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Evaluating the Bit Error Rate for M-PSK

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1. Introduction:

In a world where digital communication has evolved from being just a luxury to a necessity, and with an increasing number of clients such as Government bodies, Militaries, Schools, hospital and even banks, now depending on digitally transmitted information from long distances, it will not be incorrect to suggest that the transmission of information could be the line between life and death for most people. Hence the need for efficiently and accurately transmitted information cannot be overemphasized. Therefore, there is a need to, as accurately as possible, estimate the bit error rate of transmitted information with the aim of exploring different ways to reduce it, hence increasing the accuracy of the data being transmitted.

2. Aim:

The aim of this paper is to present an efficient way of calculating the bit error rate of an M-PSK module.

3. Method:

The method that will be invoked involves the use of grey coding to evaluate the error probability of the M-ary modulation schemes mentioned; over an AWGN channel, whereby every nearest neighbour of the constellation points defer by just one bit, thereby reducing to just one bit, the probability of erroneously interpreting the received message as one of the nearest neighbour messages to the original message. This is of course hinged on the assumption that the errors only occur between nearest neighbours. Although, this assumption is not necessarily always true, it is nonetheless a very reliable one. This method will attempt to utilise the geometric properties of the corresponding constellations with respect to its axes to estimate the bit error rate (BER) of the M-PSK detection module.

4. Analysis:

4.1 M-PSK:

Before going on to present this method, it is important to define some of the concepts that will later be referred to, as time goes. For an M-PSK scheme, the general equation for the transmitted signal is given by:

$$s(t) = s_I(t)\cos(2\pi f_c t) + s_Q(t)\sin(2\pi f_c t) \qquad (1)$$

where $s(t)$ is the transmitted message, $s_I(t)$ is the in-phase component of the baseband representation of the signal, $s_Q(t)$ is the quadrature component of the baseband representation of the signal and f_c is the carrier frequency.

Furthermore, the baseband representations of the signals can both be broken down into:

$$s_I(t) = \sum_{k=-\infty}^{\infty} \sqrt{\frac{2E_{TP}}{T}} gT(t-kT)\cos\left(I_k . \frac{2\pi}{M} + \frac{\pi}{M}\right) \qquad (2)$$

$$s_Q(t) = \sum_{k=-\infty}^{\infty} \sqrt{\frac{2E_{TP}}{T}} gT(t-kT)\sin\left(I_k . \frac{2\pi}{M} + \frac{\pi}{M}\right) \qquad (3)$$

Where E_{TP} is the symbol energy, T is the symbol period, $I_k = 0, 1, \ldots, M - 1$

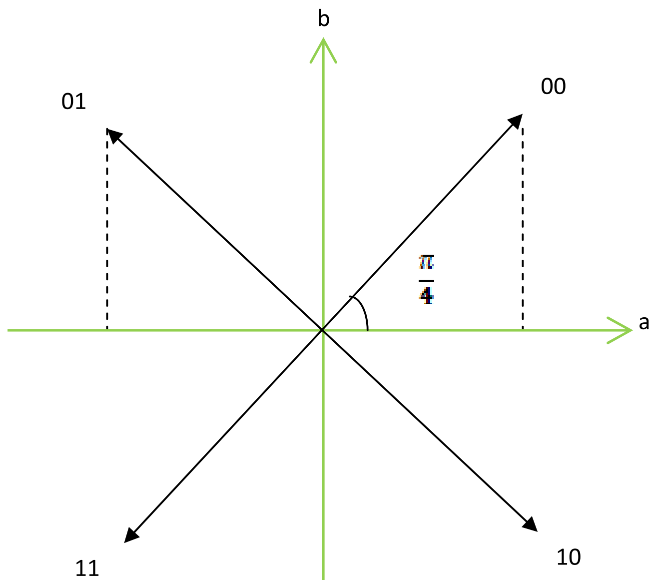
$M = 2^B$ and B is the number of bits/symbol.

