How Much Does Transmit Correlation Affect the Sum-Rate of MIMO Downlink Channels?

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Outline

- **Introduction**
- **Questions of interest in a broadcast scenario**
- **System model and multiuser scheduling schemes**
- **Capacity scaling of DPC with channel correlation**
- **Capacity scaling of beamforming with channel correlation**
- **Simulations**
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Introduction to Broadcast Channels

- Multiple antennas add tremendous value to point to point systems
- Research shifted recently to the role of multiple antennas in multiuser systems
- Broadcast scenarios (point to multi-point) are especially important because downlink scheduling is the major bottleneck for broadband wireless networks

Three Main Questions in a Broadcast Scenario (1)

Q1) Quantify the maximum sum rate possible to all users

A1) Sum-rate is achieved using dirty paper coding (DPC) (Caire and Shamai '02, Viswanath and Tse '02, Vishwanath et al. '02, Yu and Cioffi '02)

- **(-)** DPC is computationally complex at both Tx and Rx
- **(-)** Requires a great deal of Feedback (CSI for all users at Tx)

Three Main Questions in a Broadcast Scenario (2)

The second question is motivated by the drawbacks of DPC

Q2) Devise computationally efficient algorithms for capturing capacity

A2) Utilize multi-user diversity to achieve performance close to capacity

(+) Opportunist multiple random beamforming coincides asymptotically with DPC (Sharif and Hassibi '06)

$$R = M \log \log n + M \log \frac{P}{M} + o(1)$$

(+) Requires simply SINR feedback to Tx

Three Main Questions in a Broadcast Scenario (3)

Q3) With this promising performance, how does opportunist beam-forming perform under various non-idealities

A3) **(i)** Time correlation (Kountouris and Gesbert '05)

(ii) Frequency correlation (Fakhereddin, Sharif, and Hassibi'06)

(iii) Channel estimation error (Vikali, Sharif, and Hassibi '06)

(iv) Spatial correlation (D. Park and S Y. Park '05)

Main problem to be addressed:

- For a Gaussian broadcast channel, we would like to quantify the hit that transmit correlation causes to scaling laws of the sum-rate capacity. We consider DPC and various beamforming schemes.

System Model

- Base station with M antennas broadcasting to n single-antenna users
- Received signal at each antenna

$$Y_i = \sqrt{P} H_i S + W_i, \qquad i = 1, \ldots, n$$

with $E[S^* S] = 1$ and Gaussian noise $W_i \sim CN(0, I)$

- Channel H_i of i-th user is $1 \times M$ vector
  - Distributed as $CN(0, R)$; R is nonsingular with $\mathrm{tr}(R) = M$
  - Known perfectly at receiver
  - Follows a bock fading model (with coherence interval T)
  - H_i is independent from one user to another

Digression: Extreme Value Theory

- Let $x_1, x_2, \ldots, x_n$ be i.i.d random variables with pdf $f(x)$ and CDF $F(x)$. How does $\max_i x_i$ behave?
- Let z denote the limit

$$z = \lim_{x \to \infty} \frac{1 - F(x)}{f(x)}$$

then, for large n we have with high probability

$$\boxed{\max_i x_i = z \log(n)}$$

Scaling of DPC under Correlation

- Sum-rate capacity of DPC

$$R_{DPC} = E\left\{ \max_{\{P_1,\ldots,P_n,\sum P_i = P\}} \log\det\left(I + \sum_{i=1}^{n} H_i^* P_i H_i \right) \right\}.$$

- Define $H_i = H_{w_i} R^{\frac{1}{2}}$ and employ the inequality $\det(A) \leq \left(\frac{\mathrm{tr}(A)}{M}\right)^M$ to obtain

$$R_{DPC} \leq M \log\left(\frac{1}{M}\mathrm{tr}(R^{-1}) + \max_i \|H_{w_i}\|^2 \frac{P}{M} \right)$$

- For large n, $\max_i \|H_{w_i}\|^2$ behaves as $\log n$ with high probability.

