Opportunistic Beamforming with Precoding for Spatially Correlated Channels

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***Abstract*—Random beamforming (RBF) exploits multiuser diversity to increase the sum-rate capacity of MIMO broadcast channels. However, in the presence of spatial correlation between the downlink channels, multiuser diversity can not be exploited and the sum-rate suffers a signal to noise (SNR) hit. In this paper, we explore precoding techniques that minimize this hit. Basically, we derive an optimum and an approximate precoding matrix that minimizes the sum-rate hit of RBF. As a by product, we introduce a technique that evaluates the cumulative distribution function (CDF) of weighted norms of Gaussian random variables.**

I. Introduction

Multiple antennas in multiuser systems have been introduced as an effective means to boost wireless system capacity. While dirty paper coding (DPC) is known for achieving the capacity region in a broadcast scenario, it requires full feedback and it is computationally expensive [1]. Other less expensive techniques like random beamforming were able to capture most of the DPC capacity with less feedback requirements [2]. In a large user regime, the sumrate of RBF and DPC coincide at

$$R = M \log \log n + \log \frac{P}{M} + o(1)$$

where P is the transmitted power, M is the number of transmit antennas and n is the number of users. In the presence of spatial correlation between the users' channels, the sum-rate capacity experiences a hit and becomes,

$$R = M \log \log n + \log \frac{P}{M} + \log c + o(1)$$

where $\log c$ represents the hit and $c \leq 1$.

In this paper, we investigate different precoding techniques that minimize the sum-rate hit in the presence of spatial correlation. The paper is organized as follows. After the introduction in section I, we introduce the channel model in section II. Random beamforming with precoding techniques are reviewed in section III and in section IV we show our simulations results followed by our conclusions in section V.

II. Channel Model

We consider a multi-antenna Gaussian broadcast channel with n receivers equipped with one antenna and a transmitter with M antennas. The received signal at the kth user is expressed as

$$Y_k(t) = \sqrt{P} H_k S(t) + W_k, \qquad k = 1, \ldots, n, \tag{1}$$

where k denotes a user index. $S(t)$ denotes transmitted symbol and satisfies the power constraint $E\{S^* S\} \leq P$. The channel matrix H_i consists of complex Gaussian random variables $CN(0, R)$ and W_k is the additive complex Gaussian noise with $CN(0, 1)$. The covariance matrix R is a measure of the spatial correlation and is assumed to be non-singular with $tr(R) = M$.

III. Random Beamforming with Precoding

In the presence of spatial correlation, we can precode the transmitted symbol with a general matrix A before beamforming, i.e.transmit $\alpha A S(t)$. The parameter α satisfies the power constraint ($\alpha = \frac{M}{tr(AA^*)}$) and the sum-rate eventually becomes

$$R_{Prec} = M \log \log n + M \log \frac{P}{M} - h_{Prec}$$

with a hit of

$$h_{\text{Prec}} = M \log \frac{tr(AA^*)}{M} + M E \log \left(\|\phi_m\|^2_{\tilde{\Lambda}^{-1}} \right). \tag{2}$$

It should be noted that $\tilde{\Lambda}$ is constructed from the eigenvalues of $\tilde{R}$ and the effective channel gain is $\tilde{H}_k = A H_k$. Finding the optimum precoding matrix A_{opt} is challenging, but one can show that the optimum precoding matrix takes the following form

$$A_{\text{opt}} = Q_{Aopt} D_{Aopt}$$

where Q_{Aopt} is orthonormal and D_{Aopt} is a diagonal matrix with positive entries [1]. The proof of the above expression is straight forward and for brevity we omit it here. Finding Q_{Aopt} and D_{Aopt} is not easy. An intuitive choice would be to set $Q_{Aopt} = Q_R$ and optimize over D_{Aopt}. In the following sections, we examine various choices of the diagonal matrix.

