Opportunistic Random Beamforming with Precoding for Spatially Correlated Channels

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***Abstract*— It has been showed recently that dirty paper coding (DPC) achieves optimum sum-rate capacity in a multi-antenna broadcast channel with full channel state information at the transmitter. With only partial feedback, random beamforming (RBF) is able to match the sumrate of DPC for large number of users. However, in the presence of spatial correlation, RBF incurs an SNR hit as compared to DPC. In this letter, we explore precoding techniques to reduce the effect of correlation on RBF. We thus derive the optimum precoding matrix that minimizes the rate gap between DPC and RBF. Given the numerical complexity involved in calculating the optimum precoder, we derive approximate precoding matrices that are close in performance to the optimum precoder.**

I. Introduction

Consider a broadcast scenario where a base station with M antennas is to broadcast to n users each equipped with one antenna. Dirty paper coding (DPC) is a technique that maximizes the sum rate in this scenario [2], [1]. However, it requires full channel state information of all users at the base station. On the other hand, random beamforming (RBF) requires only SINR feedback and is able to match the sum rate of DPC for high number of users at a rate of [3]

$$\gamma = M \log\log n + M \log \frac{P}{M}$$

where P is the total power transmitted. For spatially correlated channels, however, the sumrate incurs a hit and it scales as

$$\gamma = M \log\log n + M \log \frac{P}{M} - M \log c \qquad (1)$$

where $c \geq 1$ is constant that depends on the eigenvalues of the channel correlation matrix R and the multiuser broadcast technique used. Specifically, the hit in the DPC case is given by $\log c = \log \det(R)^{-\frac{1}{M}}$ and for random beamforming, the hit is given by ($\|\phi\|^2_{R^{-1}} \triangleq \phi^* R^{-1}\phi$)

$$\log c = E \log \|\phi\|^2_{R^{-1}} \qquad (2)$$

where ϕ is an isotropic random beam vector.

The aim of this paper is to explore precoding techniques that will improve the performance of RBF and reduce the gap between RBF and DPC in the correlated channel case. The paper is organized as follows. After introducing the channel model, we introduce random beamforming with precoding and derive the optimum precoding matrix in Section III. Since the optimum precoder is difficult to evaluate, we derive in Section IV three approximate precoding matrices. We finally verify our results with simulations and present our conclusions.

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II. Channel and Signal Model

The signal received by the ith user is given by

$$Y_i(t) = H_i S(t) + W_i, \qquad i = 1, \ldots, n, \qquad (3)$$

where the $M \times 1$ transmit vector S is subject to the power constraint $E\{S^* S\} \leq P$ and where $W_i \sim \mathcal{C}N(0,1)$ is the additive noise. The channel H_i is a $1 \times M$ complex vector, distributed as $CN(0,R)$ independently across users. The covariance matrix R admits the eignevalue decomposition $Q_R \Lambda Q_R^*$. We also assume that $tr(R) = M$ In our simulations, we will use covariance matrices controlled by a control parameter α and taking the form (for $M = 2$ and $M = 3$ antennas)

$$R_2 = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix} \quad R_3 = \begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix} \qquad (4)$$

III. Random Beamforming with Precoding

To counter the effect of channel correlation, we introduce beamforming with precoding where the transmitter sends αAS instead of sending S. By requiring that $\alpha^2 \leq \frac{M}{\mathrm{tr}(A^*A)}$, we maintain a power constraint of P on the input. The input/output equation for this new choice of input reads

$$Y_i = \alpha H_i A S(t) + W_i$$

In other words, we are using the familiar RBF with the *effective* channel

$$\tilde{H}_i = \alpha H_i A$$

which exhibits a correlation of $\alpha^2 \tilde{R} = \alpha^2 A^* R A$. In light of (1)-(2), we see that RBF with precoding yields the sum-rate

$$\begin{aligned} \gamma_{\mathrm{PC}} &= M \log\log n + M \log \frac{P}{M} - M E \log \|\phi\|^2_{\frac{1}{\alpha^2}\tilde{R}^{-1}} \\ &= M \log\log n + M \log \frac{P}{M} - h(A) \qquad (5) \end{aligned}$$

where $h(A)$ is the hit incurred by using a precoding matrix A

$$h(A) = M \log \frac{\mathrm{tr}(A^*A)}{M} + M E \log \|\phi\|^2_{\tilde{R}^{-1}} \qquad (6)$$

where in arriving at (5) and (6), we used the fact that the choice $\alpha^2 = \frac{M}{tr(A^*A)}$ will maximize the sum-rate. The following lemma shows that the optimum A has a special structure.

