non-zero and is rather a function of the integration variable ω. Equivalently, we would like to consider the non-central quadratic form Y_{nc} defined in (1). For this case, the CDF of Y_{nc} is given in (23) as 1-D integral which can not be put in closed form. Therefore, we propose to directly approximate the integral in (23) using the saddle point (SP) technique [53] which is a well known technique for approximating integrals and has been used in numerous works before [47], [48], [49], [54], [55], [56]. The SP method has not seen widespread use in communication research, but only in specific cases, such as in the context of MIMO mutual information calculations [57], [58] or in error probability analysis [54]. To apply SP method, we seek to approximate the CDF, $F_{Y_{nC}}(y)$, as follows

$$\begin{aligned} F_{Y_{nc}}(y) &= \frac{1}{2\pi} \int \frac{e^{y(j\omega+\beta)} e^{-c(\omega)}}{(j\omega+\beta)|\mathbf{I}+(j\omega+\beta)\mathbf{\Lambda}|} dw \\ &= \frac{1}{2\pi} \int e^{s(\omega)} d\omega \end{aligned} \quad (35)$$

where

$$\begin{aligned} s(\omega) &= \ln\left[\frac{e^{y(j\omega+\beta)} e^{-c(\omega)}}{(j\omega+\beta)|\mathbf{I}+(j\omega+\beta)\mathbf{\Lambda}|}\right] \text{ and} \\ c(\omega) &= \sum_{i=1}^{M} |\bar{b}_i|^2 - \sum_{i=1}^{M} \frac{|\bar{b}_i|^2}{1+(j\omega+\beta)\lambda_i} \end{aligned} \quad (36)$$

The position of the poles of the integrand in (35) are located at $j\beta$ and $j(\beta+\lambda_i)$ and the path of integration is parallel to the real line and between $j\beta$ and $-j\infty$. When Λ has negative eigenvalues, then the path is between $j\beta$ and $j(\beta+\lambda_m)$, where λ_m is the minimum norm negative eigenvalue of Λ. For simplicity in the sequel we only deal with the case where all $\lambda_i > 0$. The idea behind the saddle point analysis is to deform the path, without crossing any poles, so that it crosses the real axis through a point, with $s'(\omega) = 0$. Then it is known that the integral close to that point will dominate the whole integral in the large M limit [53]. Thus, we differentiate $s(\omega)$ by rewriting $s(\omega)$ as $-\ln(j\omega+\beta) - \sum_{i=1}^{M} \ln[(1+\lambda_i(j\omega+\beta))] + y(j\omega+\beta) - c(\omega)$ to obtain

$$s^{'}(\omega) = \frac{-j}{(j\omega+\beta)} - \sum_{i=1}^{M} \frac{j\lambda_i}{1+\lambda_i(j\omega+\beta)} + jy + \sum_{i=1}^{M} \frac{[|\bar{b}_i|^2(-\lambda_i j)]}{(1+\lambda_i(j\omega+\beta))^2} \quad (37)$$

Thus, $s^{'}(j\omega)$ is found to be

$$s^{'}(j\omega) = \frac{-j}{(-\omega+\beta)} - \sum_{i=1}^{M} \frac{j\lambda_i}{1+\lambda_i(-\omega+\beta)} + jy - \sum_{i=1}^{M} \frac{j|\bar{b}_i|^2\lambda_i}{(1+\lambda_i(-\omega+\beta))^2} \quad (38)$$

where $\omega = j(\beta+p)$. Thus, we solve for p such that $p \in (-\infty, 0)$ by setting $s^{'}(j\omega)$ to zero as

$$-\frac{1}{(-\omega+\beta)} - \sum_{i=1}^{M} \frac{\lambda_i}{1+\lambda_i(-\omega+\beta)} + y - \sum_{i=1}^{M} \frac{|\bar{b}_i|^2\lambda_i}{(1+\lambda_i(-\omega+\beta))^2} = 0 \quad (39)$$

Equation (39) has a single real solution in the region $p \in (-\infty, 0)$ (which we denote as ω_0 such that $\omega_0 = j(\beta+p_0)$) irrespective of the values of λ_i and for any value of y (see Appendix B for a formal proof). Now, to apply the saddle point technique, we approximate $s(\omega)$ as

$$s(\omega) \approx s(\omega_0) + (\omega-\omega_0)s^{'}(\omega_0) + \frac{(\omega-\omega_0)^2}{2} s^{''}(\omega_0), \quad (40)$$

Thus, we can approximate the integration in (35) as

$$\begin{aligned} F_{Y_{nc}}(y) &\approx \frac{1}{2\pi} \int e^{s(\omega_0) + s^{'}(\omega_0)(\omega-\omega_0) + \frac{s^{''}(\omega_0)}{2}(\omega-\omega_0)^2} d\omega \\ &= \frac{1}{2\pi} e^{s(\omega_0)} \int e^{\frac{s^{''}(\omega_0)}{2}(\omega-\omega_0)^2} d\omega \\ &= \frac{1}{2\pi} e^{s(\omega_0)} \sqrt{\frac{2\pi}{|s^{''}(\omega_0)|}}, \end{aligned} \quad (41)$$

There are a few comments in order here. First, the higher order terms in the expansion of s around the saddle point can be treated performatively, and provide corrections to leading order $O(1/M)$. Hence, in the large M limit, the above result becomes exact.

Second, the above calculations provide an accurate analysis of the asymptotic behavior for the CDF. To analyze the CCDF it is more accurate to work directly with the Fourier expression for $1-u(y)$ given in (15) [4] The analysis can then be go on as above with the only difference being that $1/(j\omega+\beta)$ is replaced by $1/(-j\omega+\beta)$ and in all other terms $\beta \rightarrow -\beta$. In addition, now the saddle point solution is sought for $\omega \in j(\beta, \beta+\lambda_{min})$, where λ_{min} is the minimum positive eigenvalue of Λ.