4.1.1 Computing BER for M-PSK:

To estimate the BER for M-PSK, we use mathematical induction method by starting with the simple case first. Naturally, 4-PSK should be the first choice as it is the lowest M-PSK beyond B-PSK. The number of bits per symbol is given by:

$$B = log_2 M = log_2 4 = 2$$

Using grey coding, the constellation plot is given by:



4-PSK

If a_0a_1 represent the first and second bit of each constellation point respectively, then it can be observed that the value for a_0 can be decided by what side of the ‘a-axis’ that it lies (i.e. a_0 is 0 at the top side of the ‘a-axis’ and it is 1 at the bottom side of the ‘a-axis’). Likewise, the value for a_1 can be decided by its position with respect to the ‘b-axis’ (i.e. a_1 is 0 at the right side of the ‘b-axis’ and it takes the value 1 at the left side of the ‘b-axis’). Furthermore, if we define P_i as the probability of an error occurring based on the decision made with respect to the i-axis, we observe that P_a and P_b are independent, and that $P_a = P_b$ due to symmetry.

Given the symbols are equiprobable,

$P\left({}^{e}/{a_o}\right) - P_a$ (w.r.t axis a)

And

$P\left({}^{e}/{a_1}\right) = P_b$ (w.r.t axis b)

Hence, the probability of error for 4-PSK is given by:

$$\sum_{i=0}^{b-1} Pa_i \times P\left({}^{e}/{a_i}\right) \qquad 4$$

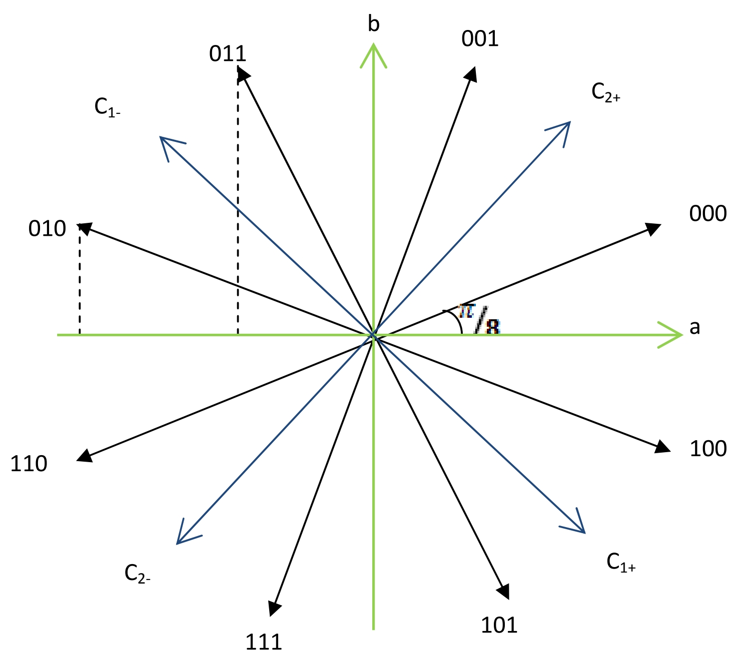
$$P_e 4PSK = \frac{1}{2} P\left({}^{e}/{a_o}\right) + \frac{1}{2} P\left({}^{e}/{a_1}\right)$$

$$= \frac{1}{2}(P_a + P_b)$$

$$= \frac{1}{2}(1 + 1)P_a \qquad 5$$

For 8-PSK,

There are 3 bits per symbol and the constellation plot is given below:



8-PSK

Let the bits in each symbol be $a_0a_1a_2$. While the values of the first two bits a_0a_1 can be decided as in the previous case, there is a new last bit a_2. For bit a_2, neither axis a nor axis b can be used to make definite decisions to determine the value it takes as was the case with 4-PSK, so a close observation of the constellation plot above shows two new independent axes C_1 and C_2, which have been introduced so as to be able to make decisions on what value a_2 takes based on the two new axes. Now, we can determine the value of a_2 without axes 'a' and 'b' as is given below:

