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Bit Error Probability Computations for M-ary Quadrature Amplitude Modulation

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Abstract

Computing the exact bit error rate (BER) for square M-ary QAM is tedious and not straightforward. However, if a generalized closed-form BER expression can be developed, then finding the BER expression will be easier. This has been achieved in [1] and will be described in more detail in this paper.

1. Introduction

M-ary Quadrature Amplitude Modulation (QAM) is a widely used modulation technique that provides high transmission rates with high bandwidth efficiency and with correct configuration, high energy efficiency. However, finding the expression of the bit error probability of M-ary QAM is not as straight forward as finding its symbol error probability. For the latter, the Union Bound can be used to have a rough estimate of the error performance of a modulation technique in terms of symbol error. But this method is not accurate and in the end the more important information is the bit error rate (BER) since a communication system ultimately sends 1's and 0's.

This paper will present the derivation of a generalized BER expression for square M-ary QAM. Note that the derivation process is explained in [1]. However, most of the intermediary steps involved in the process were not shown in [1] and this paper aims to show these steps for easier understanding of the derivation.

2. System model and assumptions

Square M-ary QAM involves the amplitude modulation of two carriers in quadrature expressed as

$$s(t) = A_c \cos(2\pi f_c)t - A_s \sin(2\pi f_c t) \qquad 0 \leq t < T \qquad (1)$$

where A_c and A_s are the signal amplitudes of the in-phase and quadrature components respectively. T is the symbol duration and f_c is the carrier frequency [1]. A_c and A_s in (1) are represented by $\log_2 M$ level amplitudes which take values of either $-(\sqrt{M}-1)d, -(\sqrt{M}-3)d, \ldots, (\sqrt{M}-3)d, (\sqrt{M}-1)d$ where d is half of the minimum distance between two symbols. Note that d can be computed as

$$d = \sqrt{\frac{3 \cdot (\log_2 \mathrm{M}) \cdot \mathrm{E_b}}{2(M-1)}} \qquad (2)$$

where $\mathrm{E_b}$ is the energy per bit.

For the discussion of this paper, a perfect 2 dimensional Gray code [2] is assumed to be used in assigning bits to each point in the QAM constellation. This assures that each symbol differs to its nearest neighbours by the minimum number of bits possible. It is also assumed that all the symbols are equiprobable. In addition, the noise to be considered in this paper is zero mean Additive White Gaussian Noise (AWGN) with variance $N_0/2$. Finally, it is assumed that there is no error contributed by carrier recovery and symbol synchronization.

3. Conventional BER derivations

The BER expressions for M-ary square QAM with M = 16, 64 and 256 are first derived using the conventional method. Using these equations, a general expression for M-ary square QAM will be derived by induction.

3.1 BER of 16-QAM

Figure 1 shows the signal constellation of a square 16-QAM where each symbol is represented by four bits constituted by the in-phase bits i_1, i_2 and quadrature bits q_1, q_2. These bits are then interleaved to form the sequence i_1, q_1, i_2, q_2 [1].

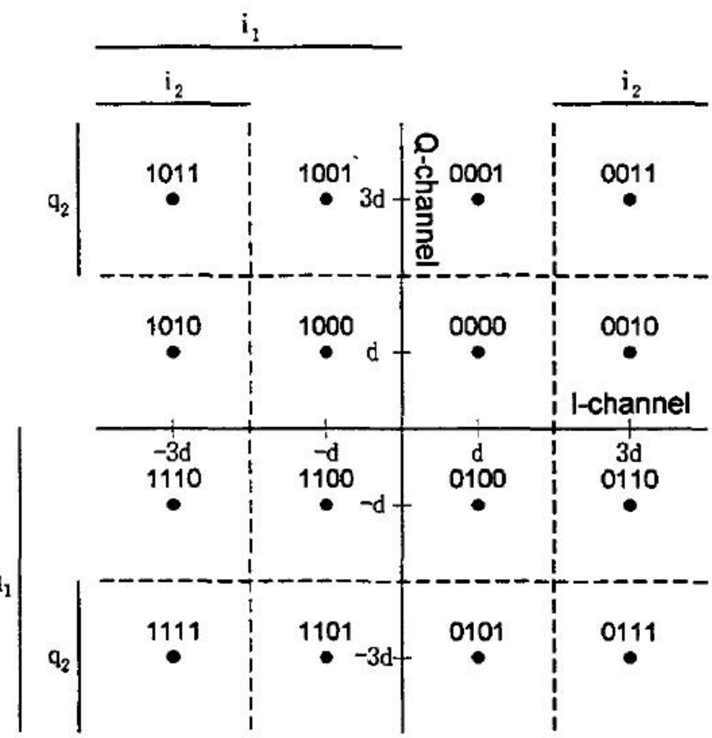

Figure 1. Signal constellation for square 16-QAM [1]

In Figure 1, the $i_k, q_k \; for \; (k = 1,2)$ labels indicate the regions where $i_k = 1$ and $q_k = 1$. These regions will help simplify the BER computations by allowing "divide and conquer" approach later on. First, the BER computation for the whole constellation is decomposed into two smaller cases. Case 1 ignores bits i_2, q_2 while focusing on i_1, q_1 (focus of section **3.2.a**). Doing this results to the simplification of Figure 1 to Figure 2.

