Q1.

a) Let $\hat{\mathbf{x}}$ be the linear minimum mean-square estimate (MMSE) of $\mathbf{x}$ given a random variable $\mathbf{y}$. Consider the random variable $\mathbf{z} = \mathbf{H}\mathbf{x}$ Can we claim that linear MMSE estimate of z given y is $\hat{z} = \mathbf{H}\hat{\mathbf{x}}$ Justify your answer by either proving it or providing a counter example.

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2 Yes. This is true. To see it, note that $\hat{z} = R_{zy} R_y^{-1} y$ Now $R_{zy} = E[z y^*] = E[Hxy^*] = H E[xy^*] = H R_{xy}$

Thus, $\hat{z} = H R_{xy} R_y^{-1} y$ 4
$= H\hat{x}$

b) Let $\hat{x}$ be the optimum minimum mean-square estimate (MMSE) of x given a random variable y. Consider the random variable $z = f(x)$. Can we claim that the MMSE estimate of z is $\hat{z} = f(\hat{x})$? Justify your answer by either proving it or providing a counter example.

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2 No. To see this, let x be a BPSK r.v. & assume that $z = x^2$. Then $z = 1$ & the best estimate of z given any variable y is $\hat{z} = 1 = z$.

Now if $y = x + v$ with $v \sim \mathcal{N}(0,1)$, then $\hat{x} = \tanh(y)$ and $f(\hat{x}) = \tanh^2(y) \neq 1$. 4

c) We have so far considered three types of estimators in the class. List these estimators and describe the advantages/disadvatages of each.

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1 MAP (+) minimizes probability of error
(−) very difficult to derive
1 (−) requires joint pdf of the variables in question

1 MMSE (+) minimizes mean square error
1 (−) difficult to derive
(−) requires joint pdf of variables

1 Linear MMSE (+) Easy to derive
(+) requires second moments only
1 (−) Has higher estimation error

7 **d)** Let x be a Gaussian random variable with mean $\bar{x}$ and variance σ_x^2. Find the expected value of the matrix

$$\mathbf{A} = (x\mathbf{C} + \mathbf{a}\mathbf{b}^*)^{-1}$$

in terms of the moments of $\bar{x}$ and variance σ_x^2. Assume that $E[\frac{1}{x}] = \frac{1}{E[x]}$ and that $E[\frac{1}{\alpha + x^2}] = \frac{1}{\alpha + E[x^2]}$

$$A = (xC + ab^*)^{-1}$$
$$= \frac{1}{x}C^{-1} - \frac{1}{x}C^{-1}a\left(1 + b^*\frac{C^{-1}}{x}a\right)^{-1}b^*\frac{C^{-1}}{x} \quad 2$$
$$= \frac{1}{x}C^{-1} - \frac{1}{x^2}\,\frac{C^{-1}ab^*C^{-1}}{1 + \frac{1}{x}b^*C^{-1}a}$$
$$= \frac{1}{x}C^{-1} - \frac{C^{-1}ab^*C^{-1}}{x^2 + x\,b^*C^{-1}a} \quad 2$$

Now take the expectation & ~~ge~~ assume ~~things can~~ that $E[f(x)] = f(E[x])$,

then

$$E[A] = \frac{1}{E[x]}C^{-1} - \frac{C^{-1}ab^*C^{-1}}{E[x^2] + E[x]\,b^*C^{-1}a} \quad 1$$
$$= \frac{1}{\bar{x}}C^{-1} - \frac{C^{-1}ab^*C^{-1}}{(\sigma_x^2 + \bar{x}^2) + \bar{x}\,b^*C^{-1}a} \quad 1$$

Replacing moments correctly 1

Q4. Consider the following random vector

$$\mathbf{z} = \begin{cases} -\mathbf{x} + \mathbf{v}_1 & \text{with probability } p \\ \mathbf{H}\mathbf{x} + \mathbf{v}_2 & \text{with probability } (1-p) \end{cases}$$

where $\mathbf{x}$, $\mathbf{v}_1$, and $\mathbf{v}_2$ are all zero mean uncorrelated variables. Also, let $\mathbf{y}$ be a zero mean random variable. $\mathbf{y}$, $\mathbf{x}$, $\mathbf{v}_1$, and $\mathbf{v}_2$ are all jointly circularly symmetric Gaussian random variables.