Third, the asymptotic evaluation of the PDF is much simpler, because of the absence of the term $1/(j\omega+\beta)$. In this case one seeks a solution between $\omega - j\beta \in (-j\infty, j\lambda_{min})$ (or $\omega - j\beta \in (j\lambda_m, \lambda_{min})$ in the presence of negative eigenvalues.)

Finally, the saddle point solution provides a robust solution even for cases where a closed form expression for $F(y)$ exists when M is not small. This is so, because

[4] Alternatively one can deform the integration contour to go above the pole at $\omega = j\beta$ above.

as seen in (32) its calculation entails the summation of a large number (M) of terms with alternating sign which becomes problematic when one needs to obtain the tails of the distribution. Simulations show very good agreement with the above analysis both for the CDF and CCDF and are reported in Section IX.

VI. The Real Quadratic Form

In this section we address the case of real quadratic form (Y_r) where $\mathbf{h}$ is a real Gaussian random vector and $\mathbf{A}$ is a symmetric matrix of size M. Proceeding as before and using the inverse Fourier Transform of the unit step, we set up the CDF for real case as

$$F_{Y_r}(y) = \frac{1}{\sqrt{(2\pi)^M}} \int_{-\infty}^{\infty} e^{-\|\mathbf{h}\|^2} u(y - \|\mathbf{h}\|_{\mathbf{A}}^2) d\mathbf{h}$$
$$= \frac{1}{(2\pi)^{\frac{M}{2}+1}} \iint_{-\infty}^{\infty} e^{-(\|\tilde{\mathbf{h}}\|^2 + \|\tilde{\mathbf{h}}\|^2_{(j\omega+\beta)\boldsymbol{\Lambda}})} d\tilde{\mathbf{h}} \frac{e^{y(j\omega+\beta)}}{j\omega+\beta} d\omega \quad (42)$$

where we have used the eigenvalue decomposition of $\mathbf{A}$ by defining $\mathbf{A} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^T$ and using $\tilde{\mathbf{h}} = \mathbf{Q}^T\mathbf{h}$. Now, the inner integral in the above is nothing but the area under a real Gaussian PDF with zero mean and covariance $(\mathbf{I} + \boldsymbol{\Lambda}(j\omega + \beta))^{-1}$. Thus, using the Gaussian PDF for the real case, we can show that

$$F_{Y_r}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{y(j\omega+\beta)}}{j\omega+\beta} \frac{1}{\sqrt{|\left(\mathbf{I} + \boldsymbol{\Lambda}\left(j\omega+\beta\right)\right)|}} d\omega \quad (43)$$

The above integral representing the CDF of the real quadratic form has no closed form and the method of partial fraction is not applicable due to presence of the root in its expression. To approximate the CDF of the real quadratic form expressed by the above integral, we will again utilize the technique of the saddle point. We start by expressing the CDf as

$$F_{Y_r}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{|\left(\mathbf{I} + \boldsymbol{\Lambda}\left(j\omega+\beta\right)\right)|}} \frac{e^{y(j\omega+\beta)}}{jw+\beta} d\omega$$
$$= \frac{1}{2\pi} \int e^{s(\omega)} d\omega \quad (44)$$

where

$$s(\omega) = \ln\left[\frac{1}{\sqrt{|\left(\mathbf{I} + \mathbf{A}\left(j\omega+\beta\right)\right)|}} \frac{e^{y(j\omega+\beta)}}{j\omega+\beta}\right]$$
$$= -\frac{1}{2}\sum_{i=1}^{M} \ln\left[1 + \lambda_i(jw+\beta)\right] + y(jw+\beta) - \ln\left(jw+\beta\right) \quad (45)$$

Again employing the approximation for $s(\omega)$ given in (40) with a solution ω_0 to achieve[5]

$$F_{Y_r}(y) \approx \frac{1}{2\pi} \int e^{s(\omega_0) + s'(\omega_0)(\omega-\omega_0) + \frac{s''(\omega_0)}{2}(\omega-\omega_0)^2} dw$$
$$= \frac{1}{2\pi} e^{s(\omega_0)} \sqrt{\frac{2\pi}{|s''(\omega_0)|}} \quad (46)$$

[5]Note that one can prove the existence of such ω_0 using the same argument utilized for the general Hermitian case.

VII. Beyond Quadratic forms of Gaussian Variables

The technique we pursued in the last sections can be applied beyond quadratic forms of Gaussian random variables. In this section, we demonstrate how our approach can be extended to a non-Gaussian variables. Specifically, we show how it can be used to characterize the 1) ratios of quadratic forms, 2) joint distributions of quadratic forms, 3) quadratic forms in isotropically distributed random vectors, and 4) alternative formulation of the Q function given by Craig [50].

A. Distribution of a ratio of Quadratic Forms

There are many applications involving quadratic forms that are encountered in signal processing and communications. For example, the SINR in random beam forming developed in [12] and moments for analyzing the behavior of ϵ-NLMS algorithm [16], [17] appear as ratio of quadratic forms.