Thus,

$$\begin{aligned} R_{DPC} &\leq M \log \left(\frac{tr(R^{-1})}{M} + \frac{P}{M} \log n \right) + \log \det R \\ &= M \log \log n + M \log \left(\frac{P}{M} \right) + M \log \sqrt[M]{\det R} \text{ for large } n \end{aligned}$$

- This is also a lower bound as it the scaling of deterministic beam forming. So, for large n

$$\boxed{R_{DPC} = M \log \log n + M \log \left(\tfrac{P}{M}\right) + M \log \sqrt[M]{\det R}}$$

- Compare with rate for spatially uncorrelated channel

$$\boxed{R_{DPC} = M \log \log n + M \log \left(\tfrac{P}{M}\right)}$$

Keep in mind that $\sqrt[M]{R} \leq \frac{\mathrm{Tr}(R)}{M} = 1$

What is Random Beam Forming?

- Choose M random orthonormal vectors ϕ_m, $m = 1, \ldots, M$ (according to an isotropic distribution)
- Construct the signal

$$S(t) = \sum_{m=1}^{M} \phi_m s_m(t), \qquad t = 1, \ldots, T$$

  where T is less than the coherence interval of the channel.

- After T channel uses we independently choose another isotropic set of orthonormal vectors $\{\phi_m\}$, and so on. So we are transmitting M random beams.
- This is a generalization of the scheme "Opportunistic Beamforming" (Viswanath et al. '02) in which only one random beam is transmitted and proportional fairness is guaranteed.

Exploit Multi-User Diversity

- Each receiver $i = 1, \ldots, n$ computes the following M SINRs

$$\text{SINR}_{i,m} = \frac{|H_i \phi_m|^2}{1/\rho + \sum_{n \neq m} |H_i \phi_n|^2}, \qquad m = 1, \ldots, M$$

  and feeds back the best SINR

- Rather than randomly assigning the beams, the transmitter assigns signal s_m to the user with the best SINR for that signal. Therefore

$$C = E \sum_{m=1}^{M} \log\left(1 + \max_{i=1,\ldots,n} \text{SINR}_{i,m}\right)$$

- Due to the symmetry of all the random variables involved:

$$C = M E \log\left(1 + \max_{i=1,\ldots,n} \text{SINR}_{i,1}\right)$$

Other Beamforming Schemes

- Random Beam forming (RBF) $S(t) = \sum_{m=1}^{M} \phi_m s_m(t)$
- RBF with Channel whitening

$$S(t) = \sum_{m=1}^{M} \sqrt{\alpha} R^{-1/2} \phi_m s_m(t)$$

- RBF with general precoding

$$S(t) = \sum_{m=1}^{M} \sqrt{\alpha} A \phi_m s_m(t)$$

- Deterministic beamforming

$$S(t) = \sum_{m=1}^{M} \phi_m s_m(t), \quad \phi_m\text{'s are fixed}$$

How to Determine Scaling of BF Schemes

1. Sum rate

$$
\begin{aligned}
R_{\mathrm{BF}} &= E \sum_{m=1}^{M} \log\left(1 + \max_{i=1,\ldots,n} \mathrm{SINR}_{i,m}\right) \\
&= M E\left(1 + \max_{i=1,\ldots,n} \mathrm{SINR}_{i,m}\right)
\end{aligned}
$$

2. To calculate expectation, condition on beams

$$
R_{\mathrm{BF}|\Phi} = M E_{H_i|\Phi}\left(1 + \max_{i=1,\ldots,n} \mathrm{SINR}_{i,m}\right)
$$

- $\mathrm{SINR}_{i,m}|\Phi$ is iid over i
- Find the distribution of $\mathrm{SINR}_{i,m}|\Phi$
- Employ extreme value theory to find $\max_{i=1,\ldots,n} \mathrm{SINR}_{i,m}$

3. Average $R_{\mathrm{BF}|\Phi}$ over Φ

Statistics of $\mathrm{SINR}_{i,m}$ (White Channel)