A. Random Beamforming with Zero Forcing

A natural choice of the precoding matrix is one which cancels the effect of the correlation, i.e.

$$\boxed{A_{\text{ZF}} = Q_R \Lambda_R^{-\frac{1}{2}}}$$

From (2), this choice would results in the following hit

$$\boxed{h_{\text{ZF}} = M \log \frac{tr(R^{-1})}{M}}$$

[1] It is shown in [6]that this intuitive choice is actually optimum

B. Random Beamforming with MMSE Precoding

The zero-forcing solution did not optimizes the sum-rate as it invested most of the input power to take care of the minimum eigenvalue. In this subsection, we consider the MMSE precoding matrix to be

$$\boxed{A_{\text{MMSE}} = Q_R(\Lambda + \beta I)^{-\frac{1}{2}}}$$

where β is a constant that we optimize to get the minimum hit. The hit in this case is given by

$$\boxed{\begin{aligned} h_{\text{MMSE}} &= M \log \frac{Tr(\Lambda + \beta I)^{-1}}{M} \\ &+ ME \log\left(1 + \beta \|\Phi_m\|^2_{\Lambda^{-1}}\right)(3) \end{aligned}}$$

To find the optimum β, we differentiate (3) with respect to β and get the following equation

$$\frac{Tr(\Lambda + \beta^* I)^{-2}}{Tr(\Lambda + \beta^* I)^{-1}} = E\left(\frac{1}{\beta + \frac{1}{\|\Phi_m\|^2_{\Lambda^{-1}}}}\right) \tag{4}$$

To solve this implicit equation, we need to find the expectation in (4) and hence the CDF of $\frac{1}{\|\phi_m\|^2_{\Lambda^{-1}}}$. In the Appendix, we evaluate the CDF of the more general form $Z = \frac{\|\phi\|^2_B}{\|\phi\|^2_C}$. By setting $B = I$ and $C = \Lambda^{-1}$, we get the desired CDF which turns out to be $G(x) = 1 - \sum_i \eta_i \left(\frac{1}{x} - \frac{1}{\lambda_i}\right)^{M-1} u\left(1 - \frac{x}{\lambda_i}\right)$, where $\eta_i = \frac{1}{\prod_{j \neq i} \frac{1}{\lambda_j} - \frac{1}{\lambda_i}}$, $\lambda_M (\lambda_M \geq \cdots \geq \lambda_1 > 0)$ are the eigenvalues of R and $u(x)$ is the unit step function. Thus, the expectation in (4) is given by

$$E\left(\frac{1}{\beta + \frac{1}{\|\Phi_m\|^2_{\Lambda^{-1}}}}\right) = \frac{1}{1 + \lambda_M} + \int_{\lambda_1}^{\lambda_M} \frac{1}{(\beta + x)^2} G(x)dx$$

C. Random Beamforming with Diagonal Precoding

We can try to improve MMSE precoding by using a more general precoding where we optimize over M unknowns instead of just one unknown (β). As shown in [6], this will yield the optimum precoding matrix. Specifically, we set

$$\boxed{A_{\text{Diag}} = Q_R D^{\frac{1}{2}}}$$

where D is a diagonal matrix with positive entries to be determined. This will result in the following hit

$$\boxed{h_{\text{Diag}} = M \log \frac{tr(D)}{M} + E \log \|\phi\|^2_{D^{-1}\Lambda^{-1}}}$$

Thus, we have a set of M parameters, $d_1, d_2, \ldots, d_M$ that we need to optimize. By taking the derivative with respect to the ith diagonal d_i element and setting it to zero, we obtain

$$\frac{1}{d_i} E\left[\frac{\frac{1}{d_i \lambda_i}|\phi(i)|^2}{\|\phi\|^2_{D^{-1}\Lambda^{-1}}}\right] = \frac{1}{tr(D)} \tag{5}$$

To solve for d_i, we need to find the expectation that appears in (5). This can be deduced from the CDF that is derived in the Appendix by setting $B = \text{diag}(0, \ldots, \frac{1}{d_i \lambda_i}, \ldots, 0)$ and $C = D^{-1}\Lambda^{-1}$. This allows us to evaluate the expression in (5) in closed form directly from the CDF as shown below:

$$E[Z_1] = \int_0^1 (1 - F_{Z_1}(z_1))dz_1.$$

D. An approximate Precoding Matrix

As proved in [6], the matrix $A_{\text{Diag}} = Q_R D^{\frac{1}{2}}$ is the optimum precoding matrix but this requires solving M non-linear equations. In this section we derive a simple and an approximate precoding matrix that avoids this. To this end, note that the difficult part in minimizing the hit is the term that depends on ϕ_m. So we rewrite the hit as