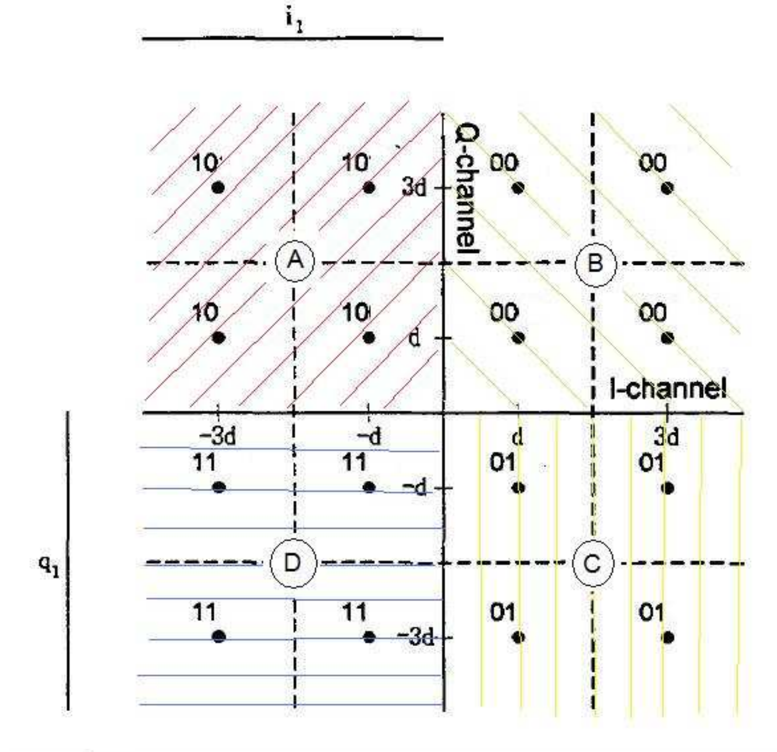

Figure 2. Signal constellation considering i_1, q_1 only

In Figure 2, the signal space is divided into four regions A, B, C and D where i_1, q_1 has fixed values 10, 00, 01 and 11 respectively. From this figure, it can also be seen that the bit decision can be made according to

$$\begin{aligned} &if\ I < 0, then\ i_1 = 1 \\ &if\ I \geq 0, then\ i_1 = 0 \\ &if\ Q < 0, then\ q_1 = 1 \\ &if\ Q \geq 0, then\ q_1 = 0\,. \end{aligned} \qquad (3)$$

Before the BER computation, recall that in the demodulator, the vector **r** is received where

$$\boldsymbol{r} = \boldsymbol{s} + \boldsymbol{n} \qquad (4)$$

and

$$\boldsymbol{s} = \begin{bmatrix} I \\ Q \end{bmatrix}\ and\ \boldsymbol{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}. \qquad (5)$$

In (5), I (in phase component) is represented by the in-phase bits i_1, q_1 and Q (quadrature component) by i_2, q_2. Also, recall that for M = 16 the half distance is given by

$$d = \sqrt{\frac{3\cdot(\log_2 \mathrm{M})\cdot \mathrm{E_b}}{2(M-1)}} = \sqrt{\frac{3\cdot(\log_2 16)\cdot \mathrm{E_b}}{2(16-1)}} = \sqrt{\frac{2\mathrm{E_b}}{5}}\ . \qquad (6)$$

Finally, we represent the bit error probability for the k[th] bit of I and Q as $P_b(k)$ e.g. the probability that there is an error in the first bits on each component I and Q is expressed as $P_b(1)$.

3.1.a. Expression for $P_b(1)$

We can simplify the problem of finding $P_b(1)$ by decomposing it further into finding $P_b(i_1)$ and $P_b(q_1)$ and then using

$$P_b(1) = \frac{1}{2}P_b(i_1) + \frac{1}{2}P_b(q_1) \qquad (7)$$

where $P_b(i_1)$ is the probability of bit error for i_1 alone and similarly, $P_b(q_1)$ is the probability of bit error for q_1 alone. Then the total bit error rate for the first bits is simply the average of the single bit errors.

3.1.a.i. Deriving $P_b(i_1)$

Fist consider the probability of error for i_1. We can compute $P_b(i_1)$ by conditional probability as

$$P_b(i_1) = \frac{1}{2}P_b(i_1 = 0) + \frac{1}{2}P_b(i_1 = 1) \qquad (8)$$

since i_1 is equally likely to be 0 or 1 (note that $P_b(i_1 = 0)$ is interpreted as probability of bit error given $i_1 = 0$).When $i_1 = 0$, the constellation point can lie in either the left or right of the 2d axis. Thus, we can also use conditional probability to get

$$P_b(i_1 = 0) = \frac{1}{2}P_b(i_1 = 0, left) + \frac{1}{2}P_b(i_1 = 0, right) \qquad (9)$$

where left means $i_2 = 0$ and right means $i_2 = 1$ (or simply look at the location of the column in the constellation; one will be on the left of the 2d axis and one will be on the right).

But we get a single bit error for the left column when $n_1 < -d$ and the right when $n_1 < -3d$. So,

$$P_b(i_1 = 0, left) = \Pr[n_1 < -d] \qquad (10)$$

$$P_b(i_1 = 0, right) = \Pr[n_1 < -3d] \qquad (11)$$

But since we are dealing with Additive White Gaussian Noise with zero mean and variance equal to $N_0/2$, we know that

$$n_1 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right) \qquad (12)$$

so,

$$\Pr[n_1 < -d] = Q\left(\frac{-d-0}{\sqrt{N_0/2}}\right) = Q\left(\frac{d}{\sqrt{N_0/2}}\right) \quad (13)$$

which means that

$$P_b(i_1 = 0, left) = \ Q\left(\frac{d}{\sqrt{N_0/2}}\right). \quad (14)$$

Similarly,

$$\Pr[n_1 < -3d] = Q\left(\frac{-3d-0}{\sqrt{N_0/2}}\right) = Q\left(\frac{3d}{\sqrt{N_0/2}}\right) \quad (15)$$

so

$$P_b(i_1 = 0, right) = \ Q\left(\frac{3d}{\sqrt{N_0/2}}\right). \quad (16)$$

Going back to (9), we now have

$$P_b(i_1 = 0) = \frac{1}{2}Q\left(\frac{d}{\sqrt{N_0/2}}\right) + \frac{1}{2}Q\left(\frac{3d}{\sqrt{N_0/2}}\right). \quad (17)$$

By symmetry, we can get

$$P_b(i_1 = 1) = \frac{1}{2}Q\left(\frac{d}{\sqrt{N_0/2}}\right) + \frac{1}{2}Q\left(\frac{3d}{\sqrt{N_0/2}}\right). \quad (18)$$

Therefore (8) becomes,

$$P_b(i_1) = \frac{1}{2}\left[\frac{1}{2}Q\left(\frac{d}{\sqrt{N_0/2}}\right) + \frac{1}{2}Q\left(\frac{3d}{\sqrt{N_0/2}}\right)\right] + \frac{1}{2}\left[\frac{1}{2}Q\left(\frac{d}{\sqrt{N_0/2}}\right) + \frac{1}{2}Q\left(\frac{3d}{\sqrt{N_0/2}}\right)\right] \quad (19)$$

$$P_b(i_1) = \frac{1}{2}\left[Q\left(\frac{d}{\sqrt{N_0/2}}\right) + Q\left(\frac{3d}{\sqrt{N_0/2}}\right)\right]. \quad (20)$$