- $\mathrm{SINR}_{i,m}$ is defined by

$$\mathrm{SINR}_{i,m} = \frac{|H_i \phi_m|^2}{1/\rho + \sum_{n \neq m} |H_i \phi_n|^2}, \qquad m = 1, \ldots, M$$

- Easy to find distribution of $\mathrm{SINR}_{i,m}|\Phi$ when H_i is white

$$\begin{aligned} f(x) &= \frac{e^{-\frac{x}{\rho}}}{(1+x)^M} \left(\frac{1}{\rho}(1+x) + M - 1 \right) \\ F(x) &= 1 - \frac{e^{-\frac{x}{\rho}}}{(1+x)^M} \end{aligned}$$

- Finding these statistics in the correlated case is challenging

Statistics of $\mathrm{SINR}_{i,m}$ Given Φ (Correlated Case)

- We can show that the CDF of SINR in the correlated case

$$F(x) = 1 - \frac{1}{2\pi^M \det(R)} \lambda_M \prod_{i=1}^{M-1} \frac{\lambda_i \lambda_M}{x(\lambda_i - \lambda_M)} e^{-\frac{1}{\rho}\frac{x}{\lambda_M}}$$

where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_M$ are the eigenvalues of the matrix

$$A = (1+x)\Lambda^{1/2}\phi_m\phi_m^*\Lambda^{1/2} - x\Lambda \qquad \rho = \frac{P}{M}$$

Note that eigenvalues are a function of x.

- pdf is given by

$$f(x) = \frac{1}{2\pi^M \det(R)} e^{-\frac{1}{\rho}\frac{x}{\lambda_M}} \prod_{i=1}^{M-1} \frac{\lambda_i \lambda_M}{x(\lambda_i - \lambda_M)} \times \left\{ \frac{1}{\rho} \frac{\|q_M\|_C^2}{\lambda_M} - \|q_M\|_B^2 - \sum_{i=1}^{M} \frac{1}{\lambda_i} \frac{\lambda_M^2 \|q_i\|_C^2 - \lambda_i^2 \|q_M\|_C^2}{x(\lambda_i - \lambda_M)} \right\}$$

where $B = \Lambda^{1/2}(\phi_m\phi_m^* - I)\Lambda^{1/2} \quad C = \Lambda^{1/2}\phi_m\phi_m^*\Lambda^{1/2}$

Scaling of Maximum SINR

- Can now show

$$\lim_{x\to\infty} \frac{1-F(x)}{f(x)} = \frac{P}{M}\frac{1}{\|\phi_m\|^2_{\Lambda^{-1}}}$$

- Using extreme value theory, we can show that for large n

$$\max_{i=1,\dots,n} \mathrm{SINR}_{i,m} = \frac{P}{M}\frac{1}{\|\phi_m\|^2_{\Lambda^{-1}}}\log n$$

- Conditional sum-rate capacity scales as

$$R_{\mathrm{BF}|\Phi} = M\log\log n + M\log\frac{P}{M} + M\log\left(\frac{1}{\|\phi_m\|^2_{\Lambda^{-1}}}\right)$$

- Sum-rate capacity of random beam-forming

$$\boxed{R_{\mathrm{RBF}} = M\log\log n + M\log\tfrac{P}{M} + ME_{\Phi}\log\left(\tfrac{1}{\|\phi_m\|^2_{\Lambda^{-1}}}\right)}$$

Averaging Over the Random Beams

- Need to obtain CDF of $\frac{1}{\|\phi_m\|^2_{\Lambda^{-1}}}$ which is challenging.
- The CDF of $y = \frac{1}{\|\phi\|^2_{\Lambda^{-1}}}$ is given by

$$\boxed{G(x) = Pr(\frac{1}{\|\phi\|^2_{\Lambda^{-1}}} < x) = 1 - \sum_i \eta_i \left(\frac{1}{x} - \frac{1}{\lambda_i(\Lambda)}\right)^{M-1} u\left(1 - \frac{x}{\lambda_i(\Lambda)}\right)}$$

where $\eta_i = \frac{1}{\prod_{j \neq i}(\frac{1}{\lambda_j(\Lambda)} - \frac{1}{\lambda_i(\Lambda)})}$

