Prove that P_N is Hermitian by $\underline{\underline{\text{induction}}}$ P_N

Assume that P_{N-1} is Hermitian. Now does that make P_N Hermitian?

$$P_N = P_{N-1} - \frac{P_{N-1} u_N^* u_N P_{N-1}}{1 + \|u_N\|^2_{P_{N-1}}}$$

Then

$$P_N^* = P_{N-1}^* - \frac{P_{N-1}^* u_N^* u_N P_{N-1}^*}{1 + \left(\|u_N\|^2_{P_{N-1}}\right)^*} \qquad \left(\|x\|_A^* = \|x\|^2_{A^*} \right)$$

$$= P_{N-1} - \frac{P_{N-1} u_N^* u_N P_{N-1}}{1 + \|u_N\|^2_{P_{N-1}}}$$

$$= P_N .$$

The starting point is also Hermitian $P_{-1} = \Pi = P_{-1}^*$.

What about the +ve definite part ?

Can we prove it by induction too ?

~~Not eas~~ If P_{N-1} is +ve definite, can we prove that P_N is +ve definite. ?

Not easy !

What is the alternative ?

Use the definition

$$P_N = \left(\Pi + H_N^{*} H_N\right)^{-1}$$

From this, we can see that P_N is Hermitian

We can also see that P_N is +ve definite. How?

Prove that if A is +ve definite, then A^{-1} is

A is +ve definite $\Rightarrow$ we can write it as

$$A = U \Lambda U^*$$

A is positive definite $\Leftrightarrow$ λ's on diagonal of $\mathbf{A}$ are +ve

Now

$$A^{-1} = U \Lambda^{-1} U^*$$

Λ^{-1} is diagonal with +ve elements, so A^{-1} is +ve definite.

Since $\Pi + H_N^* H_N > 0$, then $P_N = \left(\Pi + H_N^* H_N\right)^{-1} > 0$.