3.1.a.ii. Deriving $\boldsymbol{P_b(q_1)}$

If we rotate the whole signal constellation in Figure 1 by 90 degrees clockwise, we get q_1 with the same configuration as i_1. Thus, we can use the same method in finding $P_b(q_1)$ giving us

$$P_b(q_1) = \frac{1}{2}\left[Q\left(\frac{d}{\sqrt{N_0/2}}\right) + Q\left(\frac{3d}{\sqrt{N_0/2}}\right)\right]. \quad (21).$$

3.1.a.iii. $\boldsymbol{P_b(1)}$ computation

Finally, we can compute the probability of error for the first bits of the I and Q components using (7) to get

$$P_b(1) = \ \frac{1}{2}\left\{\frac{1}{2}\left[Q\left(\frac{d}{\sqrt{N_0/2}}\right) + Q\left(\frac{3d}{\sqrt{N_0/2}}\right)\right]\right\} + \frac{1}{2}\left\{\frac{1}{2}\left[Q\left(\frac{d}{\sqrt{N_0/2}}\right) + Q\left(\frac{3d}{\sqrt{N_0/2}}\right)\right]\right\} \quad (22)$$

$$P_b(1) = \ \frac{1}{2}\left[Q\left(\frac{d}{\sqrt{N_0/2}}\right) + Q\left(\frac{3d}{\sqrt{N_0/2}}\right)\right]. \quad (23)$$

We now express (23) in terms of the erfc function using

$$Q(x) = \frac{1}{2} erfc\left(\frac{x}{\sqrt{2}}\right). \qquad (24)$$

Upon substitution, we get

$$P_b(1) = \frac{1}{2}\left[\frac{1}{2} erfc\left(\frac{d}{\sqrt{N_0/2}} \cdot \frac{1}{\sqrt{2}}\right) + \frac{1}{2} erfc\left(\frac{3d}{\sqrt{N_0/2}} \cdot \frac{1}{\sqrt{2}}\right)\right] \qquad (25)$$

$$P_b(1) = \frac{1}{4}\left[erfc\left(\frac{d}{\sqrt{N_0}}\right) + erfc\left(\frac{3d}{\sqrt{N_0}}\right)\right]. \qquad (26)$$

But we know that $d = \sqrt{\frac{2\mathrm{E_b}}{5}}$ from (6). Thus, using this in (26) we get

$$P_b(1) = \frac{1}{4}\left[erfc\left(\frac{1}{\sqrt{N_0}} \cdot \sqrt{\frac{2\mathrm{E_b}}{5}}\right) + erfc\left(\frac{3}{\sqrt{N_0}} \cdot \sqrt{\frac{2\mathrm{E_b}}{5}}\right)\right] \qquad (27)$$

$$P_b(1) = \frac{1}{4}\left[erfc\left(\sqrt{\frac{2\mathrm{E_b}}{5N_0}}\right) + erfc\left(3\sqrt{\frac{2\mathrm{E_b}}{5N_0}}\right)\right]. \qquad (28)$$

But we also know that the SNR per bit is $r = \frac{\mathrm{E_b}}{N_0}$ so we can express $P_b(1)$ in terms of r as in

$$P_b(1) = \frac{1}{4}\left[erfc\left(\sqrt{\frac{2r}{5}}\right) + erfc\left(3\sqrt{\frac{2r}{5}}\right)\right]. \qquad (29)$$

Note that (29) is equation (5) in [1].

3.1.b. Expression for $P_b(2)$

Now, let us look at the probability of error for the second bits of I and Q. First, just as we did in 3.2.a, we disregard i_1, q_1 and consider i_2, q_2 only. To get $P_b(2)$, we take the average of the single bit error rates $P_b(i_2)$ and $P_b(q_2)$ as in

$$P_b(2) = \frac{1}{2} P_b(i_2) + \frac{1}{2} P_b(q_2)\,. \qquad (30)$$

3.1.b.i. Deriving $P_b(i_2)$

Again, we "divide and conquer" by decomposing the problem into a smaller one. So we first consider the bit error probability of i_2

$$P_b(i_2) = \frac{1}{2} P_b(i_2 = 0) + \frac{1}{2} P_b(i_2 = 1)\,. \qquad (31)$$

If $i_2 = 0,$ the point has two possible positions – left or right of the Q-channel axis. Thus we can have

$$P_b(i_2 = 0) = \frac{1}{2}P_b(i_2 = 0, left) + \frac{1}{2}P_b(i_2 = 0, right) \qquad (32)$$

where left means $i_1 = 1$ (left of Q-channel axis) and right means $i_1 = 0$ (right of Q-channel axis). But it is clear that we get an error for the points in the left column when $3d < n_1 < -d$ and the points on the right column when $d < n_1 < -3d$. So we have

$$P_b(i_2 = 0, left) = \Pr[3d < n_1 < -d] \qquad (33)$$

and

$$P_b(i_2 = 0, right) = \Pr[d < n_1 < -3d]. \qquad (34)$$

But we know that $n_1 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$ so,

$$\Pr[3d < n_1 < -d] = Q\left(\frac{-d-0}{\sqrt{N_0/2}}\right) + Q\left(\frac{3d-0}{\sqrt{N_0/2}}\right) = Q\left(\frac{d}{\sqrt{N_0/2}}\right) + Q\left(\frac{3d}{\sqrt{N_0/2}}\right). \qquad (35)$$

Thus,

$$P_b(i_2 = 0, left) = Q\left(\frac{d}{\sqrt{N_0/2}}\right) + Q\left(\frac{3d}{\sqrt{N_0/2}}\right). \qquad (36)$$

Similarly,

$$\Pr[d < n_1 < -3d] = Q\left(\frac{d-0}{\sqrt{N_0/2}}\right) + Q\left(\frac{-3d-0}{\sqrt{N_0/2}}\right) = Q\left(\frac{d}{\sqrt{N_0/2}}\right) + Q\left(\frac{3d}{\sqrt{N_0/2}}\right) \qquad (37)$$

so,

$$P_b(i_2 = 0, right) = Q\left(\frac{d}{\sqrt{N_0/2}}\right) + Q\left(\frac{3d}{\sqrt{N_0/2}}\right). \qquad (38)$$