Let's apply the technique developed above to derive the CDF of such a ratio defined as

$$Y_d = \frac{\epsilon_1 + \|\mathbf{h}\|^2_{\mathbf{B}_1}}{\epsilon_2 + \|\mathbf{h}\|^2_{\mathbf{B}_2}} \quad (47)$$

where $\mathbf{B}_1$ and $\mathbf{B}_2$ are Hermitian matrices with $\mathbf{B}_2 \geq 0$. Our approach can be easily employed to evaluate the CDF of such ratios. Note that the probability $\Pr\left\{\frac{\epsilon_1 + \|\mathbf{h}\|^2_{\mathbf{B}_1}}{\epsilon_2 + \|\mathbf{h}\|^2_{\mathbf{B}_2}} \leq x\right\}$ can be equivalently written as $\Pr\left\{\|\mathbf{h}\|_{\mathbf{B}_1 - x\mathbf{B}_2} \leq \epsilon_2 x - \epsilon_1\right\}$. Hence, by employing the expression of (32), we can immediately write

$$F_X(x) = u(\epsilon_2 x - \epsilon_1) - \sum_{l=1}^{M} \frac{\lambda_l^M(x) e^{-\frac{\epsilon_2 x - \epsilon_1}{\lambda_l}}}{|\lambda_l(x)| \prod_{i=1, i\neq l}(\lambda_l(x) - \lambda_i(x))} u\left(\frac{\epsilon_2 x - \epsilon_1}{\lambda_l}\right) \quad (48)$$

Here $\lambda_i(x)$ $(i = 1, \ldots, M)$ are the eigenvalues of $\mathbf{B}_1 - x\mathbf{B}_2$ and hence are functions of[6] x.

B. Joint Distributions of Hermitian Quadratic Forms

In this section, we show how we can use the approach presented in previous sections to find the joint distribution of several quadratic forms. We demonstrate it here for two quadratic forms, although the insights can be easily extended to a larger number of quadratic forms. Let's consider the joint CDF of the quadratic forms $\|\mathbf{h}\|^2_{\mathbf{A}} \leq y_1$ and $\|\mathbf{h}\|^2_{\mathbf{B}} \leq y_2$,

$$F_{Y_{c_1}, Y_{c_2}}(y_1, y_2) = \Pr\left\{\|\mathbf{h}\|^2_{\mathbf{A}} \leq y_1, \|\mathbf{h}\|^2_{\mathbf{B}} \leq y_2\right\} \quad (49)$$

Such a joint CDF appears when one considers the joint distribution of SINRs. For example, in [59], the performance of random beamforming in MIMO broadcast channels is studied by evaluating the joint probability

[6]The expression (48) is valid assuming that the eigenvalues of $\mathbf{B}_1 - x\mathbf{B}_2$ are distinct for each x. If not, one needs to take care of multiplicities for those values of x for which the eigenvalues $\lambda_i(x)$ are repeated.

of two such SINRs. Thus, we demonstrate that how our approach can be extended to joint distribution case. By employing the approach presented in Section III, we can write the joint CDF in (49) as

$$F_{Y_{c_1},Y_{c_2}}(y_1,y_2) =$$
$$\frac{1}{4\pi^{M+2}}\int\int_{-\infty}^{\infty}\int e^{-\|\mathbf{h}\|^2_{\mathbf{C}}}d\mathbf{h}\frac{e^{y_1(j\omega_1+\beta_1)}}{(j\omega_1+\beta_1)}d\omega_1\frac{e^{y_2(j\omega_2+\beta_2)}}{(j\omega_2+\beta_2)}d\omega_2$$

where

$$\mathbf{C} = \mathbf{I} + (j\omega_1+\beta_1)\mathbf{A} + (j\omega_2+\beta_2)\mathbf{B} \tag{50}$$

Just as we did before, we can formally think of inner integral in the above as an integral of Gaussian PDF with covariance $\mathbf{C}^{-1}$. This allows us to integrate the $\mathbf{h}$-dependent part of the integral and we can write $F_{Y_{c_1},Y_{c_2}}(y_1,y_2)$ as

$$F_{Y_{c_1},Y_{c_2}}(y_1,y_2) =$$
$$\frac{1}{4\pi^2}\int\int\frac{1}{|(\mathbf{I}+(j\omega_1+\beta_1)\mathbf{A}+(j\omega_2+\beta_2)\mathbf{B})|}\times$$
$$\frac{e^{y_1(j\omega_1+\beta_1)}}{(j\omega_1+\beta_1)}d\omega_1\frac{e^{y_2(j\omega_2+\beta_2)}}{(j\omega_2+\beta_2)}d\omega_2 \tag{51}$$

In general, we can not evaluate this integral in closed form unless $\mathbf{A}$ and $\mathbf{B}$ are diagonal (or jointly diagonalizable by an orthonormal transformation). Under this assumption, the determinant in (51) can be easily expanded and the joint CDF takes the form

$$F_{Y_{c_1},Y_{c_2}}(y_1,y_2) =$$
$$\frac{1}{4\pi^2}\int\int\frac{1}{\prod_{i=1}^{M}(1+(j\omega_1+\beta_1)a_i+(j\omega_2+\beta_2)b_i)}\times$$
$$\frac{e^{y_1(j\omega_1+\beta_1)}}{(j\omega_1+\beta_1)}d\omega_1\frac{e^{y_2(j\omega_2+\beta_2)}}{(j\omega_2+\beta_2)}d\omega_2 \tag{52}$$

where a_i and b_i are the eigenvalues of the matrices $\mathbf{A}$ and $\mathbf{B}$, respectively. Now it is straightforward to evaluate this double integral. We consider the fraction that appears in (52) as a function of $j\omega_1+\beta_1$ and expand it in a partial fraction expansion. This results in $M+1$ terms (assuming that that non of the terms are repeated). Each of these terms can be integrated with respect to ω_1 to produce $M+1$ terms that are in turn partial fractions in $j\omega_2+\beta_2$. The same process is now repeated for the ω_2 variable, arriving finally at a closed form expression for the CDF.

Remark

Any two Hermitian matrices are jointly diagonalizable but not necessarily with an orthonormal transformation. In that case, we can show that we can integrate (52) with respect to one of the ω's in (52) but not necessarily with respect to the second one. When the transformation is orthonormal, (52) can be expressed in closed form as explained above. In the non central case, the saddle point analysis can hold here as well, where now one needs to look for a saddle point for two variables ω_1 and ω_2.