$$\begin{aligned} h(t) &= M \log \frac{tr(A^*A)}{M} + ME \log \|\phi\|^2_{(A^*RA)^{-1}} \\ &= M \log \frac{tr(A^*A)}{M} + M \log tr((A^*RA)^{-1}) \\ &+ ME \log \|\phi\|^2_{\frac{(A^*RA)^{-1}}{tr(A^*RA)^{-1}}} \end{aligned} \tag{6}$$

We now minimize the sum of the first two terms of the hit and ignore the 3rd term. There are two justifications for doing so

1) The first two terms constitute an upper bound on the hit. To see this, note that

$$\begin{aligned} \log \|\phi\|^2_{\frac{(A^*RA)^{-1}}{tr(A^*RA)^{-1}}} &= E \log \|\phi\|^2_{\frac{(\tilde{\Lambda})^{-1}}{tr(\tilde{\Lambda}^{-1})}} \\ &\leq \log \|\phi\|^2 \frac{tr(\tilde{\Lambda}^{-1})}{tr(\tilde{\Lambda}^{-1})} = 0 \end{aligned}$$

where $\tilde{\Lambda}$ is the diagonal matrix of eigenvalues of A^*RA.

2) Another justification is to consider the term $\|\phi\|^2_{\frac{(\tilde{\Lambda})^{-1}}{tr(\tilde{\Lambda}^{-1})}}$ as the squared dot product of two unit norm vectors ϕ and $c = \frac{diag(\tilde{\Lambda})^{-\frac{1}{2}}}{\sqrt{tr(\tilde{\Lambda}^{-1})}}$. This squared dot product can be approximated as the squared dot product of two *uniformly* distributed unit norm vectors υ which has a CDF [5]

$$F(\upsilon) = 1 - (1 - \upsilon)^{M-1} \qquad \upsilon \in [0, 1]$$

Hence, we can approximate the expectation in (6) as

$$E \log \|\phi\|^2_{\frac{(\tilde{\Lambda})^{-1}}{tr(\tilde{\Lambda}^{-1})}} \simeq E[\log \upsilon] \tag{7}$$

$$= -\sum_{m=1}^{M-1} \frac{1}{m} \tag{8}$$

Figure (1) plots the two sides of (7) for various values of the correlation coefficient α and shows that they are almost the same.

Thus up to an almost constant term, the hit is given by

$$h(t) = M \log(\frac{tr(A^*A)}{M} + tr((A^*RA)^{-1}))$$

To minimize the hit we take the first derivative with respect to A, equate it to zero and use the eigenvalue decomposition, $R = Q_R \Lambda_R Q_R^*$, to get

$$\boxed{A_{\text{Appx}} = Q_R \Lambda_R^{-1/4}}$$

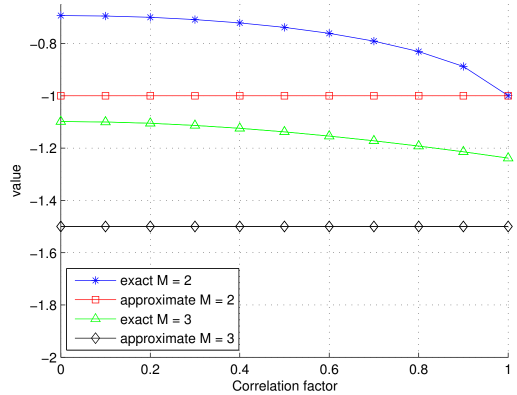

Fig. 1. Comparison between the exact and approximate terms in Equ. (7).

The resulting hit for this choice of A is given by

$$\boxed{h_{\text{Appx}} = M \log \frac{tr(\Lambda^{-\frac{1}{2}})}{M} + M \log \|\phi\|^2_{\Lambda^{-\frac{1}{2}}}}$$

IV. Simulation Results

We consider a base station having $M = 2$ and $M = 3$ antennas. The downlink channels exhibit the following correlations respectively $0 \leq \alpha < 1$

$$R_2 = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix} \quad R_3 = \begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix}$$

Figures 2 - 4 show the scaling for the different precoding techniques. As seen, diagonal precoding achieved the best sum-rate capacity with minimum hit. This is closely followed by the MMSE and the approximate precoding. Note also that zero forcing is inferior to pure RBF.