As a result of symmetry and because all the decisions made about the axes a, b, C_1 and C_2 are independent, then $P_a = P_b = P_{C1} = P_{C2}$. Also, the axes C_1 and C_2 are independent because they are orthogonal, so:

$$P\left({}^{e}/_{a_2}\right) = P_{C1}(1 - P_{C2}) + P_{C2}(1 - P_{C1})$$

But since the decision made based on axis C_1 is independent of that made based on axis C_2 (Due to orthogonality), the error probability given a_2 can be approximated to:

$$P\left({}^{e}/_{a_2}\right) \cong P_{C1} + P_{C2}$$

Then

$$Pe\ 8PSK = \frac{1}{3\left(P\left({}^{e}/_{a_0}\right) + P\left({}^{e}/_{a_1}\right) + P\left({}^{e}/_{a_2}\right)\right)}$$

$$Pe\ 8PSK = \frac{1}{3(1 + 1 + 2)P_a}$$

It can now be shown that by means of induction,

Now the BER of M-PSK can be determined by calculating P_a. To do this, we know that:

1) For any MPSK signal space representation, there are four quadrants and each quadrant contains the same number of constellation point.

2) This means there are $\frac{M}{4}$ possible values for the vertical distance of each constellation to the a-axis.
3) The value for these distances, by observing the constellation plot is given by:
4) Since all symbols are equiprobable, the probability for any given symbol over an AWGN channel is given by: $\frac{1}{M} Q\left[\sqrt{\frac{2d^2}{N_0}}\right]$

So the probability of error due to the decision made with respect to the a-axis, given any symbol is given by: .

Then,

Where d_i = distance between each symbol and the a-axis.

Now,

$$P(a) = \sum_{i=1}^{\frac{M}{4}} P\left({}^{a}/_{d_i}\right)$$

$$P(a) = \frac{4}{M}\sum_{i=1}^{\frac{M}{4}} Q\left[\sqrt{\frac{2E_{TP}}{N_0}} sin(2i-1)\frac{\pi}{M}\right] \qquad 7$$

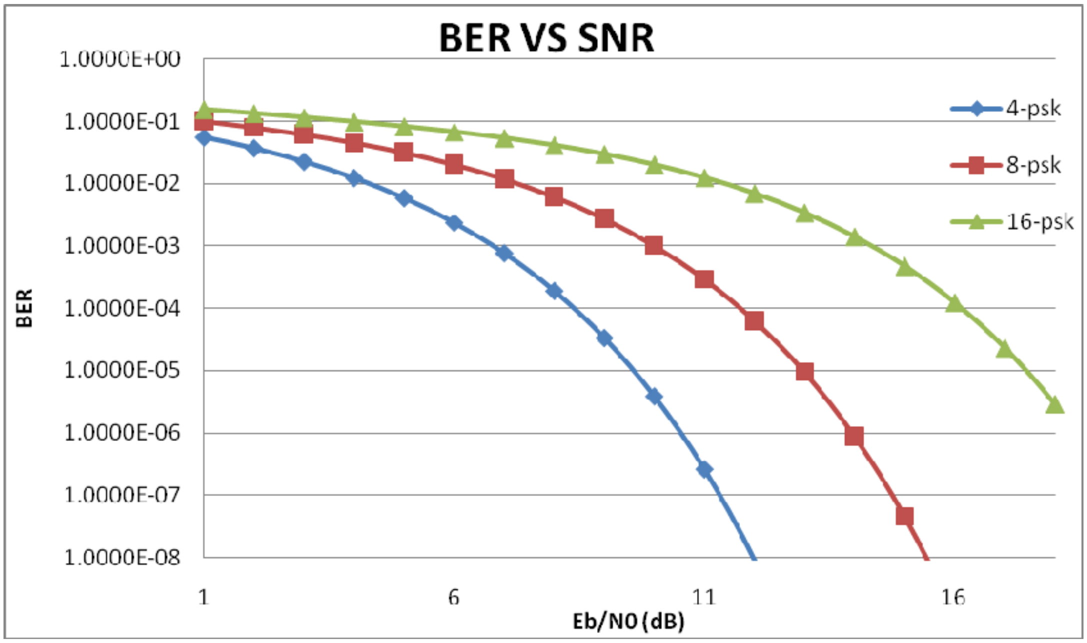
By means of substituting (7) into (6), we get the BER of MPSK as:

But

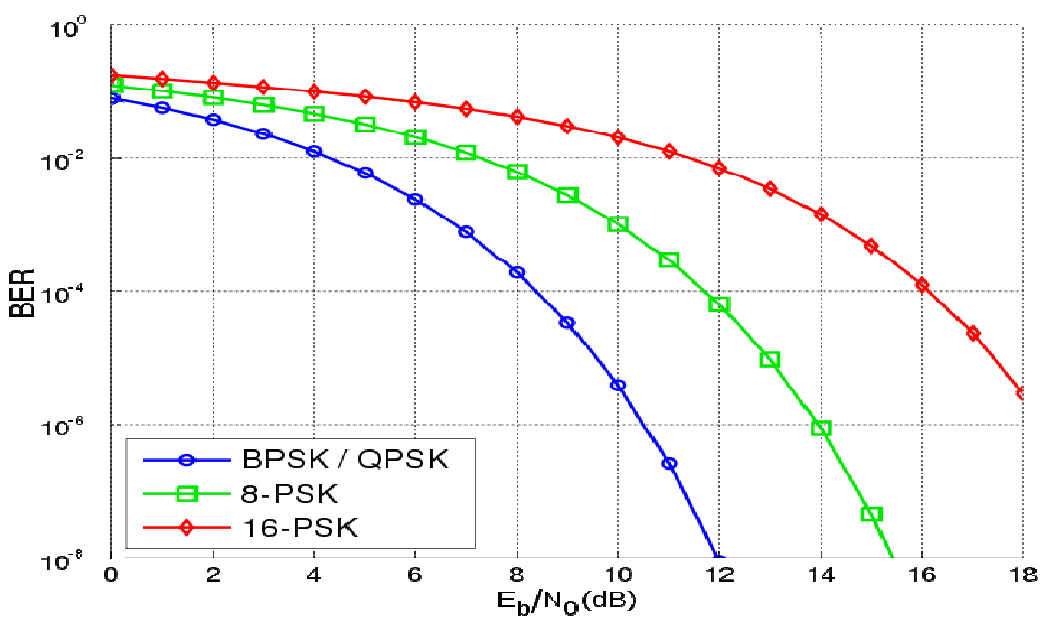
Therefore,

5 Discussion:

Although a very logical approach was employed in order to be able to calculate the BER of an M-PSK, the degree of correctness of the final result will be clearer if the result is compared to the real approximation of BER by means of plotting the BER against the SNR (E_b/N_0). This could be achieved by using MATLAB simulations to compare the results. However, due to the unavailability of MATLAB and time constraints, it has been impossible to present the simulation using MATLAB. However, I was able to use Microsoft Excel to plot the BER VS SNR of 4-PSK, 8-PSK and 16-PSK using the derived approximation, and below it; is the Real BER VS SNR for 4-PSK, 8-PSK and 16-PSK given below:



BER VS SNR of derived approximation



REAL BER VS SNR

As can be observed from the plot above, the approximation is almost exactly the same as the exact value.

Given more time, a possible extension to this project should involve the calculation of the BER of M-QAM using the same approach. With better packages such as MATLAB, it could have been easy to also compare the performances of various M-PSK modules with those of M-QAM given a fixed SNR. It is also possible to use Microsoft excel, but it will be too tedious.

6 Conclusion:

It has been shown that it is possible to estimate the BER of an M-PSK scheme by using the geometric properties of the constellation plot and this estimate has been compared with the real values for various M's by means of plotting BER VS SNR of the schemes. In conclusion, the estimation is to a very high degree, accurate.

7 References:

[1] Jianhua Lu, K.B. Letaief, Justin C-I Chuang and Ming L. Liou "M-PSK and M-QAM BER Computation Using Signal Space Concepts".

[2] Dr Tareq Al-NAffouri.

[3] John W. Craig, "A NEW, SIMPLE AND EXACT RESULT FOR CALCULATING THE PROBABILITY OF ERROR FOR TWO-DIMENSIONAL SIGNAL CONSTELLATIONS" Interstate Electronics Corporation.