C. Indefinite Quadratic Forms in Isotropic Random Vectors

Isotropic vectors and matrices play an important role in the understanding the behavior of a communication system and characterizing its limits. For example the capacity achieving signal in a multiple antenna link (with no channel information at the transmitter and receiver) is the product of an isotropic matrix with an independent diagonal real matrix [60]. Similarly, in a point-to-multipoint broadcast scenario, isotropic vectors are used as beams that carry the signal information and achieve optimal capacity in the large number of users regime [12]. In this section, we demonstrate how our approach can be used to characterize the distribution of quadratic forms in isotropic random variables. An isotropically random unitary matrix $\mathbf{\Phi}$ is a matrix with orthogonal columns and whose distribution is invariant to a pre-multiplication by a unitary matrix [60], i.e., $p(\mathbf{\Phi}) = p(U\mathbf{\Phi})$. A column of $\mathbf{\Phi}$, $\boldsymbol{\phi}$ is called an isotropic vector and has the following marginal PDF

$$p(\boldsymbol{\phi}) = \frac{\Gamma(M)}{\pi^M}\delta(1-\|\boldsymbol{\phi}\|^2) \tag{53}$$

In this section, we characterize the CDF of a weighted norm of an isotropic vector given by

$$Y_i = \|\boldsymbol{\phi}\|^2_{\mathbf{A}} \tag{54}$$

where $\mathbf{A}$ is a Hermitian matrix. To evaluate the probability $P\{Y_i \leq y\}$, we can use equivalent representation for inequality $Y_i \leq y$ as $y - \|\boldsymbol{\phi}\|^2_A \geq 0$. Thus, the CDF of Y_i can be set up as

$$\begin{aligned} F_{Y_i}(y) &= \int p(\boldsymbol{\phi})u(y-\|\boldsymbol{\phi}\|^2_{\mathbf{A}})d\boldsymbol{\phi} \\ &= \frac{\Gamma(M)}{\pi^M}\int\delta(1-\|\boldsymbol{\phi}\|^2)u(y-\|\boldsymbol{\phi}\|^2_{\mathbf{A}})d\boldsymbol{\phi} \end{aligned} \tag{55}$$

Now, we employ the integral representation of the step function (14) and a similar representation of delta function [60]

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{x(\alpha+j\omega)}d\omega \tag{56}$$

to obtain[7]

$$\begin{aligned} F_{Y_i}(y) = \frac{\Gamma(M)}{4\pi^{M+2}}e^{\alpha}\int\int\int e^{-\boldsymbol{\phi}^H(\alpha I+A(j\omega_1+\beta)-j\omega_2 I)\boldsymbol{\phi}}d\boldsymbol{\phi} \\ \times e^{-j\omega_2}d\omega_2\frac{e^{(j\omega_1+\beta)y}}{(j\omega_1+\beta)}d\omega_1 \end{aligned} \tag{57}$$

By inspecting the inner integral, we note that it is similar to the Gaussian density integral which allow us to simplify $F_{Y_i}(y)$ as

$$F_{Y_i}(y) =$$
$$\frac{\Gamma(M)e^{\alpha}}{4\pi^{M+2}}\int\int\frac{e^{-j\omega_2}e^{(j\omega_1+\beta)y}d\omega_2 d\omega_1}{|\alpha\mathbf{I}+\mathbf{A}(j\omega_1+\beta)-j\omega_2\mathbf{I}|(j\omega_1+\beta)} \tag{58}$$

[7]The representation of delta function is valid for any $\alpha > 0$.

The determinant in (58) can be expressed in terms of the eigenvalues of $\mathbf{A}$ and expanded using partial fraction as[8]

$$\frac{1}{\prod_{i=1}^{M}\left(\alpha+\lambda_i(j\omega_1+\beta)-j\omega_2\right)} = \frac{1}{(j\omega_1+\beta)^{M-1}}\sum_{i=1}^{M}\frac{\eta_i}{(\alpha+\lambda_i(j\omega_1+\beta)-j\omega_2)} \quad (59)$$

where $\eta_i = \frac{1}{\prod_{k\neq i}(\lambda_k-\lambda_i)}$. We now use residue theory to evaluate the integral with respect to ω_2 as

$$\frac{1}{2\pi}\int d\omega_2 \frac{e^{-j\omega_2}}{\det\left(\alpha I+(j\omega_1+\beta)A-j\omega_2 I\right)} = \sum_{i=1}^{M}\eta_i e^{-\alpha-\lambda_i(j\omega_1+\beta)} \quad (60)$$

where α is chosen such that $\alpha+\lambda_i>0$. We can thus write

$$F_{Y_i}(y) = \frac{\Gamma(M)}{2\pi}\int_{-\infty}^{\infty}\sum_{i=1}^{M}\eta_i e^{-\lambda_i(j\omega_1+\beta)}\frac{e^{(j\omega_1+\beta)y}}{(j\omega_1+\beta)^M}d\omega_1 \quad (61)$$

Note that the CDF is now independent of the constant α as it should. Using residue theory again, we can show that

$$F_{Y_i}(y) = \sum_{i=1}^{M}\eta_i(y-\lambda_i)^{M-1}u(y-\lambda_i) \quad (62)$$

Remark
Just as in the Gaussian case, the same approach could be used to characterize the CDF of joint distribution of quadratic forms in isotropic random variables and the ratios of such forms.