Lemma: The optimum precoding matrix A_{opt} can be written as

$$A_{opt} = Q_{Aopt} D^{\frac{1}{2}}_{Aopt}$$

where Q_{Aopt} is an orthonormal matrix and $D^{\frac{1}{2}}_{Aopt}$ is a diagonal matrix with positive entries.

Proof: Consider the precoding hit $h(A)$ in (6) and let $\tilde{R} = \tilde{Q}\tilde{\Lambda}\tilde{Q}^*$ denote the eigenvalue decomposition of $\tilde{R}$. Since ϕ is isotropic, it is invariant under multiplication by the orthonormal matrix $\tilde{Q}$. Thus[1],

$$E\|\phi\|^2_{\tilde{R}^{-1}} \quad = \quad E\|\phi\|^2_{\tilde{Q}\tilde{\Lambda}^{-1}\tilde{Q}^*} = E\|\tilde{Q}^*\phi\|^2_{\tilde{\Lambda}^{-1}} = E\|\phi\|^2_{\tilde{\Lambda}^{-1}}$$

Hence, the hit can be written as

$$h(A) = M\log\frac{\mathrm{tr}(A^*A)}{M} + ME\log\|\phi\|^2_{\tilde{\Lambda}^{-1}}$$

Now the first term of the hit depends on $\mathrm{tr}(A^*A)$ and hence $\mathrm{tr}(AA^*)$. The second term depends on the eigenvalues of $\tilde{R}$, i.e. of A^*RA, or equivalently the eigenvalues of RAA^*. So both terms of the hit are determined by AA^*. One choice of the optimum matrix A_{opt} is thus $A_{opt} = Q_{A_{opt}}D^{\frac{1}{2}}_{A_{opt}}$ where $Q_{A_{opt}}$ is orthonormal and $D_{A_{opt}}$ is diagonal with positive entries. This proves the lemma.

A. Determining Q_{opt}

An intuitive choice is to set $Q_{opt} = Q_R$. In the following, using an approach inspired by [5], we show that this choice is actually optimum. To this end, let Π_l be a diagonal matrix with all $1'$s on the diagonal except for a -1 at the lth entry and define $A_l = Q\Pi_l D^{\frac{1}{2}}$. This induces the effective correlation $\tilde{R}_l$. The hit that results by using either of the precoding matrices A or A_l is the same. To see this, note that $\mathrm{tr}(A^*A) = \mathrm{tr}(A_l^*A_l) = \mathrm{Tr}(D)$. Moreover,

$$\|\phi\|^2_{\tilde{R}_l^{-1}} = \|\phi\|^2_{D^{-\frac{1}{2}}\Pi_l Q R^{-1} Q^* \Pi_l D^{-\frac{1}{2}}} = \|\Pi_l D^{-\frac{1}{2}}\phi\|^2_{QRQ^*}$$

Note however that the distribution of ϕ is unchanged by the changing the sign of the lth entry. Hence,

$$\begin{aligned} E\log\|\phi\|^2_{\tilde{R}_l^{-1}} &= E\log\|\Pi_l D^{-\frac{1}{2}}\phi\|^2_{QRQ^*} \\ &= E\log\|D^{-\frac{1}{2}}\phi\|^2_{QRQ^*} = E\log\|\phi\|^2_{\tilde{R}^{-1}} \end{aligned}$$

Thus, both terms of the hits are the same and $h(\tilde{A}) = h(A)$. Using Jensen's inequality, we can show that

$$\frac{1}{2}\log\|\phi\|^2_{\tilde{R}^{-1}} + \frac{1}{2}\log\|\phi\|^2_{\tilde{R}_l^{-1}} \geq \log\|\phi\|^2_{(\frac{1}{2}\tilde{R}+\frac{1}{2}\tilde{R}_l)^{-1}}$$

It thus follows that

$$\begin{aligned} h(A) &= \frac{1}{2}h(A) + \frac{1}{2}h(A_l) \\ &\geq M\log\frac{\mathrm{tr}(D)}{M} + ME\log\|\phi\|^2_{(\frac{1}{2}\tilde{R}+\frac{1}{2}\tilde{R}_l)^{-1}} \end{aligned}$$

Note that the weight matrix above can be rewritten as

$$\frac{1}{2}\tilde{R} + \frac{1}{2}\tilde{R}_l = \frac{1}{2}Q^*RQ + \frac{1}{2}\Pi_l Q^* R^{-1} Q \Pi_l$$

From the right side, we see that the weight matrix has entries entries equal to those of Q^*RQ *except* the off diagonals lying on the lth column or lth row which are zero. This argument can be repeated for $l = 1, \ldots, M$. Hence, nulling the off diagonal entries of QRQ^* can only reduce the hit. Thus, QRQ^* should be diagonal, i.e. $Q = Q_R$.