1) Find the linear mean square estimator of $\mathbf{z}$ given $\mathbf{y}$ in terms of the linear estimators of $\mathbf{x}$, $\mathbf{v}_1$, and $\mathbf{v}_2$ given $\mathbf{y}$

2) Find the optimum mean-square estimate of $\mathbf{z}$ given $\mathbf{y}$. What do you conclude?

3) Is $\mathbf{z}$ a Gaussian random variable?

Q1

5 1) The linear. mse est. of z given y is

$$\hat{z} = R_{zy} R_y^{-1} y \quad 1$$

because z & y are zero mean random variables.

Now

$$R_{zy} = p\,E[(-x+v_1)y^*] + (1-p)\,E[(Hx+v_2)y^*] + (1-p)\,E[v_2 y^*]$$

$$= -p\,E[xy^*] + pE[v_1 y^*] + (1-p)H\,E[xy^*] + (1-p)\,R_{v_2 y} \quad 2$$

$$= -p\,R_{xy} + p\,R_{v_1 y} + (1-p)H\,R_{xy} + (1-p)\,R_{v_2 y}$$

$$\Rightarrow \hat{z} = R_{zy} R_y^{-1} y$$

$$= -p\,R_{xy}R_y^{-1}y + p\,R_{v_1 y}R_y^{-1}y + (1-p)HR_{xy}R_y^{-1}y + (1-p)\,R_{v_2 y}R_y^{-1}y \quad 1$$

$$= -p\hat{x} + p\,\hat{v}_1 + (1-p)H\hat{x} + (1-p)\,\hat{v}_2$$

$$= \big(H - p(H+I)\big)\,\hat{x} + p\,\hat{v}_1 + (1-p)\,\hat{v}_2 \quad 1$$

where $\hat{x}$, $\hat{v}_1$, & $\hat{v}_2$ are lin. estimates of x, v_1, v_2 given y, respectively

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2)

$$\hat{z} = E[z|y] \quad 1$$

$$= p\,E[-x+v_1|y] + (1-p)\,E[Hx+v_2|y] \quad 1$$

$$= -p\,E[x|y] + pE[v_1|y] + (1-p)HE[x|y] + (1-p)E[v_2|y] \quad 1$$

Since $x, v_1, v_2, \& y$ are jointly Gaussian, the linear estimates of these variables given y coincide with the optimum estimates 2

So

$$\hat{z} = \big(H - p(H+I)\big)\,\hat{x} + p\hat{v}_1 + (1-p)\hat{v}_2$$

so lin. & opt. are 1

$\Rightarrow$ so lin. the same

where
$$\hat{x} = R_{xy}R_y^{-1}y$$
$$\hat{v}_1 = R_{v_1 y}R_y^{-1}y$$
$$\hat{v}_2 = R_{v_2 y}R_y^{-1}y.$$

3) Although the linear & optimum estimates of z are the same, z is not a Gaussian random variable. To see this, we can consider the special case when $x, v_1, \& v_2$ are scalars & $H=1$. Then, it is easy to show that

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$$f(z) = \frac{p\, e^{-\frac{z^2}{2(\sigma_x^2+\sigma_{v_1}^2)}}}{\sqrt{2\pi(\sigma_x^2+\sigma_{v_1}^2)}} + \frac{(1-p)}{\sqrt{2\pi(\sigma_x^2+\sigma_{v_2}^2)}} \cdot e^{-\frac{z^2}{2(\sigma_x^2+\sigma_{v_2}^2)}}$$

3

which is not Gaussian unless $\sigma_{v_1}^2 = \sigma_{v_2}^2$.

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4) $$\hat{z} = R_{zx} R_x^{-1} x. \quad 1$$

$$R_{zx} = p\, E[(-x+v_1)x^*] + (1-p)\, E[(Hx+v_2)x^*] \quad 1$$

$$= -p\, E[xx^*] + p\, E[v_1 x^*] + (1-p) E[Hxx^*] + (1-p) E[v_2 x$$

1

$$= -pR_x + (1-p) H R_x$$

$$R_{zx} = (-p+H-Hp) R_x \quad 1$$

$$\Rightarrow \hat{z} = R_{zx} R_x^{-1} x$$

$$= (-p+H-Hp) R_x R_x^{-1} x$$

$$\hat{z} = (-p+H-Hp)x \quad 2$$

Now, mean square error.