D. Alternative Proof of Craig's Formula for Q function

In this section, we use the approach outlined in this paper to provide an alternative proof for the Craig's formula for the Q function [50] which is given in Equ. (8). This formulation is convenient because the argument of Q appears in the integrand as opposed to being part of the integration limits. These formulations were then used by Alouini and Simon in [61] to present a unified performance analysis of digital communications over generalized fading channels. In the following, we show how to derive these representations in a natural manner. The 1-dimensional Q function is the probability that the real Gaussian variable $Y\sim\mathcal{N}(0,1)$ satisfies

$$Q(x) = P\{y>x\}$$

Using the unit step function, this can be written as

$$Q(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\frac{e^{(y-x)(j\omega+\beta)}}{j\omega+\beta}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{y^2}{2}}dy$$

[8] If the λ_i's are not all distinct, we can proceed as we did in Section IV.

or upon completing the squares,

$$Q(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\frac{e^{-x(j\omega+\beta)}}{j\omega+\beta}\frac{1}{\sqrt{2\pi}}e^{\frac{1}{2}(j\omega+\beta)^2} \times\int_{-\infty}^{\infty}e^{-\frac{1}{2}(y^2-2y(j\omega+\beta)+(j\omega+\beta)^2)}dy$$

and by realizing that the inner integral sums out to unity (the integral is the area under the Gaussian PDF with "mean" $j\omega+\beta$ and variance 1)

$$Q(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-x(j\omega+\beta)+\frac{1}{2}(j\omega+\beta)^2}}{j\omega+\beta}d\omega$$

Now introduce the change of variables $\omega=\beta\tan\theta$, then $d\omega=\beta(1+\tan^2\theta)$ and $Q(x)$ becomes

$$Q(x) = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{-\beta(1+j\tan\theta)x+\frac{1}{2}\beta^2(1+j\tan\theta)^2}(1-j\tan\theta)d\theta$$
$$= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{(-\beta x+\frac{1}{2}\beta^2-\frac{1}{2}\beta^2\tan^2\theta)+j(-\beta x\tan\theta+\beta^2\tan\theta)}(1-j\tan\theta)d\theta$$

Now assume that $x>0$ and set $beta=x>0$. Then

$$Q(x) = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{-\frac{x^2}{2}(1+\tan^2\theta)}(1-j\tan\theta)d\theta$$

The imaginary part is odd and hence integrates to zero while the even part can be simplified to

$$Q(x) = \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}}e^{-\frac{x^2}{2\sin\theta}}d\theta$$

Contrast this approach with the original approach of Craig in [50] which requires the use of polar transformation. Our approach achieved the same result by a simple and elegant way. The same method can actually be used to rederive Craigs formulas for Q^2 and Q^3 in contrast to [50], [61], [62] which each had to use different method to derive alternative formulas for $Q^i (i=1,2,3)$. However, due to the lack of space, we limit the derivation here to the Q-function.

VIII. Outage Probability in Multiuser Beamforming for Correlated LOS Channels

In this section, we show how to use our approach to deal with the evaluation of outage probability in multiuser beamforming for correlated LOS channels. In this context, consider a multi-antenna Gaussian broadcast channel with one transmitter (base station) equipped with M antennas and K users (receivers) each equipped with one antenna. The transmitter chooses M orthonormal beam vectors $\boldsymbol{\phi}_m$ (each of size $M\times 1$). These beams are then used to transmit the symbols $s_1(t), s_2(t),\cdots,s_M(t)$ by constructing the $M\times 1$ transmitted vector $\mathbf{s}(t)=\sum_{m=1}^{M}\boldsymbol{\phi}_m(t)s_m(t)$. The received signal r_i at the i^{th} receiver is given by

$$r_i(t) = \sqrt{P_i}\mathbf{h}_i^H(t)\mathbf{s}(t)+v_i(t) \quad (63)$$

where P_i is the received SNR at the i^{th} receiver, $\mathbf{h}_i(t)$ is the $M\times 1$ channel vector associated between the base station and the i^{th} receiver such that $\mathbf{h}_i(t)$ is distributed as $\mathcal{CN}(\mathbf{b}, \mathbf{R})$. The term $v_i(t)$ represents additive complex Gaussian noise with zero mean and unit variance, that is, $v_i \sim \mathcal{CN}(0, 1)$. Since, transmitted symbols s_i are independently and identically distributed to different users, we have $E[\mathbf{s}\mathbf{s}^H] = \frac{1}{M}\mathbf{I}$ and $P_i = P, \forall\ i$. After incorporating expression for $\mathbf{s}(t)$ in (63) and dropping the time index t for simplicity, we can set up the received signal as $r_i = \sqrt{P}\sum_{m=1}^{M}\mathbf{h}_i^H\boldsymbol{\phi}_m s_m + v_i$. Thus, the SINR associated with the m^{th} beam at the i^{th} receiver is given by

$$\mathrm{SINR}_{i,m} = \frac{|\mathbf{h}_i^H\boldsymbol{\phi}_m|^2}{\frac{M}{P} + \sum_{k=1,k\neq m}^{M}|\mathbf{h}_i^H\boldsymbol{\phi}_k|^2} \tag{64}$$

which can be reformulated in indefinite quadratic form as

$$\mathrm{SINR}_{i,m} = \frac{||\mathbf{h}_i||^2_{\boldsymbol{\phi}_m\boldsymbol{\phi}_m^H}}{\frac{M}{P} + ||\mathbf{h}_i||^2_{\boldsymbol{\Phi}\boldsymbol{\Phi}^H - \boldsymbol{\phi}_m\boldsymbol{\phi}_m^H}} = \frac{||\mathbf{h}_i||^2_{\boldsymbol{\phi}_m\boldsymbol{\phi}_m^H}}{\frac{M}{P} + ||\mathbf{h}_i||^2_{\mathbf{I} - \boldsymbol{\phi}_m\boldsymbol{\phi}_m^H}} \tag{65}$$

where $\boldsymbol{\Phi} = [\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \cdots, \boldsymbol{\phi}_M]$ is an $M \times M$ matrix and $\boldsymbol{\Phi}\boldsymbol{\Phi}^H = \mathbf{I}$ as the beam vectors are orthonormal. Our aim is to derive the probability of outage that the $\mathrm{SINR}_{i,m}$ is less than certain threshold ξ, that is, $\Pr(\mathrm{SINR}_{i,m} < \xi)$. For that, we have investigated the scenario of zero mean case[9], that is, when the channel vector has zero mean vector, that is, $\mathbf{b} = \mathbf{0}$.

