Performance Analysis over fading channels

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Abstract

The project presents a performance analysis over fading channels. It's presented an alternative representation for the Gaussian Q-function that will be used for this analysis. The project focused in one kind of a fading channel, the Rayleigh Fading and two kinds of modulations.

1 – Introduction

Usually in the study of probability of error we only consider the effect of a white Gaussian Noise on the carrier transmitted. However, to make this analysis even closer to the reality, we can also consider the effect of fading channels, which is going to attenuate the amplitude of carrier. To calculate the average probability of error, we will face the Gaussian Q-function. In this paper we will see how an alternative way of representing this function can help to evaluate the probability of symbol or bit error.

2 – Alternative representation of the Q function

The Gaussian Q-function is most used to characterize the error probability performance of digital signals communicated over an AWGN channel. Q(x) is defined as the complement of the CDF corresponding to the normalized Gaussian random variable X. And it's given by:

$$Q(x)=\int_{x}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)dy \qquad (1)$$

However, this representation suffers from two disadvantages involving the lower and the upper limit. The infinitive upper limit is not good for the computer evaluation, when using numerical integral evaluation or algorithm techniques.

The second, and more critical, disadvantage is the presence of a variable on the lower limit. For the pure AWGN channel, that does not make a great impact, which is a reason why the classical form of the Q-function has been used for the performance evaluation over the years.

However, for fading channels, that is going to be shown further in this paper, the Q-function also depends on other parameters. Thus, to evaluate the average error probability in the presence of a fading, one must average the Q-function over the fading amplitudes distributions. So, it will be more desirable to have a form independent of other variables in the integral limits and also don't have an infinitive limit. Thus, with those two properties is going to be easy to evaluate the integral by some mathematical manipulation.

This alternative representation, which is going to be use in this paper, is given by:

$$Q(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \exp\left(-\frac{x^2}{2sin^2\Theta}\right)d\Theta \tag{2}$$

And the alternative representation of Q^2 given:

$$Q^2(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{4}} \exp\left(-\frac{x^2}{2sin^2\Theta}\right)d\Theta \tag{3}$$

3 – Fading Channel

Usually in the 242 class, we're used to work with AWGN channels, which mean channels that only suffer the impact of noise. But now, it's going to be study fading channels, which besides the effect of noise, it's affected by the fading amplitude α in the carrier. Where α is a RV with mean-square value $\Omega = \overline{\alpha^2}$ and PDF $\rho_\alpha(\alpha)$ which is dependent on the nature of the radio propagation environment.

So, the signal after passing through the fading channel, it is perturbed by additive white Gaussian noise, which we assume to be independent of the fading amplitude α and characterized by a one-sided spectral density N_0. Equivalently, the received instantaneous signal power is modulated by α^2. Thus, we define the instantaneous signal-to-noise power(SNR) per symbol by $\gamma = \alpha^2 E_b/N_0$ and the average SNR per symbol by $\bar{\gamma} = \Omega E_b/N_0$.

In a nutshell, with only a Gaussian noise n, the output y on the receiver would be given by y=s + n , where s was the symbol supposed sent. However, now we have the carrier attenuated by fading amplitude α. So the output would be y = α s + n. So, the average probability of error would be the expected value of the probability of error taken over the RV α. Thus, the probability of error in a given by:

$$P_s(E) = \int_0^\infty P_s(E;\gamma)\,\rho_\gamma(\gamma)d\gamma \quad (4)$$

4 – Rayleigh

There are several kinds of distributions that characterize multipath fading channel. In this paper, we will study the Rayleigh fading channel. The Rayleigh distribution is used to model multipath fading with no direct line-of-sight(LOS) path. In this case, the PDF of the fading amplitude α is given by:

$$p_\alpha(\alpha) = \frac{2\alpha}{\Omega}\exp\left(-\frac{\alpha^2}{\Omega}\right) \quad (5)$$

Introducing a change of variables we can obtain the PDF in relation to the SNR.

$$\rho_\gamma(\gamma) = \frac{1}{\bar{\gamma}}exp\left(-\frac{\gamma}{\bar{\gamma}}\right) \quad (6)$$

And the MGF for this model is given by:

$$M_\gamma(-s) = \frac{1}{1+s\bar{\gamma}} \quad (7)$$

5 – Useful expressions

In this section some useful expressions for evaluating average error probability performance will be shown. These expressions, which the new representation of Q-Function are used, are going to be used later in this paper for the calculus of probability of error of some types of modulation.

First, we will evaluate an integral which the integrand consists of the product of the Gaussian Q-function and fading PDF, that is:

$$I = \int_0^{\infty} Q\left(a\sqrt{\gamma}\right)\rho_\gamma(\gamma)d\gamma \tag{8}$$

Where a is a constant that depends on the modulation, and γ is the SNR. If we use the classical interpretation of the Q-function (1) to evaluate this integral, we will find some difficulties because of the presence of γ in the lower limit. However, if we use the alternative form (2), it will be a simpler form, and we will be able to change the order of the integrals for instance. So, replacing the Q-function for the alternative integral we have:

$$I = \int_0^{\infty} \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \exp\left(-\frac{a^2\gamma}{2sin^2\Theta}\right)d\Theta\, \rho_\gamma(\gamma)d\gamma \tag{9}$$

Now, if a constant in the limits, we can change the order of the integrals:

$$I = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\left[\int_0^{\infty} \exp\left(-\frac{a^2\gamma}{2sin^2\Theta}\right)\rho_\gamma(\gamma)d\gamma\right]d\Theta \tag{10}$$