Going back to (32), we now have

$$P_b(i_2 = 0) = \frac{1}{2}\left[Q\left(\frac{d}{\sqrt{N_0/2}}\right) + Q\left(\frac{3d}{\sqrt{N_0/2}}\right)\right] + \frac{1}{2}\left[Q\left(\frac{d}{\sqrt{N_0/2}}\right) + Q\left(\frac{3d}{\sqrt{N_0/2}}\right)\right] \qquad (39)$$

which simplifies to

$$P_b(i_2 = 0) = Q\left(\frac{d}{\sqrt{N_0/2}}\right) + Q\left(\frac{3d}{\sqrt{N_0/2}}\right). \qquad (40)$$

Next we look for $P_b(i_2 = 1)$ by using

$$P_b(i_2 = 1) = \frac{1}{2}P_b(i_2 = 1, left) + \frac{1}{2}P_b(i_2 = 1, right) \qquad (41)$$

where left means $i_1 = 1$ (left of -2d axis) and right means $i_1 = 0$ (left of -2d axis). But we get an error for the points on the left column when $d < n_1 < 5d$ and $-5d < n_1 < -d$ for the points on the right column. This can be expressed as,

$$P_b(i_2 = 1, left) = \Pr[d < n_1 < 5d]\ . \qquad (42)$$

and

$$P_b(i_2 = 1, right) = \Pr[-5d < n_1 < -d] = Q\left(\frac{d}{\sqrt{N_0/2}}\right) - Q\left(\frac{5d}{\sqrt{N_0/2}}\right).. \qquad (43)$$

But since $n_1 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$ we can get

$$\Pr[d < n_1 < 5d] = Q\left(\frac{d-0}{\sqrt{N_0/2}}\right) - Q\left(\frac{5d-0}{\sqrt{N_0/2}}\right) = Q\left(\frac{d}{\sqrt{N_0/2}}\right) - Q\left(\frac{5d}{\sqrt{N_0/2}}\right). \quad (44)$$

Thus,

$$P_b(i_2 = 1, left) = Q\left(\frac{d}{\sqrt{N_0/2}}\right) - Q\left(\frac{5d}{\sqrt{N_0/2}}\right). \quad (45)$$

Similarly,

$$\Pr[-5d < n_1 < -d] = Q\left(\frac{-d-0}{\sqrt{N_0/2}}\right) - Q\left(\frac{-5d-0}{\sqrt{N_0/2}}\right) = Q\left(\frac{d}{\sqrt{N_0/2}}\right) - Q\left(\frac{5d}{\sqrt{N_0/2}}\right) \quad (46)$$

giving us

$$P_b(i_2 = 1, right) = Q\left(\frac{d}{\sqrt{N_0/2}}\right) - Q\left(\frac{5d}{\sqrt{N_0/2}}\right). \quad (47)$$

Now that we have (45) and (47), we can find (41) as

$$P_b(i_2 = 1) = \frac{1}{2}\left[Q\left(\frac{d}{\sqrt{N_0/2}}\right) - Q\left(\frac{5d}{\sqrt{N_0/2}}\right)\right] + \frac{1}{2}\left[Q\left(\frac{d}{\sqrt{N_0/2}}\right) - Q\left(\frac{5d}{\sqrt{N_0/2}}\right)\right] \quad (48)$$

and

$$P_b(i_2 = 1) = Q\left(\frac{d}{\sqrt{N_0/2}}\right) - Q\left(\frac{5d}{\sqrt{N_0/2}}\right). \quad (49)$$

We now use (40) and (49) in (31) to get

$$P_b(i_2) = \frac{1}{2}\left[Q\left(\frac{d}{\sqrt{N_0/2}}\right) + Q\left(\frac{3d}{\sqrt{N_0/2}}\right)\right] + \frac{1}{2}\left[Q\left(\frac{d}{\sqrt{N_0/2}}\right) - Q\left(\frac{5d}{\sqrt{N_0/2}}\right)\right] \quad (50)$$

$$P_b(i_2) = \frac{1}{2}\left[2 \cdot Q\left(\frac{d}{\sqrt{N_0/2}}\right) + Q\left(\frac{3d}{\sqrt{N_0/2}}\right) - Q\left(\frac{5d}{\sqrt{N_0/2}}\right)\right] \quad (51)$$

3.1.b.ii. Deriving $\boldsymbol{P_b(q_2)}$

If we rotate the whole signal constellation 90 degrees clockwise, q_2 will be in the same configuration as i_2 as in the previous section. Thus, we will get the the same computation and values leading to

$$P_b(q_2) = \frac{1}{2}\left[2 \cdot Q\left(\frac{d}{\sqrt{N_0/2}}\right) + Q\left(\frac{3d}{\sqrt{N_0/2}}\right) - Q\left(\frac{5d}{\sqrt{N_0/2}}\right)\right]. \quad (52)$$

3.1.b.iii. $\boldsymbol{P_b(2)}$ computations

Finally, we can compute the probability of error for the second bits of the I and Q components using (51) and (52) in (30) as

$$P_b(2) = \frac{1}{2}\left\{\frac{1}{2}\left[2 \cdot Q\left(\frac{d}{\sqrt{N_0/2}}\right) + Q\left(\frac{3d}{\sqrt{N_0/2}}\right) - Q\left(\frac{5d}{\sqrt{N_0/2}}\right)\right]\right\} + \frac{1}{2}\left\{\frac{1}{2}\left[2 \cdot Q\left(\frac{d}{\sqrt{N_0/2}}\right) + Q\left(\frac{3d}{\sqrt{N_0/2}}\right) - Q\left(\frac{5d}{\sqrt{N_0/2}}\right)\right]\right\} \quad (53)$$

which simplifies to

$$P_b(2) = \frac{1}{2}\left[2 \cdot Q\left(\frac{d}{\sqrt{N_0/2}}\right) + Q\left(\frac{3d}{\sqrt{N_0/2}}\right) - Q\left(\frac{5d}{\sqrt{N_0/2}}\right)\right]. \quad (54)$$