To proceed further, we formulate the $\Pr(\mathrm{SINR}_{i,m} < \xi)$ using the step function representation and the PDF of $\mathbf{h}_i$ as follows:

$$\begin{aligned}
&\Pr(\mathrm{SINR}_{i,m} < \xi) \\
&= \Pr\left(\frac{M}{P}\xi + ||\mathbf{h}_i||^2_{\xi\mathbf{I} - (\xi+1)\boldsymbol{\phi}_m\boldsymbol{\phi}_m^H} > 0\right) \\
&= \int\int \frac{e^{-||\mathbf{h}_i||^2_{\mathbf{R}^{-1} - \xi(j\omega+\beta)\mathbf{I} + (\xi+1)(j\omega+\beta)\boldsymbol{\phi}_m\boldsymbol{\phi}_m^H}}\, e^{\frac{M}{P}\xi(j\omega+\beta)} d\mathbf{h}_i\ d\omega}{2\pi^{M+1}|\mathbf{R}|(j\omega+\beta)} \\
&= \frac{1}{2\pi|\mathbf{R}|}\int \frac{(j\omega+\beta)^{-1} e^{\frac{M}{P}\xi(j\omega+\beta)}\ d\omega}{|\mathbf{R}^{-1} - \xi(j\omega+\beta)\mathbf{I} + (\xi+1)(j\omega+\beta)\boldsymbol{\phi}_m\boldsymbol{\phi}_m^H|}
\end{aligned} \tag{66}$$

The above integral can be solved using the approach given in Section IV. Thus, finally the probability of outage for zero mean case is found to be

$$\Pr(\mathrm{SINR}_{i,m} < \xi) = 1 - \frac{\lambda_M}{|\mathbf{R}|}\Pi_{i=1}^{M-1}\frac{\lambda_i\lambda_M}{\xi(\lambda_i - \lambda_M)} e^{-\frac{M}{P}\frac{\xi}{\lambda_M}} \tag{67}$$

where $\lambda_i (i = 1, \cdots, M)$ are the eigenvalues of matrix $\mathbf{A}$ which is defined as

$$\mathbf{A} = (1+\xi)\boldsymbol{\Lambda}^{1/2}\bar{\boldsymbol{\phi}}_m\bar{\boldsymbol{\phi}}_m^H\boldsymbol{\Lambda}^{1/2} - \xi\boldsymbol{\Lambda} \tag{68}$$

and $\bar{\boldsymbol{\phi}}_m = \mathbf{U}\boldsymbol{\phi}_m$ is the transformed version of $\boldsymbol{\phi}_m$ with $\mathbf{U}$ as the eigenvector matrix obtained from the eigenvalue decomposition of $\mathbf{R}^{-1}$, that is, $\mathbf{R}^{-1} = \mathbf{U}^H\boldsymbol{\Lambda}^{-1}\mathbf{U}$.

[9]The scenario of non-zero mean case be evaluated using the procedure outlined in Section V

IX. Simulation Results

In this section, simulation results are presented to validate our theoretical results[10]. The simulated results are obtained by averaging over 10000 independent experiments. Throughout the simulation, the random variables Y_{nc}, Y_c, Y_r, Y_d and Y_i are obtained via correlated circular complex Gaussian vector $\mathbf{h}$ of length $M = 4$, where $\mathbf{h}$ is generated with the correlation matrix with entries $R_{i,j} = \alpha_c^{|i-j|}$ with correlation factor[11] α_c $(0 < \alpha_c < 1)$. The aim of our simulations is to validate the derived analytical results. Specifically, the objectives of our simulation experiments is to validate the analytical expressions for the following tasks

1) The CDF of central case (i.e., CDF of Y_c) for distinct and repeated eigenvalues,
2) To compare the CDF of Y_{nc} obtained via our saddle point approach and the Raphaeli's series approximation [38],
3) The CDF of real case (i.e., CDF of Y_r) using saddle point technique,
4) The CDF of ratio of indefinite quadratic forms (i.e., CDF of Y_d),
5) The CDF of indefinite quadratic form in isotropic random vector (i.e., CDF of Y_i).
6) The Probability of outage in random beamforming (i.e., $\Pr(\mathrm{SINR}_{i,m} < \xi)$).

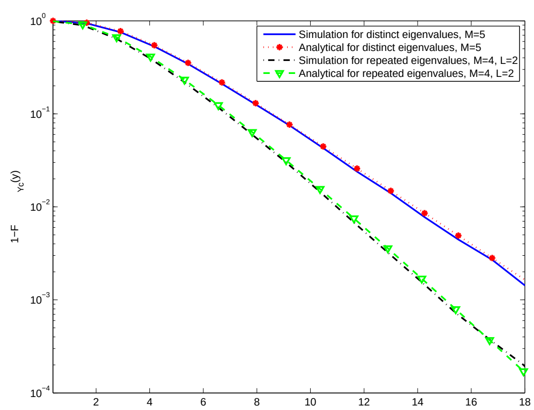

Fig. 1. The CCDF of Y_c for distinct and repeated eigenvalues.

We have plotted the CCDF (i.e., 1-CDF) in order to have better visualization of the results. In Fig. 1, the CCDF of central case (i.e., for Y_c) with distinct and repeated eigenvalues are plotted and compared with their respective simulation results. For the case of distinct eigenvalue, We set $M = 5$ for the distinct eigenvalue case and $M = 5$ with multiplicity $L = 2$ for the repeated eigenvalue case. An excellent match between theory and simulation can be observed from the reported results.