V. Conclusion

In this paper we suggested precoding techniques that counter the effect of correlation on the sum-rate of RBF. Specifically, we showed that in the presence of spatial correlation, RBF incurs a hit. We also showed that RBF with MMSE and diagonal loading reduced the SNR hit and improved the sum-rate capacity as compared to pure RBF and RBF with zero forcing was inferior to pure RBF. Furthermore, we introduced a rather simple, direct and an approximate precoding technique that matched the performance of MMSE precoding.

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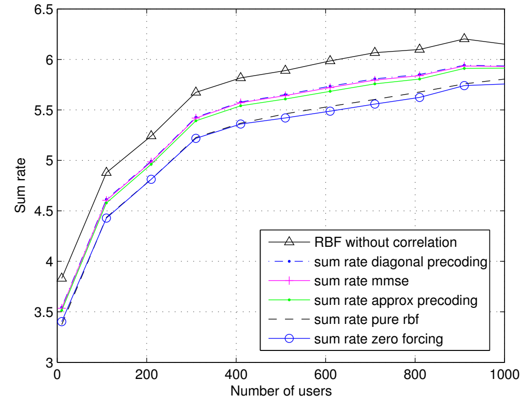

Fig. 2. Sum-rate versus the number of users in a system with M = 2, P = 10 and α = 0.5.

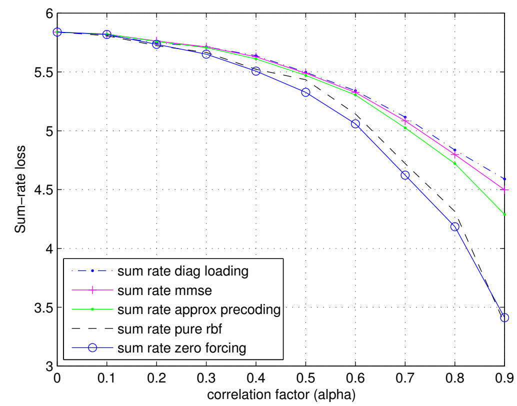

Fig. 3. Sum-rate loss versus correlation factor for a system with M = 2, P=10 and n = 400.

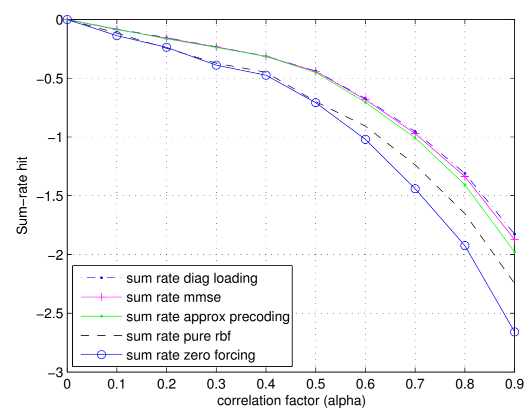

Fig. 4. Sum-rate loss versus correlation factor for a system with M = 3, P=10 and n = 100.

VI. Appendix

In this section, we evaluate the CDF of a general quantity

$$Z = \frac{\|\phi\|_B^2}{\|\phi\|_C^2}$$

where B and C are diagonal matrices. To this end, note that the inequality $Z \leq x$ can be written as $\|\phi_m\|_{xC-B}^2 \geq 0$. The CDF is then given by

$$\begin{aligned} P\{Z \leq x\} &= \int_{\|\phi_m\|_{xC-B}^2 \geq 0} p(\phi) d\phi \\ &= \int p(\phi) u(\|\phi_m\|_{xC-B}^2) d\phi \end{aligned} \tag{9}$$

where $p(\phi)$ is the pdf of ϕ defined in (4) and $u(x)$ is the step function. This integral is difficult to calculate due the inequality constraint (the unit step function) and due to the delta function. To go around this, we use the following unit step representation [4]