Using $Q(x) = \frac{1}{2} erfc\left(\frac{x}{\sqrt{2}}\right)$ in (54), we get,

$$P_b(2) = \frac{1}{2}\left[2 \cdot \frac{1}{2} erfc\left(\frac{d}{\sqrt{N_0/2}} \cdot \frac{1}{\sqrt{2}}\right) + \frac{1}{2} erfc\left(\frac{3d}{\sqrt{N_0/2}} \cdot \frac{1}{\sqrt{2}}\right) - \frac{1}{2} erfc\left(\frac{5d}{\sqrt{N_0/2}} \cdot \frac{1}{\sqrt{2}}\right)\right] \quad (55)$$

and

$$P_b(2) = \frac{1}{4}\left[2 \cdot erfc\left(\frac{d}{\sqrt{N_0}}\right) + erfc\left(\frac{3d}{\sqrt{N_0}}\right) - erfc\left(\frac{5d}{\sqrt{N_0}}\right)\right]. \quad (56)$$

Using $d = \sqrt{\frac{2\mathrm{E_b}}{5}}$ in $P_b(2)$, we get

$$P_b(2) = \frac{1}{4}\left[2 \cdot erfc\left(\frac{1}{\sqrt{N_0}} \cdot \sqrt{\frac{2\mathrm{E_b}}{5}}\right) + erfc\left(\frac{3}{\sqrt{N_0}} \cdot \sqrt{\frac{2\mathrm{E_b}}{5}}\right) - erfc\left(\frac{5}{\sqrt{N_0}} \cdot \sqrt{\frac{2\mathrm{E_b}}{5}}\right)\right] \quad (57)$$

and

$$P_b(2) = \frac{1}{4}\left[2 \cdot erfc\left(\sqrt{\frac{2\mathrm{E_b}}{5N_0}}\right) + erfc\left(3\sqrt{\frac{2\mathrm{E_b}}{5N_0}}\right) - erfc\left(5\sqrt{\frac{2\mathrm{E_b}}{5N_0}}\right)\right]. \quad (58)$$

But we know that the SNR per bit is $r = \frac{\mathrm{E_b}}{N_0}$ so we can express $P_b(2)$ in terms of r as

$$P_b(2) = \frac{1}{4}\left[2 \cdot erfc\left(\sqrt{\frac{2r}{5}}\right) + erfc\left(3\sqrt{\frac{2r}{5}}\right) - erfc\left(5\sqrt{\frac{2r}{5}}\right)\right] \quad (59)$$

Note that this is equation (8) in [1].

3.1.c. Exact expression for square 16-QAM bit error rate

Now that we have $P_b(1)$ and $P_b(2)$, we can compute for the bit error probability for square 16-QAM by taking the average of the conditional error probabilities using

$$P_b = \frac{1}{2}\sum_{k=1}^{2} P_b(k). \quad (60)$$

Note that this is equation 9 in [1].

3.2. BER of 64-QAM

Figure 3 shows the signal constellation for a square 64-QAM where each symbol is represented by six bits interleaved – $i_1, q_1, i_2, q_2, i_3, q_3$. Also, note that

$$d = \sqrt{\frac{3\cdot(\log_2 \mathrm{M})\cdot \mathrm{E_b}}{2(M-1)}} = \sqrt{\frac{3\cdot(\log_2 64)\cdot \mathrm{E_b}}{2(64-1)}} = \sqrt{\frac{\mathrm{E_b}}{7}}. \qquad (61)$$