[10]All the MATLAB codes related to the simulations presented in this work are provided on the author′s website.

[11]The case $\alpha_c = 0$ corresponds to the white case while $\alpha_c = 1$ corresponds to the fully correlated case

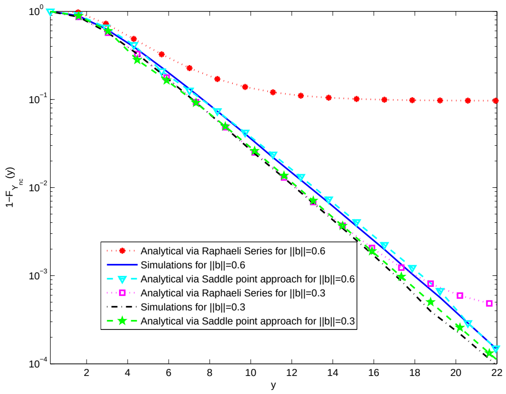

Fig. 2. The CCDF of Y_{nc} with $M = 4$ for $\|\mathbf{b}\| = 0.3$ and 0.6.

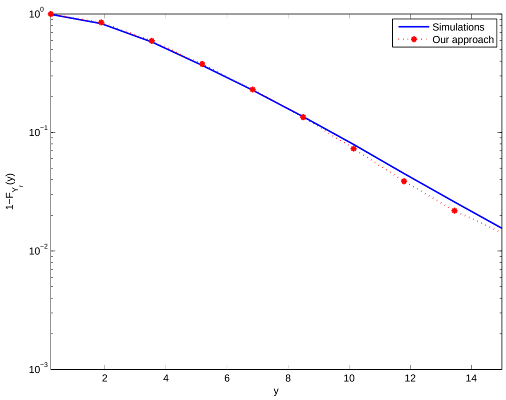

Fig. 3. The CCDF of Y_r for $M = 5$

Next, in Fig. 2, we compare the CCDF of Y_{nc} obtained via our approach (using saddle point technique) with the one obtained using the Raphaeli's series approximation [38] for two different values of mean vector $\mathbf{b}$, that is, for $\mathbf{b} = [.15\ .15\ .15\ .15]$ (i.e. $\|\mathbf{b}\|$ equals to 0.3) and $\mathbf{b} = [.3\ .3\ .3\ .3]$ (i.e. $\|\mathbf{b}\|$ equals to 0.6). It can be easily seen that the Raphaeli's series approximation works well only for lower values of y but it gives very poor estimate of the CCDF near tail especially for the second case[12]. On the other hand, our approach is good and consistent for both the cases. More importantly, if we contrast the computational complexity of the two approaches, our approach is much simpler as compared to that of the Raphaeli's approach as it requires multiple summations including an infinite summation [38]. The CCDF of the real quadratic form (Y_r) is investigated in Fig. 3 via saddle point approximation and again a very close match of analytical and simulation results is obtained. Next, the CCDF of a ratio of indefinite quadratic forms (Y_d) and isotropic random vector (Y_i) are compared with the one via simulation in Fig. 4 and Fig. 5, respectively. Finally, the probability of outage in random beamforming is compared via simulations for both zero mean and non-zero mean channel vector (with $\|\mathbf{b}\| = 2.5$) in Fig. 6. In summary, an excellent match between theory and simulation is observed for all the derived analytical results.

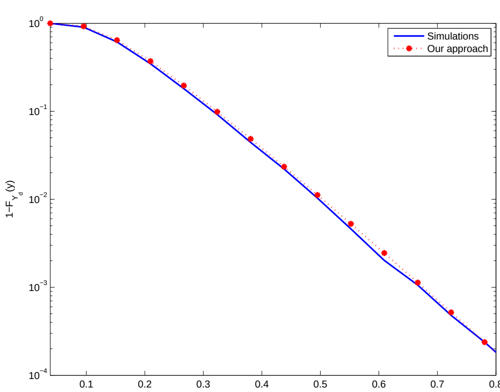

Fig. 4. The CCDF of Y_d for $M = 5$, $\epsilon_1 = 1$, $\epsilon_2 = 0.01$, $\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{I}$.

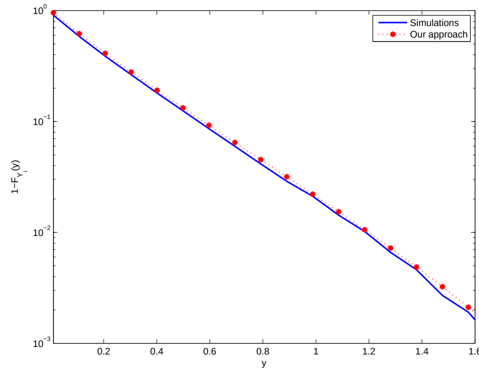

Fig. 5. The CCDF of Y_i for $M = 5$

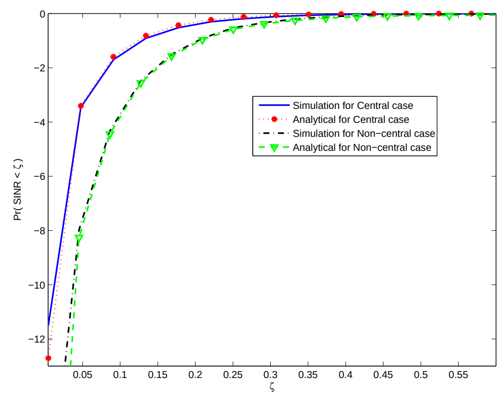

Fig. 6. Probability of outage for random beamforming (dB).