$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{x(j\omega_1+\beta_1)}}{j\omega_1 + \beta}$$

which is valid for any $\beta_1 > 0$. We can thus write

$$u(\|\phi_m\|_{xC-B}^2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{(\|\phi_m\|_{xC-B}^2)(j\omega_1+\beta_1)}}{j\omega_1 + \beta_1} d\omega_1$$

We can also replace the delta function with a similar integral representation

$$p(\phi) = \frac{\Gamma(M)}{\pi^M} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega_2(\|\phi\|^2-1)} d\omega_2$$

We thus have the following integral representation of the CDF of Z

$$\begin{aligned} P\{r \leq x\} &= \frac{\Gamma(M)}{4\pi^{M+2}} \times \\ &\int_{-\infty}^{\infty} d\omega_1 \frac{1}{j\omega_1 + \beta_1} \int_{-\infty}^{\infty} d\omega_2 e^{-j\omega_2} \\ &\int d\phi e^{-\phi^*((B-xC)(j\omega_1+\beta_1)-j\omega_2 I)\phi} \end{aligned}$$

By inspecting the inner integral, we note that it is similar to the Gaussian density integral. Specifically, we have

$$\frac{1}{\pi^M} \int d\phi e^{-\phi^*((B-xC)(j\omega_1+\beta_1)-j\omega_2 I)\phi} = \frac{1}{\det((B-xC)(j\omega_1+1)-j\omega_2 I)}$$

This allows us to write

$$P\{Z \leq x\} = \frac{\Gamma(M)}{4\pi^{M+2}} \int d\omega_1 \frac{1}{j\omega_1 + \beta_1} \int d\omega_2 \frac{e^{-j\omega_2}}{\det((j\omega_1+\beta_1)(B-xC)-j\omega_2 I)} \tag{10}$$

We now turn our attention to the integral with respect to ω_2. To evaluate this integral, we use partial fraction expansion to represent the determinant as

$$\begin{aligned} &\frac{1}{\det((j\omega_1+\beta_1)(B-xC)-j\omega_2 I)} \\ &= \frac{1}{\prod_{i=1}^M ((b_i - c_i x)(j\omega_1+\beta_1) - j\omega_2)} \\ &= \frac{1}{(j\omega_1+\beta_1)^{M-1}} \times \\ &\sum_{i=1}^M \frac{\eta_i}{((b_i - c_i x)(j\omega_1+\beta_1) - j\omega_2)} \end{aligned} \tag{11}$$

where $\eta_i = \frac{1}{\prod_{k \neq i}((b_k - b_i) - (c_k - c_i)x)}$. This expansion is valid assuming that $(b_k - b_i)^2 + (c_k - c_i)^2 \neq 0$. We can now residue theory to evaluate the integral with respect to ω_2 as

$$\frac{1}{2\pi} \int d\omega_2 \frac{e^{-j\omega_2}}{\det((j\omega_1+\beta_1)(B-xC)-j\omega_2 I)} = \sum_{i=1}^M \eta_i e^{(b_i - c_i x)(j\omega_1+\beta_1)} u(b_i - c_i x)$$

We can thus write

$$\begin{aligned} P\{Z \leq x\} &= \frac{\Gamma(M)}{4\pi^{M+2}} \int_{-\infty}^{\infty} d\omega_1 \frac{1}{(j\omega_1+\beta_1)^M} \\ &\sum_{i=1}^M \eta_i e^{(a_i - b_i x)(j\omega_1+\beta_1)} u(a_i - b_i x) \end{aligned}$$

We can now use residue theory to show that

$$\begin{aligned} P\{Z \leq x\} &= \frac{\Gamma(M)}{4\pi^{M+2}} \sum_{i=1}^M \eta_i u(b_i - c_i x) \\ &\int_{-\infty}^{\infty} d\omega_1 \frac{e^{(j\omega_1+\beta_1)(b_i - c_i x)}}{(j\omega_1+\beta_1)^M} \\ &= \sum_{i=1}^M \eta_i (b_i - c_i x)^{M-1} u(b_i - c_i x) u(b_i - c_i x) \end{aligned}$$

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