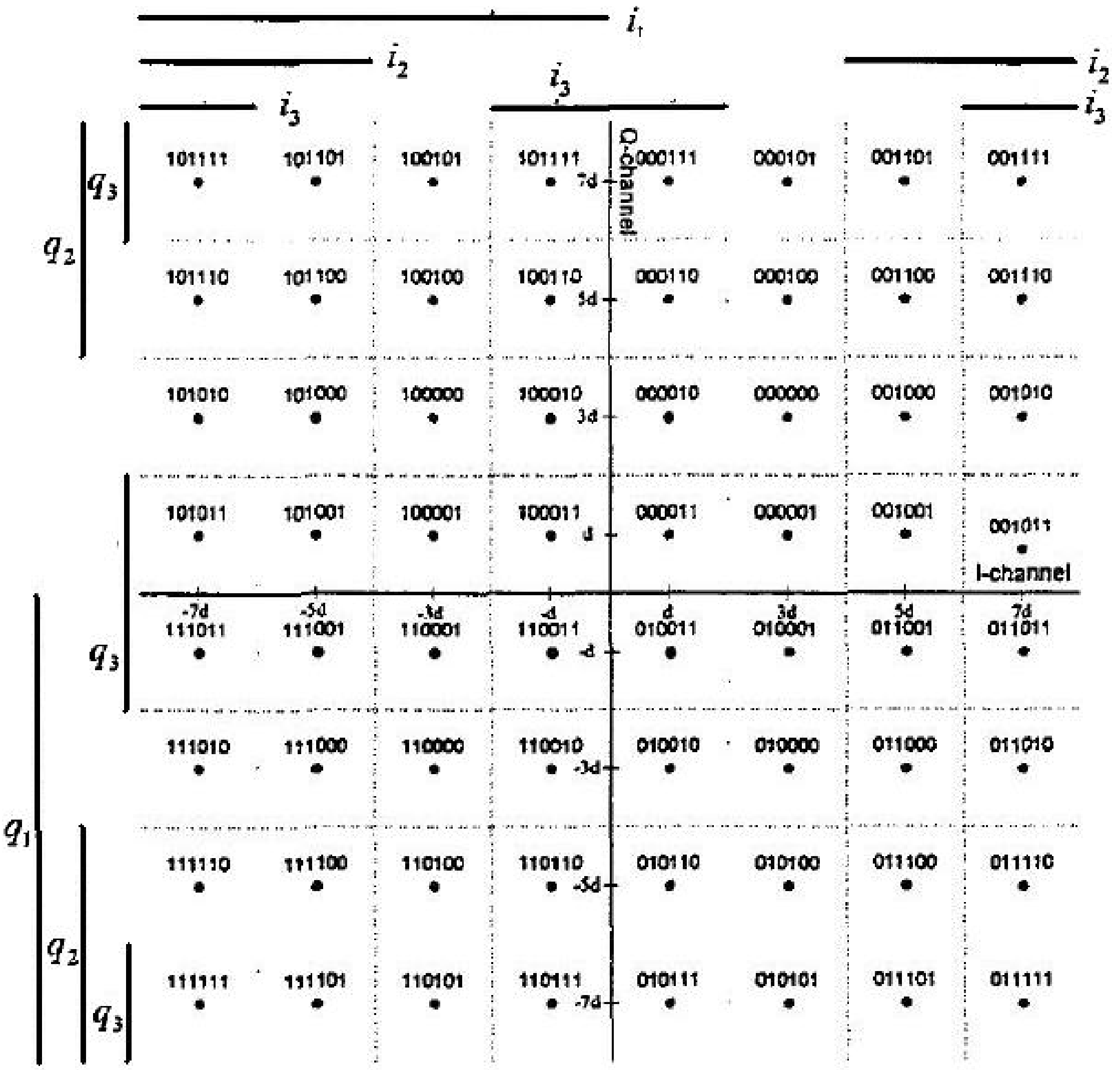

Figure 3. Signal constellation of square 64-QAM [1]

We can find the probability of bit error expression by implementing the same method we used in the previous section but we consider three cases this time – $P_b(1)$, $P_b(2)$ and $P_b(3)$ where $P_b(k)$ indicates the probability that the kth bit of I and Q are in error.

The resulting expressions for $P_b(1)$, $P_b(2)$ and $P_b(3)$ are

$$P_b(1) = \frac{1}{8}\left[erfc\left(\sqrt{\frac{r}{7}}\right) + erfc\left(3\sqrt{\frac{r}{7}}\right) + erfc\left(5\sqrt{\frac{r}{7}}\right) + erfc\left(7\sqrt{\frac{r}{7}}\right)\right], \qquad (62)$$

$$P_b(2) = \frac{1}{8}\left[2 \cdot erfc\left(\sqrt{\frac{r}{7}}\right) + 2 \cdot erfc\left(3\sqrt{\frac{r}{7}}\right) + erfc\left(5\sqrt{\frac{r}{7}}\right) + erfc\left(7\sqrt{\frac{r}{7}}\right) - erfc\left(9\sqrt{\frac{r}{7}}\right) - erfc\left(11\sqrt{\frac{r}{7}}\right)\right], \qquad (63)$$

and

$$P_b(3) = \frac{1}{8}\left[4 \cdot erfc\left(\sqrt{\frac{r}{7}}\right) + 3 \cdot erfc\left(3\sqrt{\frac{r}{7}}\right) - 3 \cdot erfc\left(5\sqrt{\frac{r}{7}}\right) - 2 \cdot erfc\left(7\sqrt{\frac{r}{7}}\right) + 2 \cdot erfc\left(9\sqrt{\frac{r}{7}}\right) + erfc\left(11\sqrt{\frac{r}{7}}\right) - erfc\left(13\sqrt{\frac{r}{7}}\right)\right] \qquad (64)$$

which are equations (11), (13) and (15) respectively, taken from [1]. And just as in the previous section, we get the final BER by averaging using

$$P_b = \frac{1}{3}\sum_{k=1}^{3} P_b(k). \qquad (65)$$

3.3. BER of 256-QAM

For 256-QAM, each symbol is represented by eight interleaved bits – $i_1, q_1, i_2, q_2, i_3, q_3, i_4, q_4$. Also, note that

$$d = \sqrt{\frac{3\cdot(\log_2 \mathrm{M})\cdot \mathrm{E_b}}{2(M-1)}} = \sqrt{\frac{3\cdot(\log_2 256)\cdot \mathrm{E_b}}{2(256-1)}} = \sqrt{\frac{4\mathrm{E_b}}{85}}. \qquad (66)$$

And just as in section 3.1, we can derive the expressions for $P_b(1)$, $P_b(2)$, $P_b(3)$ and $P_b(4)$ as

$$P_b(1) = \frac{1}{16}\left[erfc\left(\sqrt{\frac{5r}{48}}\right) + erfc\left(3\sqrt{\frac{5r}{48}}\right) + erfc\left(5\sqrt{\frac{5r}{48}}\right) + erfc\left(7\sqrt{\frac{5r}{48}}\right) + erfc\left(9\sqrt{\frac{5r}{48}}\right) + erfc\left(11\sqrt{\frac{5r}{48}}\right) + erfc\left(13\sqrt{\frac{5r}{48}}\right) + erfc\left(15\sqrt{\frac{5r}{48}}\right)\right], \qquad (67)$$

$$P_b(2) = \frac{1}{16}\left[2 \cdot erfc\left(\sqrt{\frac{5r}{48}}\right) + 2 \cdot erfc\left(3\sqrt{\frac{5r}{48}}\right) + 2 \cdot erfc\left(5\sqrt{\frac{5r}{48}}\right) + 2 \cdot erfc\left(7\sqrt{\frac{5r}{48}}\right) + erfc\left(9\sqrt{\frac{5r}{48}}\right) + erfc\left(11\sqrt{\frac{5r}{48}}\right) + erfc\left(13\sqrt{\frac{5r}{48}}\right) + erfc\left(15\sqrt{\frac{5r}{48}}\right) - erfc\left(17\sqrt{\frac{5r}{48}}\right) - erfc\left(19\sqrt{\frac{5r}{48}}\right) - erfc\left(21\sqrt{\frac{5r}{48}}\right) - erfc\left(23\sqrt{\frac{5r}{48}}\right)\right], \qquad (68)$$

$$P_b(3) = \frac{1}{16}\left[4 \cdot erfc\left(\sqrt{\frac{5r}{48}}\right) + 4 \cdot erfc\left(3\sqrt{\frac{5r}{48}}\right) + 3 \cdot erfc\left(5\sqrt{\frac{5r}{48}}\right) + 3 \cdot erfc\left(7\sqrt{\frac{5r}{48}}\right) - 3 \cdot erfc\left(9\sqrt{\frac{5r}{48}}\right) - 3 \cdot erfc\left(11\sqrt{\frac{5r}{48}}\right) - 2 \cdot erfc\left(13\sqrt{\frac{5r}{48}}\right) - 2 \cdot erfc\left(15\sqrt{\frac{5r}{48}}\right) + 2 \cdot erfc\left(17\sqrt{\frac{5r}{48}}\right) + 2 \cdot erfc\left(19\sqrt{\frac{5r}{48}}\right) + erfc\left(21\sqrt{\frac{5r}{48}}\right) + erfc\left(23\sqrt{\frac{5r}{48}}\right) - erfc\left(25\sqrt{\frac{5r}{48}}\right) - erfc\left(27\sqrt{\frac{5r}{48}}\right)\right], \qquad (69)$$