B. Determining D_{opt}

We have so far established that $A_{\mathrm{opt}} = Q_R D^{\frac{1}{2}}_{opt}$ where D_{opt} is a diagonal matrix to be determined. The hit in this case is given by

$$h(A_{\mathrm{opt}}) = M\log\frac{\mathrm{tr}(D_{\mathrm{opt}})}{M} + E\log\|\phi\|^2_{D^{-1}_{\mathrm{opt}}\Lambda^{-1}}$$

Taking the derivative with respect to ith diagonal element of D_{opt}, d_i and setting it to zero, yields

$$\frac{1}{d_i}E\left[\frac{\frac{1}{d_i\lambda_i}|\phi(i)|^2}{\|\phi\|^2_{D^{-1}_{\mathrm{opt}}\Lambda^{-1}}}\right] = \frac{1}{tr(D_{\mathrm{opt}})} \qquad (7)$$

where $\phi(i)$ is the ith element of ϕ and where in arriving at (7), we exchanged the differentiation and expectation operations. Thus, we have a set of M implicit equations for $d_1, d_2, \ldots, d_M$. We can solve these equations numerically provided we first obtain the expectation of the random variable Z_1 that appears in (7). We can show (see [4]) that for diagonal matrices B and C, the CDF of the more general random variable $Z = \frac{\|\phi\|^2_B}{\|\phi\|^2_C}$ is given by[2]

$$F_Z(x) = \sum_{i=1}^{M}\frac{(a_i - b_i x)^{M-1}}{\prod_{k\neq i}((a_k - a_i) - (b_k - b_i)x)}u(-a_i + b_i x)$$

where $u(\dot{})$ is the unit step function. By setting $B = \mathrm{diag}(0, \ldots, \frac{1}{d_i\lambda_i}, \ldots, 0)$ and $C = D^{-1}\Lambda^{-1}$, we obtain the CDF of Z_1. With the support of Z_1 over the interval $(0,1)$, it is expectation is given by

$$E[Z_1] = \int_0^1 (1 - F_{Z_1}(z_1))dz_1$$

IV. Approximate Precoding Matrices

As seen above, to obtain the optimum precoding matrix, we need to simultaneously solve M nonlinear equations. We thus derive in this section some approximate precoding matrices. An intuitive choice of the precoder is the zero forcing one $A_{\mathrm{ZF}} = Q_R\Lambda^{-\frac{1}{2}}$, which produces the hit $h_{\mathrm{ZF}} = M\log\frac{tr(R^{-1})}{M}$. This performs worse than RBF as demonstrated by simulations. The MMSE precoder $A_{\mathrm{MMSE}} = Q_R(\Lambda + \beta I)^{-\frac{1}{2}}$ is a special case of the optimum precoder that requires a 1-dimensional optimization. It is easy that β is obtained by solving one fixed point equation

$$\frac{\mathrm{tr}(\Lambda + \beta I)^{-2}}{\mathrm{tr}(\Lambda + \beta I)^{-1}} = E\left(\frac{1}{\beta + \frac{1}{\|\phi\|^2_{\Lambda^{-1}}}}\right) \qquad (8)$$

The third precoder is obtained by minimizing an upper bound on the hit. To this end, note that the the difficult part in minimizing the hit is the term that depends on ϕ. So we rewrite this hit as

$$\begin{aligned} h(A) &= M\log\frac{\mathrm{tr}(A^*A)}{M} + ME\log\|\phi\|^2_{(A^*RA)^{-1}} \\ &= M\log\frac{\mathrm{tr}(A^*A)}{M} + M\log\mathrm{tr}((A^*RA)^{-1}) \\ &+ ME\log\|\phi\|^2_{\frac{(A^*RA)^{-1}}{\mathrm{tr}(A^*RA)^{-1}}} \end{aligned}$$

[1] It is easy to see that for any matrices G and F and vector a, $\|a\|^2_{G^*FG} = \|Ga\|^2_F$.

[2] In [4], we derived the CDF $\|\phi\|^2_A$. The derivation of a ratio of such quantities is more challenging but is omitted for brevity.