- Use CDF to show that

$$\begin{aligned} R_{RBF} &= M \log\log n + M \log \frac{P}{M} + \\ &\log \lambda_1(\Lambda) + \sum_{i=1}^{M} \eta_i \log\left(\frac{\lambda_i}{\lambda_1}\right) \sum_{k=1}^{M-1} \frac{1}{k+2} (\frac{-1}{\lambda_i})^{M-1-k} \left.\frac{1}{y^{k+2}}\right|_{\lambda_1}^{\lambda_i} \end{aligned}$$

Sum rate of Deterministic Beam Forming

- Sum-rate of deterministic beam forming

$$R_{BF-D} = M \log\log n + M \log \frac{P}{M} + \sum_{i=1}^{M} \log\left(\frac{1}{\phi_i^* U^* \Lambda^{-1} U \phi_i}\right)$$

$U^* \Lambda^{-1} U$ is the eigenvalue decomposition of R^{-1}

- Special case: $U\phi_i$'s are the columns of identity matrix

$$\boxed{R_{BF-D} = M \log\log n + M \log \tfrac{P}{M} + M \log \sqrt[M]{\det R}}$$

Since $\mathrm{tr}(R) = M$, the geometric mean satisfies $\det(R) \leq 1$

- Scaling coincides with (D. Park and S Y. Park '05) which focused on the M = 2 case

Sum rate of RBF with Channel Whitening

- For random beam forming with channel whitening,

$$S(t) = \sum_{m=1}^{M} \sqrt{\alpha} R^{-1/2} \phi_m s_m(t)$$

- Set $\alpha = \frac{\mathrm{tr}(R^{-1})}{M}$ to guarantee $E[S^* S] \leq 1$

- Scaling becomes the same as for white channel case with reduced signal power

$$\boxed{R_{BF-W} = M \log\log n + M \log \frac{P}{M} + M \log \frac{M}{\mathrm{tr}(R^{-1})}}$$

Simulations

- Consider a base station with $M = 2$ and $M = 3$ antennas
- The corresponding correlation matrix is

$$R = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix}$$

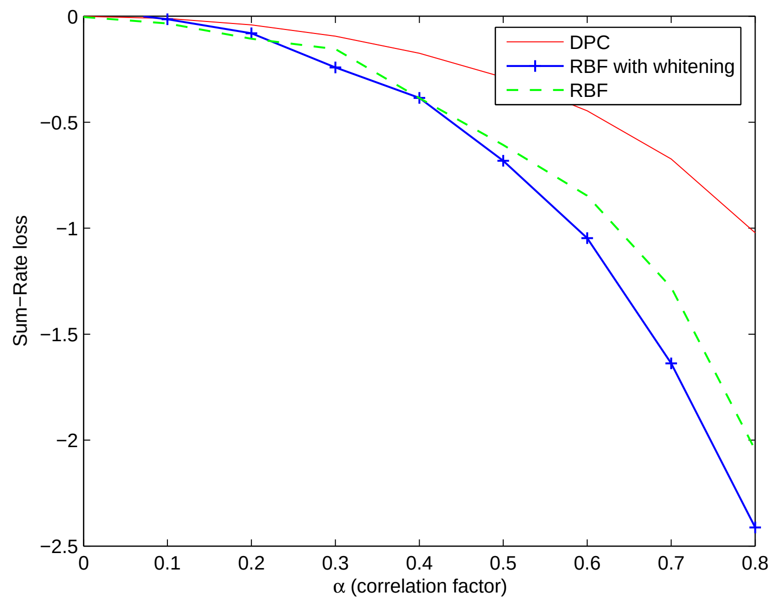

Figure 1: Sum-rate loss versus the correlation factor α for a system with $M = 2$ and $n = 100$.