and

$$P_b(4) = \frac{1}{16}\left[8 \cdot erfc\left(\sqrt{\frac{5r}{48}}\right) + 7 \cdot erfc\left(3\sqrt{\frac{5r}{48}}\right) - 7 \cdot erfc\left(5\sqrt{\frac{5r}{48}}\right) - 6 \cdot erfc\left(7\sqrt{\frac{5r}{48}}\right) + 6 \cdot erfc\left(9\sqrt{\frac{5r}{48}}\right) + 5 \cdot erfc\left(11\sqrt{\frac{5r}{48}}\right) - 5 \cdot erfc\left(13\sqrt{\frac{5r}{48}}\right) - 4 \cdot erfc\left(15\sqrt{\frac{5r}{48}}\right) + 4 \cdot erfc\left(17\sqrt{\frac{5r}{48}}\right) + 3 \cdot erfc\left(19\sqrt{\frac{5r}{48}}\right) - 3 \cdot erfc\left(21\sqrt{\frac{5r}{48}}\right) - 2 \cdot erfc\left(23\sqrt{\frac{5r}{48}}\right) + 2 \cdot erfc\left(25\sqrt{\frac{5r}{48}}\right) - erfc\left(27\sqrt{\frac{5r}{48}}\right)\right] \qquad (70)$$

which are equations (18), (20), (22) and (24) in [1], respectively.

Finally, the final BER for 256-QAM is computed using equation (25) in [1] given by

$$P_b = \frac{1}{4}\sum_{k=1}^{4} P_b(k). \qquad (71)$$

4. General BER Expression for Square M-ary QAM

Now that we have the expressions for the conditional probabilities for M = 16, 64 and 256, we can use induction to determine a regular pattern in the expressions. The first observation is that for M = 16, 64 and 256, each conditional probability $P_b(k)$ has a factor of $\frac{1}{\sqrt{M}}$.

For the next discussion, let us use the following notation (for conciseness and this will help see the patterns easier later on):

$$P_b(k) = \frac{1}{\sqrt{M}}\left[\sum R \cdot erfc\left(S\sqrt{\frac{3(\log_2 M)\cdot r}{2(M-1)}}\right)\right] = \frac{1}{\sqrt{M}}\sum\langle R, S\rangle \qquad (72)$$

so that we can represent for example for M = 16,

$$P_b(2) = \frac{1}{4}\left[2 \cdot erfc\left(\sqrt{\frac{2\mathrm{E_b}}{5N_0}}\right) + erfc\left(3\sqrt{\frac{2\mathrm{E_b}}{5N_0}}\right) - erfc\left(5\sqrt{\frac{2\mathrm{E_b}}{5N_0}}\right)\right] = \frac{1}{4}[\langle 2,1\rangle + \langle 1,3\rangle + \langle -1,5\rangle]. \quad (73)$$

Thus, if we temporarily disregard the factor, we can represent $P_b(2)$ or any $P_b(k)$ with just the $\langle R, S\rangle$ values as shown in the next tables.

Table 1. $\langle R, S\rangle$ components of $P_b(1)$ for M = 16, 64 and 256

k = 1		
M = 16	**M = 64**	**M = 256**
1,1	1,1	1,1
1,3	1,3	1,3
	1,5	1,5
	1,7	1,7
		1,9
		1,11
		1,13
		1,15

From Table 1, we can observe that for k = 1, R is always equal to 1 and there are $\sqrt{M}/2$ terms. Also, the S values are always increasing as 1, 3, 5, 7... for k = 1.

Table 2. $\langle R, S\rangle$ components of $P_b(\log_2 \sqrt{M})$ for M = 16, 64 and 256

k = $\log_2 \sqrt{M}$		
M = 16 (k = 2)	**M = 64 (k = 3)**	**M = 256 (k = 4)**
2,1	4,1	8,1
1,3	3,3	7,3
-1,5	-3,5	-7,5
	-2,7	-6,7
	2,9	6,9
	1,11	5,11
	-1,13	-5,13
		-4,15
		4,17
		3,19
		-3,21
		-2,23
		2,25
		1,27
		-1,29

For $k = \log_2 \sqrt{M}$ (highest value of k for any M), we observe that the S values are increasing as 1, 3, 5, 7... which is similar to that in $k = 1$. Another observation is that R starts with $\sqrt{M}/2$ and decreases (absolutely) and it continuously changes sign until it reaches -1 [1].

Table 3. $\langle R, S \rangle$ components of $P_b(k)$ for M = 16, 64 and 256 for other values of k

M = 64 (k = 2)	M = 256 (k = 2)	M = 256 (k = 3)
2,1	2,1	4,1
2,3	2,3	4,3
1,5	2,5	3,5
1,7	2,7	3,7
-1,9	1,9	-3,9
-1, 11	1,11	-3,11
	1,13	-2,13
	1,15	-2,15
	-1,17	2,17
	-1,19	2,19
	-1,21	1,21
	-1,23	1,23
		-1,25
		-1,27

For other values of k, similar (but not exactly the same) rules apply, that is S increases as 1, 3, 5, 7... and R starts with $\sqrt{M}/2$ and decreases (absolutely) and it continuously changes sign until it reaches -1. Each term for each conditional probability $P_b(k)$ can be indexed by $j = 0, 1, 2 \ldots (1 - 2^{-k}) \cdot \sqrt{M} - 1$, where $(1 - 2^{-k}) \cdot \sqrt{M}$ is the actual number of terms per conditional probability. As we can see from Table 3, the pattern for M = 256, k = 2 is slightly different due to the fact that $P_b(2)$ has $(1 - 2^{-2}) \cdot \sqrt{256} = 12$ terms and R has to start with $\sqrt{M}/2$ and end with -1.