It's good to remember that the MGF of a RV, like γ, is given by:

$$M_\gamma(s) \triangleq \int_0^{\infty} e^{s\gamma}\,\rho_\gamma(\gamma)d\gamma \tag{11}$$

Therefore we can replace the term in brackets by the MGF of γ. So, we have that

$$I = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} M_\gamma\left(-\frac{a^2}{2sin^2\Theta}\right)d\Theta \tag{12}$$

And for the Rayleigh distribution we substitute (7) in (12), and it gives:

$$I \triangleq I_r(a,\bar{\gamma}) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\left(1+\frac{a^2\bar{\gamma}}{2sin^2\Theta}\right)^{-1}d\Theta = \frac{1}{2}\left(1-\sqrt{\frac{a^2\bar{\gamma}/2}{1+a^2\bar{\gamma}/2}}\right) \tag{13}$$

So now, we going to repeat this analysis for Q^2. So now:

$$I = \int_0^{\infty} Q^2\left(a\sqrt{\gamma}\right)\rho_\gamma(\gamma)d\gamma \tag{14}$$

Using the alternative representation of Q^2function(3), we have:

$$I = \int_0^{\infty}\frac{1}{\pi}\int_0^{\frac{\pi}{4}}\exp\left(-\frac{a^2\gamma}{2sin^2\Theta}\right)d\Theta\,\rho_\gamma(\gamma)d\gamma \tag{15}$$

And then, using the previous approach for the Rayleigh distribution we have:

$$I \triangleq I_r(a,\bar{\gamma}) = \frac{1}{\pi}\int_0^{\frac{\pi}{4}}\left(1+\frac{a^2\bar{\gamma}}{2sin^2\Theta}\right)^{-1}d\Theta \tag{16}$$

Solving the integral we get:

$$I \triangleq I_r(a,\bar{\gamma}) = \frac{1}{4}\left[1-\sqrt{\frac{a^2\bar{\gamma}/2}{1+a^2\bar{\gamma}/2}}\left(\frac{4}{\pi}tan^{-1}\sqrt{\frac{1+a^2\bar{\gamma}/2}{a^2\bar{\gamma}/2}}\right)\right] \tag{17}$$

6 – PAM

For M-PAM SEP over the AWGN is given by

$$P_s(E) = 2\left(\frac{M-1}{M}\right) Q\left(\sqrt{\frac{6E_s}{N_o(M^2-1)}}\right) \tag{18}$$

Now if we consider the Rayleigh fading channel, the SNR instead of be E_s/N_0 will be $\gamma log_2 M$, given that $E_b = E_s/log_2 M$. So substituting in (18) and using (4) , we obtain:

$$P_s(E) = 2\left(\frac{M-1}{M}\right)\int_0^{\infty} Q\left(\sqrt{\frac{6\,\gamma log_2 M}{(M^2-1)}}\right)\rho_\gamma(\gamma)d\gamma \tag{19}$$

So using the result (13), with $a^2 = 6\ log_2 M/(M^2-1)$:

$$P_s(E) = \left(\frac{M-1}{M}\right)\left(1-\sqrt{\frac{3\bar{\gamma}_s}{M^2-1+3\bar{\gamma}_s}}\right) \tag{20}$$

7 – QAM

For M-PAM SEP over the AWGN is given by

$$P_s(E) = 4\left(\frac{\sqrt{M}-1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3E_s}{N_o(M-1)}}\right) - 4\left(\frac{\sqrt{M}-1}{\sqrt{M}}\right)^2 Q^2\left(\sqrt{\frac{3E_s}{N_o(M-1)}}\right) \tag{21}$$

To obtain the SEP on the Rayleigh fading channel we substitute E_s/N_0 for $\gamma log_2 M$.

$$P_s(E) = 4\left(\frac{\sqrt{M}-1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3\gamma log_2 M}{(M-1)}}\right) - 4\left(\frac{\sqrt{M}-1}{\sqrt{M}}\right)^2 Q^2\left(\sqrt{\frac{3\gamma log_2 M}{(M-1)}}\right) \tag{22}$$

Applying (13) on the Q and (17) on the Q^2 function, with $a^2 = 3\, log_2 M/(M-1)$:

$$P_s(E) = 2\left(\frac{\sqrt{M}-1}{\sqrt{M}}\right)\left(1-\sqrt{\frac{1.5\bar{\gamma}_s}{(M-1+1.5\bar{\gamma}_s)}}\right)- \left(\frac{\sqrt{M}-1}{\sqrt{M}}\right)^2\left[1-\sqrt{\frac{1.5\bar{\gamma}_s}{M-1+1.5\bar{\gamma}_s}}\left(\frac{4}{\pi}tan^{-1}\sqrt{\frac{M-1+1.5\bar{\gamma}_s}{1.5\bar{\gamma}_s}}\right)\right] \quad (23)$$

8 – Conclusion

With this paper, we conclude that the alternative representation of the Q function can, indeed, help evaluating probability of error, especially over fading chanels, where the fading amplitude is a random variable, which would be in one of the limits in the integral of the traditional form of the Q function. On the other hand, the alternative representation, which the limits of the integral are not variable dependent, make the math manipulations easier and therefore, help to achieve some expressions for the Average probability of error.

9 – References

[1] SIMON, M., ALOUINI, M., *Digital Communication over Fading Channels, Second Edition*. Wiley, 2004.

[2] SIMON, M., ALOUINI, M., *A Unified Approach to the Performance Analysis of Digital Communication over Generalized Fading Channels*. IEEE, 1998.