We now minimize the sum of the first two terms of the hit and ignore the 3rd term. There are two justifications for doing so. Thus, note that the first two terms constitute an upper bound on the hit because

$$\log \|\phi\|^2_{\frac{(A^*RA)^{-1}}{\mathrm{tr}(A^*RA)^{-1}}} = \log \|\phi\|^2_{\frac{(\tilde{\Lambda})^{-1}}{\mathrm{tr}(\tilde{\Lambda}^{-1})}} \leq \log \|\phi\|^2 \frac{\mathrm{tr}(\tilde{\Lambda}^{-1})}{\mathrm{tr}(\tilde{\Lambda}^{-1})} = 0$$

where $\tilde{\Lambda}$ is the diagonal matrix of eigenvalues of A^*RA.

$$h(A) = M \log \frac{\mathrm{tr}(A^*A)}{M} + \log tr(A^*RA)^{-1}$$

Taking the first derivative with respect to A and setting the result to zero yields $\frac{2}{\mathrm{tr}(A^*A)} A = \frac{2}{\mathrm{tr}(A^*RA)^{-2}} RA(A^*RA)^{-1}$, or

$$AA^*RAA^* = \frac{\mathrm{tr}(A^*A)}{\mathrm{tr}(A^*RA)^{-1}} I \qquad (9)$$

So AA^* is a left and a right inverse of R. Using the fact that $R = Q_R \Lambda Q_R^*$, we can show that the following choice satisfies (9).

$$A_{\mathrm{Appx}} = Q_R \Lambda^{-1/4}$$

V. Simulations

We consider a broadcast scenario with a base station having $M = 2$ and $M = 3$ antennas. The channels exhibit correlations matrices (4) parameterized by $0 \leq \alpha < 1$. We evaluate in Figure 1 the sum-rate of RBF and RBF with precoding (optimum, MMSE, ZF, and approximate) for $\alpha = .5$ and $M = 2$. Figures 1 and 2 show the hit incurred by these techniques for various degrees of correlation for $M = 2$ and $M = 3$, respectively. We note that optimum precoding outperforms other precoding techniques (as expected) closely followed by MMSE precoder while zero-forcing precoding is inferior to RBF.

VI. Conclusion

In this paper, we considered random beamforming in a spatially correlated regime. While RFB matches DPC for uncorrelated channels (in the large number of users regime), it incurs an SNR hit in the presence of correlation. The paper suggested precoding techniques as a way to counter the effect of correlation. Specifically, we derived the optimum precoder and three approximate precoders. Apart from zero forcing, the precoders obtained all outperform RBF and manage to reduce its gap with DPC.

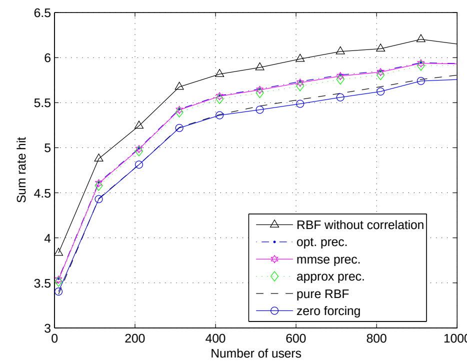

Fig. 1. Sum-rate versus the number of users in a system with M = 2, P = 10 and α = 0.5

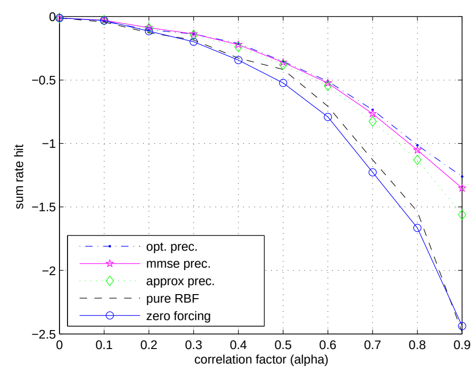

Fig. 2. Sum-rate loss versus correlation factor α in a system with M = 2, P=10 and n=400

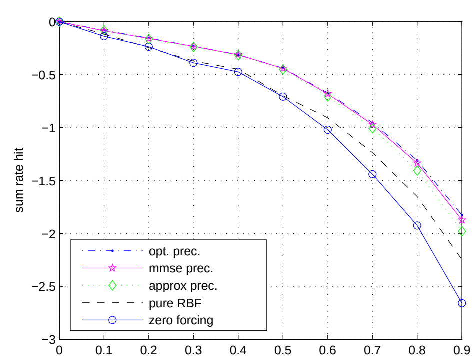

Fig. 3. Sum-rate loss versus correlation factor α in a system with M = 3, P=10 and n=100

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