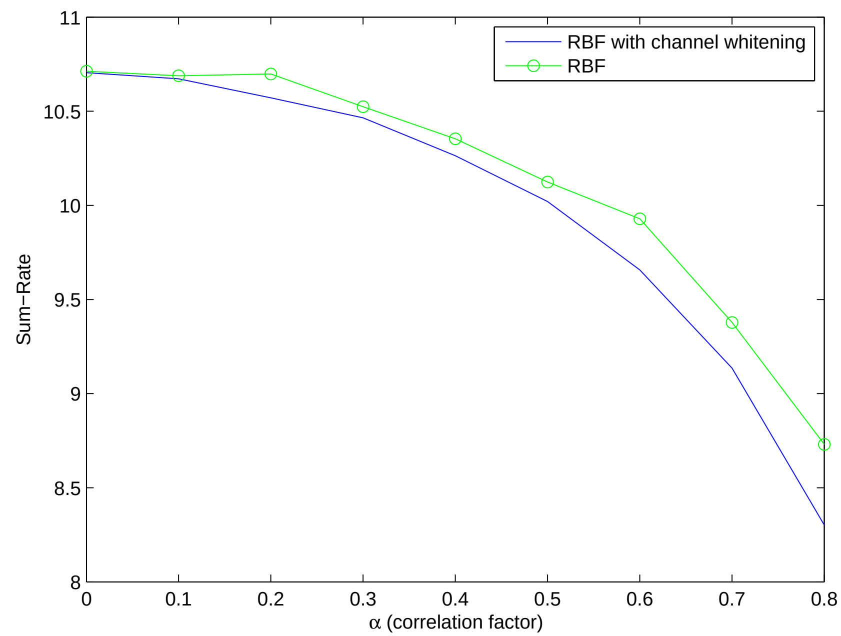

Figure 2: Sum-rate versus the correlation factor α for a system with $M = 2$, $P = 10$, and $n = 100$.

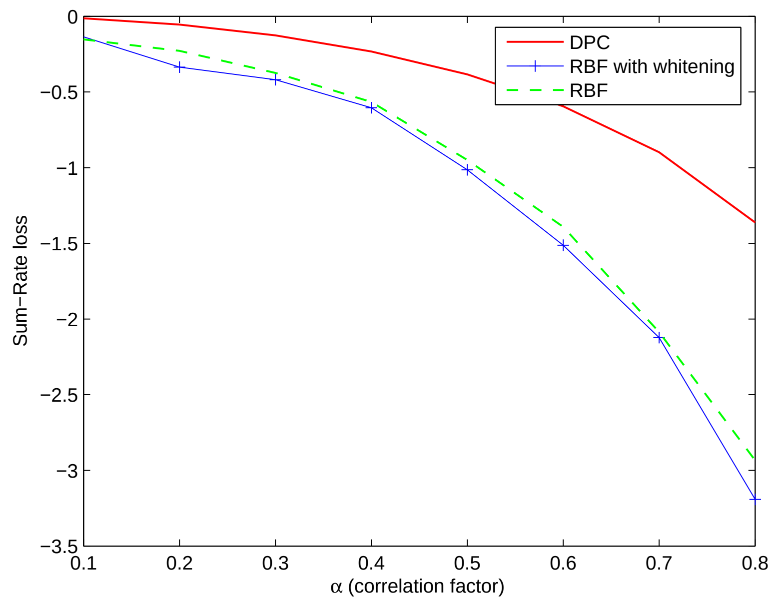

Figure 3: Sum-rate loss versus the correlation factor α for a system with $M = 3$ and $n = 100$.