$$\tilde{z} = z - \hat{z}$$

$$= p(-x+v_1) + (1-p)(Hx+v_2) - [-px + H(1-p)x]$$

$$= p(-x+v_1+x) + (1-p)(Hx+v_2) - H(1-p)x$$

$$= pv_1 + (1-p)(Hx - Hx + v_2)$$

$$\tilde{z} = pv_1 + (1-p)v_2$$

$$E[\tilde{z}^2] = \mathrm{Tr}[R_z - \mathbb{K} R_x \mathbb{K}^*]$$

where $\mathbb{K} = (-p\mathbb{I} + H(1-p))$

$$R_z = E[z z^*]$$

$$R_z = p(R_x + R_{v_1}) + (1-p)[H R_x H^* + R_{v_2}]$$

Q2.

a) Let x abd y be random variables such that

$$y = x + v$$

where x is BPSK and v has the following distribution

$$v \text{ is } \begin{cases} \text{zero mean Gaussian with variance } \frac{3}{4} \text{ if } x = 1 \\ \text{zero mean Gaussian with variance } \frac{1}{4} \text{ if } x = -1 \end{cases}$$

i) Find the optimum mean-square estimate of y given x and find the corresponding minimum mean square error.

i) Find the optimum mean-square estimate of x given y

(7)

Q2 i) $E[y|x] = E[x+v|x]$ (1) (1)

(4)

$= x + E[v|x]$

$= x + E[v]$ (1) since v & x are independent

$= x$ (1) since x is zero mean

The mmse is

(3) $E[(y-\hat{y})^2] = E[(y-x)^2]$ (1)

(1)

$= E[(v)^2]$

$= \sigma_v^2$ (1)

Q2 13

ii) The opt. est. of x given y is

$$\hat{x} = E[x|y] \quad 1$$

To find it, we need to find

$$f_{x|y}(x|y) = \frac{f_{x,y}(x,y)}{f_y(y)} \quad 1$$

$$= \frac{f_{y|x}(y|x) f(x)}{f_y(y)} \quad 1$$

Now x is BPSK, so

$$f_x(x) = \frac{1}{2}\delta(x-1) + \frac{1}{2}\delta(x+1) \quad 1$$

$$f_y(y) = \cancel{\frac{1}{2\sqrt{6\pi\sigma^2}} e^{\frac{-(y-1)}{6\sigma^2}} + \frac{1}{2}\frac{1}{\sqrt{}}}$$

$$= \frac{1}{2\sqrt{6\pi\sigma^2}} e^{\frac{-(y-1)^2}{6\sigma^2}} + \frac{1}{2\sqrt{2\pi\sigma^2}} e^{\frac{-(y+1)}{2\sigma^2}} \qquad \sigma^2 = 1/4 \quad 2$$

$$f_{x,y}(x,y) = f_x(x)\, f_{y|x}(y|x) \quad 1$$

$$= f_x(1)\, f_{y|1}(y|1) + f_x(-1)\, f_{y|-1}(y|-1)$$

$$= \frac{1}{2}\delta(x-1)\frac{1}{\sqrt{6\pi\sigma^2}} e^{\frac{-(y-1)^2}{6\sigma^2}} + \frac{1}{2}\delta(x+1)\frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(y+1)}{2\sigma^2}} \quad 1$$

so

$$f_{x|y}(x|y) = \frac{\frac{1}{2}\frac{1}{\sqrt{6\pi\sigma^2}} e^{-\frac{(y-1)^2}{6\sigma^2}}}{\frac{1}{2\sqrt{6\pi\sigma^2}} e^{-\frac{(y-1)^2}{6\sigma^2}} + \frac{1}{2\sqrt{2\pi\sigma^2}} e^{-\frac{(y+1)^2}{2\sigma^2}}} \delta(x-1)$$

2

$$+ \frac{\frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y+1)^2}{2\sigma^2}}}{\frac{1}{2\sqrt{6\pi\sigma^2}} e^{-\frac{(y-1)^2}{6\sigma^2}} + \frac{1}{2\sqrt{2\pi\sigma^2}} e^{-\frac{(y+1)^2}{2\sigma^2}}} \delta(x+1)$$

Thus,

$$E[x|y] = \frac{\frac{1}{\sqrt{3}} e^{-\frac{(y-1)^2}{6\sigma^2}}}{\frac{1}{\sqrt{3}} e^{-\frac{(y-1)^2}{6\sigma^2}} + e^{-\frac{(y+1)^2}{2\sigma^2}}} - \frac{e^{-\frac{(y+1)^2}{2\sigma^2}}}{\frac{1}{\sqrt{3}} e^{-\frac{(y-1)^2}{6\sigma^2}} + e^{-\frac{(y+1)^2}{2\sigma^2}}}.$$

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