X. Conclusion

In this work, we provide a unified framework to characterize the statistical behavior (PDF and CDF) of quadratic forms in Gaussian random variables.

[12]In fact, the performance of the Raphaeli's series further degrades for larger $\|b\|$ even by increasing the number of terms in the series summation.

Our approach is unified in nature as it applies to definite/indefinite forms in Gaussian random variables (real or complex, central and non-central). The main idea is to replace the inequalities that defines the CDF in terms of unit step function and to represent the latter in terms of its Fourier transform. This allow us to integrate the effect of the Gaussian variable, leaving us with a one-dimensional integral that we can evaluate in closed form or approximate using the saddle point theorem. The saddle point approach can be directly extended to cases where the correlation matrix itself or even the mean vector $\mathbf{b}$ is random. We also show how our approach can be easily extended beyond quadratic forms such as ratios of quadratic forms, joint distribution of quadratic forms, and indefinite quadratic forms in isotropic random vectors. Simulation results support our theoretical developments.

APPENDIX A: Evaluation of Integral in Equ. (21)

By examining (21), we note that the inner integral looks like a Gaussian integral. Intuition suggests that this integral can be written as

$$\frac{1}{\pi^M}\int_{-\infty}^{\infty} e^{-(\tilde{\mathbf{h}}+\tilde{\mathbf{b}})^H\mathbf{B}(\tilde{\mathbf{h}}+\tilde{\mathbf{b}})}d\tilde{\mathbf{h}} = \frac{1}{\prod_{i=1}^{M}\left(1+\lambda_i(j\omega+\beta)\right)} = \frac{1}{|\mathbf{I}+\mathbf{\Lambda}(j\omega+\beta)|} \quad (69)$$

The above result can be verified easily. To see this, consider the integral in (69) which can be decomposed as

$$\frac{1}{\pi^M}\int_{-\infty}^{\infty} e^{-(\tilde{\mathbf{h}}+\tilde{\mathbf{b}})^H(\mathbf{I}+\mathbf{\Lambda}(j\omega+\beta))(\tilde{\mathbf{h}}+\tilde{\mathbf{b}})}d\tilde{\mathbf{h}} = \frac{1}{\pi^M}\prod_{i=1}^{M}\int_{-\infty}^{\infty} e^{-(1+\beta\lambda_i+\lambda_i j\omega)|\tilde{h}_i+\tilde{b}_i|^2}d\tilde{h}_i, \quad (70)$$

where λ_i is the i^{th} diagonal element in matrix Λ, $\tilde{h}_i$ is the i^{th} element in vector $\tilde{\mathbf{h}}$, and $\tilde{b}_i$ is the i^{th} element in vector $\tilde{\mathbf{b}}$. For each i, we can choose β such that $1+\beta\lambda_i>0$. With this choice of β, it is easy to see that [52]

$$\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-(1+\beta\lambda_i+\lambda_i j\omega)|\tilde{h}_i+\tilde{b}_i|^2}d\tilde{h}_i = \frac{1}{1+\lambda_i(j\omega+\beta)}$$

Thus, by using the above, we finally arrive at the result reported in (69).

APPENDIX B: Proof for the existence of the solution for (39)

To prove this, we define a function $g(w)$ as

$$g(w) = \frac{1}{(-\omega+\beta)} + \sum_{i=1}^{M}\frac{1}{\mu_i+(-w+\beta)} + \sum_{i=1}^{M}\frac{|\bar{b}_i|^2\mu_i}{\left[\mu_i+(-w+\beta)\right]^2} \quad (71)$$

where $\mu_i = \frac{1}{\lambda_i}$. Thus, it suffices to show that $\exists w_0$ s.t. $g(w_0)=y$ for any values of y and λ_i , $i=1\ldots M$. It is important to note that $\omega > j\beta$ for CCDF and $\omega < j\beta$ for CDF.

Let λ_1 be the largest positive eigenvalue and λ_M be the smallest negative eigenvalue (i.e. the largest negative eigenvalue in absolute value). If the matrix $\mathbf{A}$ is positive definite, i.e. the eigenvalues are all positive, the above result that establishes the existence of the saddle point still holds with some minor modifications in the proof. By studying the $g(\omega)$ asymptotically, it can be noted that

$$\lim_{w\to(\mu_1+\beta)^-} g(w) = +\infty \quad \text{and} \quad \lim_{w\to(\mu_M+\beta)^+} g(w) = -\infty \quad (72)$$

Thus, based on this and the fact that $g(w)$ is continuous in the interval $(\mu_M+\beta, \mu_1+\beta)$; we conclude that given any value of y, $\exists$ a value $w_0 \in (\mu_M+\beta, \mu_1+\beta)$ where $g(.)$ intersects the horizontal line y. In fact, we consider the interval $(-\infty, \mu_1+\beta)$ which results in

$$\lim_{w\to-\infty} g(w) = 0 \quad \text{and} \quad \lim_{w\to(\mu_1+\beta)^-} g(w) = +\infty \quad (73)$$

Again, this together with the continuity of $g(w)$ on the interval $(-\infty, \mu_1+\beta)$ guarantees the existence of a value $w_0 \in (-\infty, \mu_1+\beta)$ where $g(.)$ intersects the horizontal line y . This is the case since $\mathbf{A}$ being positive definite necessitates that $y>0$.

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Application	References
Power allocation and optimal combining for multiple antennas	[63], [64]
Performance analysis of beamforming techniques	[65]–[66]
Outage analysis	[67], [68]
Performance analysis of sensor networks	[69]
Performance analysis of the adaptive filtering algorithms	[15]–[17],[70]

TABLE I

LIST OF SOME SPECIFIC APPLICATIONS OF INDEFINITE QUADRATIC FORMS IN COMMUNICATIONS.