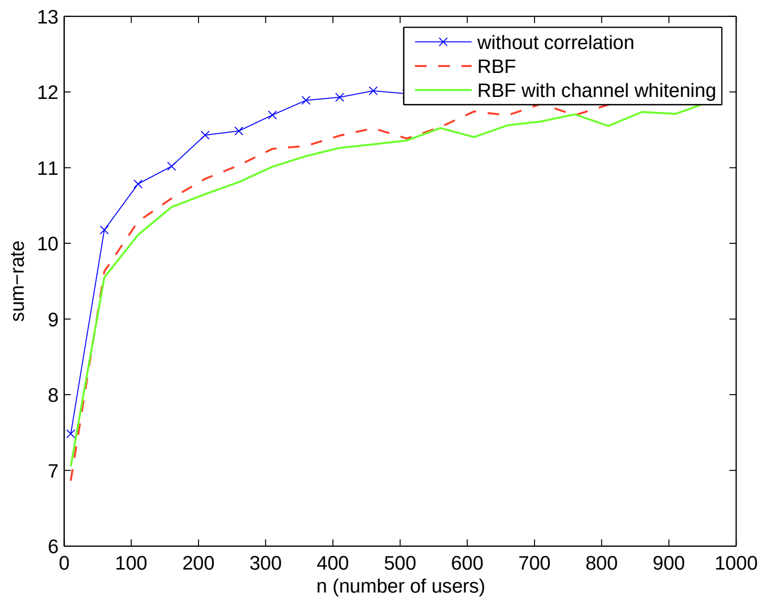

Figure 4: Sum-rate versus the number of users in a system with $M = 2$ and $\alpha = 0.5$

Conclusion

- Studied the effect of spatial correlation on various multiuser schemes for MIMO broadcast channels.
- Considered DPC and random, deterministic, and channel whitening schemes.
- All these techniques exhibit the same scaling for iid channels

$$\boxed{R_{\text{sum-rate}} = M \log\log n + M \log \tfrac{P}{M}}$$

- In the presence of correlation between transmit antennas, scaling is

$$\boxed{R_{\text{sum-rate}} = M \log\log n + M \log \tfrac{P}{M} + M \log c}$$

  The constant $0 < c \leq 1$ depends on the scheduling scheme and the eigenvalues of the correlation matrix R.

Recent Results:

Scaling Laws of Group Broadcast Channels

- K groups of users
- Each group of users is interested in the same data
- Worst user of each group is the bottle neck
- Worst user is difficult to define in the multi-antenna case
- For K groups with n users each, we show that capacity scales like

$$\boxed{C = K \frac{P}{n^{\frac{1}{M}}}}$$

- We show that to have a constant rate, M should grow at least as fast as $\log n$
- This is a joint work with Amir Dana and Babak Hassibi, Cal Tech.

Extra Slide: Finding the Distribution of $SINR$

- Consider the $SINR$ for the first beam

$$\mathrm{SINR}_{i,1} = \frac{|H_i \phi_1|^2}{1/\rho + \sum_{n=2}^{M} |H_i \phi_n|^2},$$

- Define S by

$$S = -\frac{x}{\rho} + H_i^* ((1+x)\phi_1 \phi_1^* - xI) H_i$$

Then

$$\begin{aligned} P(\mathrm{SINR}_{i,1} > x) = P(S > 0) &= \int_{-\infty}^{\infty} P(H_i) u(S) dH_i \\ &= \frac{1}{\pi^M \det(R)} \int_{-\infty}^{\infty} e^{-H_i^* R^{-1} H_i} u(S) dH_i \end{aligned}$$

- To evaluate integral, use the integral representation of unit step

$$u(S) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{(j\omega+\beta)S}}{j\omega + \beta} d\omega$$

- Desired probability becomes

$$
\begin{aligned}
& P(\ \text{SINR}_{i,1} > x) \\
= \quad & \frac{1}{2\pi^{M+1}\det(R)} \int_{-\infty}^{\infty} d\omega \frac{1}{j\omega+\beta} \int_{-\infty}^{\infty} dH_i e^{(j\omega+\beta)S - H_i^* R^{-1} H_i} \\
= \quad & \frac{1}{2\pi^{M+1}\det(R)} \int_{-\infty}^{\infty} d\omega \frac{e^{-(j\omega+\beta)\frac{x}{\rho}}}{j\omega+\beta} \int_{-\infty}^{\infty} dH_i e^{-H_i^* \tilde{R}^{-1} H_i} \\
= \quad & \frac{1}{2\pi^{M+1}\det(R)} \int_{-\infty}^{\infty} d\omega \frac{e^{-(j\omega+\beta)\frac{x}{\rho}}}{j\omega+\beta} \frac{1}{\det(\tilde{R})}
\end{aligned}
$$