More generally, the following formula for conditional bit error probability is derived:

$$P_b(k) = \frac{1}{\sqrt{M}} \sum_{j=0}^{(1-2^{-k})\cdot\sqrt{M}-1} \left[(-1)^{\left\lfloor \frac{j \cdot 2^{k-1}}{\sqrt{M}} \right\rfloor} \cdot \left(2^{k-1} - \left\lfloor \frac{j \cdot 2^{k-1}}{\sqrt{M}} - \frac{1}{2} \right\rfloor \right) \cdot erfc\left((2 \cdot j + 1) \sqrt{\frac{3(\log_2 M) \cdot r}{2(M-1)}} \right) \right]. \quad (74)$$

where $\lfloor x \rfloor$ denotes the largest integer not grater than x; also called floor function. Note that this is equation (27) from [1].

Given this formula, the conditional probabilities can be computed. Finally, the exact Bit Error Rate for a square M-QAM where $M = 2^N$ and N is even can be obtained by averaging (equation (28) in [1]),

$$P_b = \frac{1}{\log_2 \sqrt{M}} \sum_{k=1}^{\log_2 \sqrt{M}} P_b(k). \qquad (75)$$

5. Observations

Given (74) and (75) we can now get an expression for a square M-ary QAM system where $M = 2^N$ and N is even.

For M = 4, equation (75) reduces to the BER of quadrature phase shift keying (QPSK) signal [1].

$$P_b = \frac{1}{\log_2 \sqrt{4}} \sum_{k=1}^{\log_2 \sqrt{4}} P_b(k) \qquad (76)$$

$$P_b(1) = \frac{1}{\sqrt{4}} \sum_{j=0}^{(1-2^{-1})\cdot\sqrt{4}-1} \left[(-1)^{\left\lfloor \frac{j\cdot 2^{1-1}}{\sqrt{4}} \right\rfloor} \cdot \left(2^{1-1} - \left\lfloor \frac{j\cdot 2^{1-1}}{\sqrt{4}} - \frac{1}{2} \right\rfloor \right) \cdot erfc\left((2\cdot j+1)\sqrt{\frac{3(\log_2 4)\cdot r}{2(4-1)}} \right) \right]$$
(77)

$$P_b(1) = \frac{1}{2} \sum_{j=0}^{0} \left[(-1)^{\left\lfloor \frac{j}{2} \right\rfloor} \cdot \left(1 - \left\lfloor \frac{j}{2} - \frac{1}{2} \right\rfloor \right) \cdot erfc\left((2\cdot j+1)\sqrt{r} \right) \right] = \frac{1}{2}(-1)^0 \cdot (1-0) \cdot erfc\left((2\cdot 0+1)\sqrt{r} \right) \quad (78)$$

$$P_b = \frac{1}{2} erfc(\sqrt{r}) \qquad (79)$$

Equation (79) is equation (29) in [1] (with a difference in the radical sign, possibly a typographical error in the paper) and to check for correctness, we do a conventional derivation of P_b.

Figure 4 below shows the constellation for M = 4.

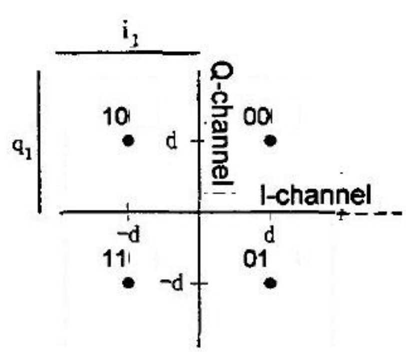

Figure 4. Signal constellation for M = 4

Using the method we applied in section 3.1, we get

$$P_b = \frac{1}{2}P_b(i_1) + \frac{1}{2}P_b(q_1). \qquad (80)$$

But

$$P_b(i_1) = \Pr\,[n_1 > d] \qquad (81)$$

and

$$P_b(q_1) = \Pr\,[n_1 < -d]. \qquad (82)$$

But since $n_1 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$ we can get

$$\Pr[n_1 > d] = Q\left(\frac{d-0}{\sqrt{N_0/2}}\right) = Q\left(\frac{d}{\sqrt{N_0/2}}\right) = P_b(i_1) \qquad (83)$$

and

$$\Pr[n_1 < -d] = Q\left(\frac{-d-0}{\sqrt{N_0/2}}\right) = Q\left(\frac{d}{\sqrt{N_0/2}}\right) = P_b(q_1). \qquad (84)$$

Thus,

$$P_b = \frac{1}{2}Q\left(\frac{d}{\sqrt{N_0/2}}\right) + \frac{1}{2}Q\left(\frac{d}{\sqrt{N_0/2}}\right) = Q\left(\frac{d}{\sqrt{N_0/2}}\right). \qquad (85)$$

And finally, expressing (85) in terms of erfc and using

$$d = \sqrt{\frac{3\cdot(\log_2 \mathrm{M})\cdot \mathrm{E_b}}{2(M-1)}} = \sqrt{\frac{3\cdot(\log_2 4)\cdot \mathrm{E_b}}{2(4-1)}} = \sqrt{\mathrm{E_b}} \qquad (86)$$

we get

$$P_b = Q\left(\sqrt{\frac{\mathrm{E_b}}{N_0/2}}\right) = \frac{1}{2}erfc(\sqrt{r}) \qquad (87)$$

which verifies our result in (79).

As an additional observation, we note that the derivation of (74) and (75) is highly dependent on the bit assignments for each point in the constellation. Thus, changing the assignments would change the probability of bit error. Finally, we also observe that it is helpful to use bit assignments that result to symmetry among the bit level decision regions since symmetries simplify the actual computations (at least for the conventional method).

6. Conclusion

This paper was able to describe the derivation in [1] of a closed BER expression for a coherent Gray coded M-ary square QAM signal in an AWGN channel. The method involved the derivation of the BER expressions for M = 16, 64 and 256 using the conventional method and then using the derived expressions in the induction technique to arrive at a generalized BER expression. We have seen from section 3.1 that deriving the exact BER expression is very tedious and not straightforward. But using the derived expressions (74) and (75) makes the computations much more convenient as demonstrated in section 5.

7. References

[1] D. Yoon; K. Cho; J. Lee, “Bit Error Probability of M-ary Quadrature Amplitude Modulation” Vehicular Technology Conference, Sept. 2000.

[2] W.J. Weber, III, “Differential Encoding for Multiple Amplitude and Phase Shift Keying Systems,” IEEE Trans. Commun., vol. 26, pp. 385-391, March 1978.